Two types of analyses to highlight difficulties in elementary algebra

ISABELLE DEMONTY
University of Liège – Belgium
Isabelle.demonty@ulg.ac.be

Abstract

«What makes the comprehension of elementary algebra such a difficult task for most students?» This is the general problematic that is treated in this article. More specifically, this article presents two types of analyses of the results obtained by beginners of algebra students to a question that implies algebraic transformation. The first analyse proposes an interpretation of the difficulties of the nine items and the second one aims to describe students' profiles. On theses bases, several action paths that aim to help the teachers in this field will be briefly presented.

The purpose of this article is to give pieces of information about the large and complex problematic below: 'What makes the comprehension of elementary algebra such a difficult task for most students?'

This question will be highlighted by reporting results of items concerning simplifying algebraic expressions collected from an external mathematical assessment.

Two types of analyses will be presented. The first one describes the scores by items and the second one tends to describe students' profiles. On these bases, the article will present some action paths that aim to help the teachers in this field.

CONTEXT OF THE RESEARCH

In February 2008, an external assessment was submitted to all the students in French community who were in grade 2, 5 and 8. We collected the data of a representative sample of 1,853 students who were in grade 8. These students came from 100 schools. In every school, one classroom was selected.

In French community of Belgium, the learning of algebra begins in grade 7 with the algebraic transformations and equations solving. At the moment of the test, students assessed have approximately one year and a half of algebra practice.

This external evaluation has two complementary aims:

- To give the political decision-makers information onto the characteristics of our educational system;
- To give teachers an opportunity to compare the results of their students with those obtained by all the students in French Community of Belgium.

After the test, in May 2008, a document that presents the results of the students in French Community of Belgium was sent to the teachers and the political decision-makers. A few months later, in September 2008, teachers received didactical paths to give them ideas in the mathematical topics which were the most difficult for students: algebraic transformation in the field of the numbers and deductive reasoning in geometry.

In this article, we describe the results obtained by students to a question that was compound by 9 items focused on algebraic transformations.

RESULTS

Analysis by item

We can organize the nine items in three sets according to the mathematical procedure which must be used in the transformations.

Three items uses a sum of terms.

$$5a - a$$
$$a^2 + a^2$$
$$2a - 7 + a$$

One item uses a product of factors.

$$7a \,.\, 2a$$

The five others items uses the distributive law in simple cases or in more complicated ones.

$$-2(a + 3)$$
$$4a(3+5a)$$
$$(n + 2) \,.\, 2 + 2$$
$$a - (2 - a)$$
$$(b + 5) \,.\, (b + 6)$$

The table 1 presents the proportion of students who have reduced correctly each of the items.

TABLE 1: Results by items

7a . 2a	79%
5a – a	76%
2a – 7 + a	57%
-2 (a + 3)	55%
4a . (3 + 5a)	47%
$a^2 + a^2$	42%
(b + 5) (b + 7)	42%
(n + 2) . 2 + 2	29%
a – (2 – a)	25%

The results are very different from an item to another. According to Sfard's approach (1991), we can interpret these differences: the students can reduce the two easier expressions with a procedural approach: the reduced expressions have no obvious operation sign.

For the seven others expressions, students have to manage with structural view of the algebraic expressions. We can separate these seven expressions in two categories: the first fifth ones ask for realising one transformation and, in the two more difficult cases, students have to realise two transformations.

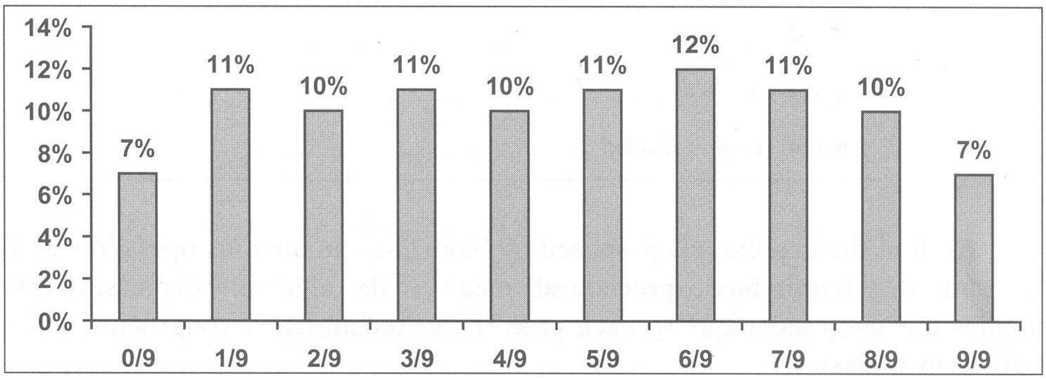
This analysis could let us think that two major difficulties must be resolved in order to respond correctly to all the items: first of all, viewing algebraic expressions from a structural point of view and after this, being able to organize two algebraic transformations.

Analysis by students' profiles

Is it possible to infirmed this interpretation by referring to the students' profiles? If it is the case, there must be an important number of students who reduce correctly two expressions and seven expressions.

The graph below shows the proportion of students who have reduced correctly one expression, two expressions and so one.

GRAPH 1: score of the students to the question



The two suggested analyses seem to lead to different results: the proportion of students in each category is very close except for the very competent students (those who obtained 9/9) and the very weak ones (those who obtained 0/9).

We wanted to go further in the profile's students analysis and to focus in more details on the profiles of the students who have reduced correctly two expressions and seven expressions: do the students who have obtained 2 points have the correct responses for 5a – a and 7a . 2a ? And what about students who failed at only two items: are these two items the most complicated ones, those where two transformations must be done?

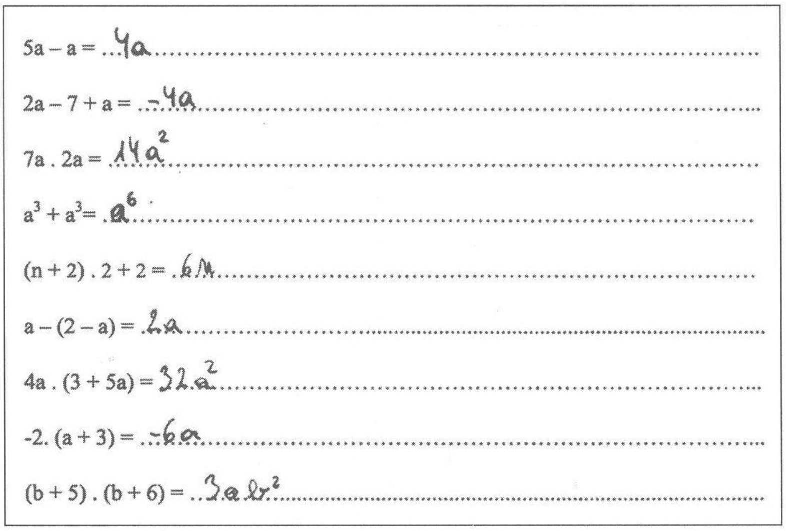
In the two cases, the most frequently observed profile is effectively congruent with the items analysis (20% for 2/9 and 30% for 7/9). But in the two cases, more than 20 different profiles were observed in few cases (less than 4% of students) and these profiles seem to be really haphazard.

How to explain this diversity of cases? An analysis of the responses of 86 students from four classrooms can help to understand better the difficulties of the students. To illustrate this, we suggest to analyse the responses proposed by three students who were in the same classroom: Samir, Laetitia and Nicolas.

■ *Samir: every reduced expression must be without any operation sign*

The figure bellow presents the responses proposed by Samir for the nine items.

FIGURE 1: Samir's production


$5a - a = 4a$
$2a - 7 + a = -4a$
$7a \, . \, 2a = 14a^2$
$a^3 + a^3 = a^6$
$(n + 2) \, . \, 2 + 2 = 6n$
$a - (2 - a) = 2a$
$4a \, . \, (3 + 5a) = 32a^2$
$-2. \, (a + 3) = -6a$
$(b + 5) \, . \, (b + 6) = 3ab^2$

Each of the expressions proposed by Samir has no obvious operation sign. It seems that Samir has a procedural vision of the algebraic expression and applies the same technique in each case. This student has a congruent profile with item analysis.

■ *Laetitia : a good mastery of the last techniques learned*

The case of Laetitia who is in the same classroom is quite different, as shown in figure 2.

FIGURE 2: Laetitia's production.

$5a - a = 4a$

$2a - 7 + a = 2a - 7a = -5a$

$7a . 2a = 14a$

$a^3 + a^3 = a^6$

$(n + 2) . 2 + 2 = 2n + 4 + 2 = 2n + 6$

$a - (2 - a) = -2a + a^2$

$4a . (3 + 5a) = 12a + 20a$

$-2. (a + 3) = -2a - 6$

$(b + 5) . (b + 6) = b^2 + 6b + 5b + 30 = b^2 + 11b + 30$

Laetitia has reduced correctly three algebraic expressions: the simplest one and two of the most complicated ones. It seems that Laetitia has «forgotten» the elementary rules but can manage the transformations that have just been learned in her mathematical course. It seems also that she over generalises the distributive rule to expressions that do not use this rule: the expression «*a – (2-a)*» is treated in the same way than the expression «*-a . (2-a)*».

■ *Nicolas: difficulties seem to appear when several techniques have to be managed*

$5a - a = 4a$

$2a - 7 + a = -7 + 3a$

$7a . 2a = 14a^2$

$a^3 + a^3 = 2a^3$

$(n + 2) . 2 + 2 = 2n . 2 = 4n + 2$

$a - (2 - a) = -2$

$4a . (3 + 5a) = 12a + 20a^2$

$-2. (a + 3) = -2a + -6$

$(b + 5) . (b + 6) = b^2 + 6b + 5b + 30 = b^2 + 11b + 30$

The case of Nicolas is interesting too: Nicolas seems to master correctly the sum of terms, the product of factors and the distributive law in both simple and more complicated cases. But he fails when two of these procedures have to be applied in the same case. To reduce «$(n+2).2+2$», he doesn't hesitate to transform n+2 by 2n even if, in simple cases like «$2a-7+3a$» or «$b^2+11b+30$», he doesn't make this mistake.

These three examples of students' productions lead us to conclude that in a same classroom, the difficulties of students and their mistakes are very different from one case to another one. If Samir has to manage with a conceptual change of his view of the expressions, Laetitia seems to need to review basic transformations that she learned previously in her mathematical course. In the case of Nicolas, it seems that even if the techniques are mastered, he can't mobilise them into more difficult cases.

CONCLUSION

Several authors leaded research on difficulties of young students in simplifying algebraic expressions.

According to Greeno (1982, in Kieran 1992), «*beginning algebra students are consistent neither in their approach to the testing of conditions before performing some operation nor in their process of performing the operations. For example, they might simplify $4(6x-3y)+5x$ as $4(6x-3y+5x)$ on one occasion, but do something else o another one*» (p. 397).

Wenger (1987) has observed similar results even with more advanced learners who didn't seem to see the right things when they analysed the expressions.

Sackur, Drouhard, Maurel and Pecal (1997) explain this situation in this way: when they transform expressions, students don't try to make sense: an algebraic transformation is correct only if it is consistent with THE rule they learned. For example, they explain that the expression «$(a+b)^2$» is not equal to «a^2+b^2» because »the double product» is missing. From their point of view, the respective value of the two expressions isn't a convincing explanation. Eisenberg and Dreyfus (1988) suggest that the teaching of algebra has a responsibility in this situation: according to these authors, a recent trend in algebra leaded to focus teaching on procedures rather than on underlying structure. In this situation, students consider algebra as simply a compendium of rules of procedures that don't have to be understood.

To help student in this field, Sackur *et al.* (1997) advise them to give a sense to algebraic transformation and to consider that an expression like «$a.(2a+3)$» has a numerical value. This numerical value depends on the value assigned to «a». In this perspective, a transformation will be considered as correct only if the numerical value of the expression does not change if the transformation is applied, for all the possible value assigned to a..

In the context of the research project, we proposed, a few months after the test, didactical paths to help teachers in the teaching of early algebra. Two approaches were proposed. They both aim to make student realise algebraic transformations in a context where they have a sense.

The first direction proposed was the analysis of numerical patterns in a context of generalisation or in a context of a proof. If most teachers realise this sort of activity in order to introduce algebra, they don't use to explore them afterward. Among these situations, some can be used when the algebra transformations have been taught, in order to apply them in context that making them sense.

The second proposed direction consists in an individual interview directly focused on algebraic expressions. This technique was elaborated by Sackur *et al.* (1997). The principle of the interview is the following one: the interviewer proposes an algebraic expression to a student and asks him to write down an expression that is not equal to it. On this basis, the student has to elaborate reasoning because he has never learned a rule to find an expression that is different to another one. After this, the interviewer asks the student if the expression proposed will always be different from the first one. This interview aims to convince student that an algebraic transformation is correct if the two expressions are the same for all possible values assigned to the letter (students often explain the correctness of the transformation referring to a rule they learned). Sackur *et al.* (1997) precise that this interview is not convenient for all the students, because but can permit certain students to better understand what they make when they transform algebraic expressions.

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