A method for identification of non-linear multi-degree-of-freedom systems

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Abstract: System identification methods for non-linear aeroelastic systems could find uses in many aeroelastic applications such as validating finite element models and tracking the stability of aircraft during flight flutter testing. The effectiveness of existing non-linear system identification techniques is limited by various factors such as the complexity of the system under investigation and the type of non-linearities present. In this work, a new approach is introduced which can identify multi-degree-of-freedom systems featuring any type of non-linear function, including discontinuous functions. The method is shown to yield accurate identification of three mathematical models of aeroelastic systems containing a wide range of structural non-linearities.

Keywords: aeroelasticity, non-linear systems, multi-degree-of-freedom systems, system identification, bilinear stiffness, cubic stiffness, freeplay stiffness, hysteresis

NOTATION

$a_{1,2}$	aerodynamic lift curve slope for a 3 DOF wing model
$b_{1,2}$	aerodynamic wing moment curve slope for a 3 DOF wing model
c	damping coefficient in a 1 DOF aeroelastic model
$c_{1,2}$	aerodynamic control surface moment curve slope for a 3 DOF wing model
$\mathbf{C}$	damping matrix
$\hat{\mathbf{C}}$	damping matrix pre-multiplied by the inverse of the mass matrix
$\mathbf{C}_{\text{aero}}$	aerodynamic damping matrix
$\hat{\mathbf{C}}_{\text{id}}$	identified $\hat{\mathbf{C}}$
$\mathbf{C}_{\text{struct}}$	structural damping matrix
$\boldsymbol{f}$	restoring force vector
$\boldsymbol{F}$	excitation force vector
$\hat{\boldsymbol{F}}$	excitation force vector pre-multiplied by the inverse of the mass matrix
$\boldsymbol{h}$	generalized restoring force vector
I_γ, $I_{\gamma\theta}$, etc.	second moments of area of a 3 DOF wing model
$k_{1,2}$	stiffness coefficients in a 1 DOF system
k_{a}	aerodynamic stiffness contribution
k_{s}	structural stiffness contribution
k_γ, k_θ, k_β	structural coefficients in a 3 DOF wing model
$\mathbf{K}$	stiffness matrix
$\hat{\mathbf{K}}$	stiffness matrix pre-multiplied by the inverse of the mass matrix
$\mathbf{K}_{\text{aero}}$	aerodynamic stiffness matrix
$\hat{\mathbf{K}}_{\text{id}}$	identified $\hat{\mathbf{K}}$
$\mathbf{K}_{\text{struct}}$	structural stiffness matrix
m	mass coefficient in a 1 DOF model
$\mathbf{M}$	mass matrix
$\mathbf{M}_{\text{u}}$	diagonal modal mass matrix
$\mathbf{M}_{\dot{\theta}}$, $\mathbf{M}_{\dot{\beta}}$	unsteady aerodynamic derivatives
$\boldsymbol{N}$	non-linear vector (including linear contributions)
$\hat{\boldsymbol{N}}$	purely non-linear vector
$\boldsymbol{q}$	displacement vector
r	number of modal coordinates
$\mathbf{R}$	pseudo-inverse of $\boldsymbol{\phi}$
s	span of a 3 DOF wing model
t	time
u	modal coordinates
u_{g}	excitation function in a 1 DOF model
V	velocity
x_{f}	position of the flexural axis in a 3 DOF wing model
x_{ff}, y_{ff}	points of application of the excitation force in a 3 DOF wing model
y	time-dependent variable in a 1 DOF model
β, γ, θ	control surface, heave and pitch coordinates in a 3 DOF wing model
ρ	air density
$\boldsymbol{\phi}$	modal matrix

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1 INTRODUCTION

The use of system identification methods to identify frequencies, dampings and mode shapes is commonplace in the aerospace industry. Such methods are used to analyse ground vibration test data in order to validate finite element models, and also during flight flutter testing to track the stability of aircraft as the flight envelope is expanded. There is a vast literature related to the identification of linear systems and a wide range of methods have been implemented in the aerospace field. However, the identification of systems that contain non-linearities is not yet at a stage where an accurate model of a real full-sized structure, e.g. an aircraft, could be estimated.

There already exist methods like the NARMAX model [**1**, **2**], higher order spectra [**3**] and the restoring force method [**4**, **5**], which can identify aeroelastic systems given the inputs and outputs. However, these methods have still not reached the level of maturity necessary to allow their application to general aeroelastic systems. Both NARMAX and the higher order spectra method are incapable of identifying systems with discontinuous non-linearities, such as bilinear stiffness or freeplay, which are common in aeroelastic systems. The restoring force method does not share this limitation, but its application to multi-degree-of-freedom systems is still problematic.

A further consideration that must be made is whether the identification process is parametric. The analysis of an identified system is much simpler when the terms in the model resulting from the identification process are parametric, i.e. model explicitly the non-linearities present in the system. However, both NARMAX and the restoring force method yield better results when using non-parametric as well as parametric terms. Hence the resulting model contains terms without any physical meaning.

The effects of structural, aerodynamic and, in particular, control system non-linearities upon the aeroelastic behaviour of aircraft are becoming of increasing concern. Recent emphasis has been devoted to the study and prediction of limit cycle oscillations (LCO). Although unsteady computational fluid dynamics (CFD) codes are being developed to model non-linear aeroelastic behaviour, their efficient use is a long way off, and for the foreseeable future there will be a requirement to estimate the parameters of non-linear systems.

This paper presents a method for the identification of non-linear multiple degree-of-freedom (DOF) systems with any type of non-linearity. Although the method is general, the application described here is suited particularly to the identification of aeroelastic systems. A number of simulated examples are given that demonstrate the use of the method. In reference [**6**], a demonstration of the use of the proposed method in conjunction with gust load prediction methods can be found.

2 MOTIVATION

Aeroelastic systems are usually described by the general equation

$$\mathbf{M}\ddot{\boldsymbol{q}} + (\mathbf{C}_{\text{aero}} + \mathbf{C}_{\text{struct}})\dot{\boldsymbol{q}} + (\mathbf{K}_{\text{aero}} + \mathbf{K}_{\text{struct}})\boldsymbol{q} = \boldsymbol{F} \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the mass, damping and stiffness matrices respectively, $\boldsymbol{q}$ is the displacement vector and $\boldsymbol{F}$ is the excitation force vector. The subscripts denote whether the matrices are due to structural or aerodynamic functions.

When using the restoring force method, the above equations are rewritten as

$$\mathbf{M}\ddot{\boldsymbol{q}} + \boldsymbol{f}(\dot{\boldsymbol{q}}, \boldsymbol{q}) = \boldsymbol{F}(t) \tag{2}$$

where $\boldsymbol{f}(\dot{\boldsymbol{q}}, \boldsymbol{q})$ is the restoring force of the system. For a single DOF system with known mass, the restoring force can be expressed in terms of the inertial force and the excitation. It is then straightforward to plot and curve-fit the restoring force surface [**4**].

For a multiple DOF system the process needs to take place in modal space. By setting $\boldsymbol{q} = \Phi\boldsymbol{u}$, where $\boldsymbol{u}$ is the modal displacement vector and Φ is the $(m \times r)$ modal matrix (r being the number of modes to be considered), substituting in equation (2) and premultipyling by Φ^{T}, the restoring force equation becomes (see reference [**5**])

$$\boldsymbol{h}(\dot{\boldsymbol{u}}, \boldsymbol{u}) = \Phi^{\mathrm{T}}\boldsymbol{F} - \mathbf{M}_{\mathrm{u}}\mathbf{R}\ddot{\boldsymbol{q}} \tag{3}$$

where $\boldsymbol{h} = \Phi^{\mathrm{T}}\boldsymbol{F}(t)$ is the generalized restoring force vector, $\mathbf{M}_{\mathrm{u}} = \Phi^{\mathrm{T}}\mathbf{M}\Phi$ is the diagonal modal mass matrix and $\mathbf{R} = [\Phi^{\mathrm{T}}\Phi]^{-1}\Phi^{\mathrm{T}}$. The modal restoring force can be estimated provided estimates for the generalized mass and modal matrices exist and the generalized displacements have been obtained from acceleration—or, indeed, acceleration, velocity and position [**7**]—measurements. Obviously, the process of obtaining the generalized mass and modal matrices requires a further identification analysis to be carried out before forming the restoring force surfaces, and this is by no means straightforward, particularly on non-linear systems.

The method proposed here attempts to evade some of the above difficulties, while maintaining the flexibility of the restoring force method to be able to deal with all types of non-linearity. Use is made of the fact that at an arbitrary response level, the restoring force due to the non-linearity is constant. The approach estimates the exact equation of motion of the system by curve-fitting the response at this chosen response level. A simple demonstration of the method is given in the following simulated example.

3 A SIMPLE APPLICATION

Consider a single degree-of-freedom system with a cubic stiffness non-linearity and assume that the position and type of the non-linearity and also the number of modes (one) are known. The equation of motion for this system is

$$m\ddot{y} + c\dot{y} + k_1 y + k_2 y^3 = u_g(t) \tag{4}$$

where m is the mass, c is the damping coefficient, k_1 is the linear stiffness coefficient and k_2 is the non-linear stiffness coefficient. Since it is known that the non-linear term depends on y, the identification process begins with isolating time instances where y has some given value. At this level, the non-linear term has a constant value due to its dependence on y. Thus, the equation of motion for the system can be written as

$$\begin{pmatrix} \ddot{y}(t_1) & \dot{y}(t_1) & 1 \\ \ddot{y}(t_2) & \dot{y}(t_2) & 1 \\ \vdots & \vdots & \vdots \\ \ddot{y}(t_n) & \dot{y}(t_n) & 1 \end{pmatrix} \begin{Bmatrix} m \\ c \\ N \end{Bmatrix} = \begin{Bmatrix} u_g(t_1) \\ \vdots \\ u_g(t_n) \end{Bmatrix} \tag{5}$$

where the non-linear constant term $N = k_1 y + k_2 y^3$ and $t_1, \ldots, t_n$ are the instances in time that correspond to the chosen response level. Notice that y itself does not appear in the equations since, having a constant value, it would render the left-hand side matrix singular. Equation (5) can be solved using a least-squares process to give m, c and N.

The equation of motion can then be rearranged in the form

$$N(t) = -m\ddot{y} - c\dot{y} + u_g(t) \tag{6}$$

to give the values of N at all time steps.

The result of the identification process is the values of the mass and damping coefficient as well as the stiffness for all time steps. A characteristic of this approach, which differs from others, is that the linear and non-linear parts of the stiffness have been merged together in one function, N. The response of the system can be found through the use of this combined function.

However, should these elements need to be determined, N can be discretized for the jth level as

$$N_j = k_1 y_j + \hat{N}_j$$

Then, if the type of non linearity in $\hat{N}$ is known, the linear and non-linear parts can be separated by means of curve-fitting. For instance, if for the present example it is known that the non-linear term is cubic, then it will also be known that

$$N_j = k_1 y_j + k_2 y_j^3$$

or

$$\begin{Bmatrix} N_1 \\ N_2 \\ \vdots \\ N_n \end{Bmatrix} = \begin{pmatrix} y_1 & y_1^3 \\ y_2 & y_2^3 \\ \vdots & \vdots \\ y_n & y_n^3 \end{pmatrix} \begin{Bmatrix} k_1 \\ k_2 \end{Bmatrix}$$

Alternatively, if the non-linear function, $\hat{N}$, is unknown but differentiable, then the combined function N can be split into the linear and non-linear parts by differentiating it twice with respect to y, which eliminates the linear part. If the result is then integrated while setting the constants of integration to zero, the purely non-linear part of N is obtained, i.e.

$$\hat{N} = \iint \frac{d^2 N}{dy^2}\, dy\, dy \tag{7}$$

Subtracting $\hat{N}$ from N gives the linear stiffness variation and a linear curve fit will yield the linear stiffness coefficient. It should be noted, though, that differentiation introduces additional numerical errors.

To illustrate the complete procedure numerically, the excitation force, $u_g(t)$, was taken to be a sine sweep and the system parameters were set at $m = 1.2$, $c = 0.7$, $k_1 = 5.8 \times 10^3$ and $k_2 = 1.16 \times 10^9$. Figure 1 shows the constant level displacement points that were used to start the analysis. Parameter estimates of $m = 1.199\,988$, $c = 0.700\,016$, $k_1 = 5.799\,974 \times 10^3$ and $k_2 = 1.159\,989 \times 10^9$ were found. Figure 2 shows the true and estimated cubic stiffness values. It can be seen that, for this simple case, very good quality estimates were found.

4 PROCEDURE

The previous example demonstrated the rationale behind the proposed method. However, in order to apply it to more realistic systems, various refinements are needed. The first crucial refinement is to multiply the equations of motion throughout by the inverse of the mass matrix, which has the effect of ensuring that the excitation term appears in all the equations of motion. Thus the equations become

$$\ddot{\boldsymbol{q}} + \mathbf{M}^{-1}\mathbf{C}\dot{\boldsymbol{q}} + \mathbf{M}^{-1}\mathbf{K}\boldsymbol{q} = \mathbf{M}^{-1}\boldsymbol{F} \tag{8}$$

but now $\mathbf{M}^{-1}\boldsymbol{F}$ must also be treated as an unknown. The only term in equation (8) that is completely known is the acceleration $\ddot{\boldsymbol{q}}$. As a consequence, equation (5) is replaced by

$$\begin{pmatrix} \dot{q}_1(t_1) & \ldots & \dot{q}_m(t_1) & q_1(t_1) & \ldots & q_m(t_1) & f_x(t_1) & 1 \\ \dot{q}_1(t_2) & \ldots & \dot{q}_m(t_2) & q_1(t_2) & \ldots & q_m(t_2) & f_x(t_2) & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dot{q}_1(t_n) & \ldots & \dot{q}_m(t_n) & q_1(t_n) & \ldots & q_m(t_n) & f_x(t_n) & 1 \end{pmatrix}$$

$$\times \begin{Bmatrix} C_{i1} \\ \vdots \\ \hat{C}_{im} \\ \hat{K}_{i1} \\ \vdots \\ \hat{K}_{im} \\ \hat{F}_i \\ N_i \end{Bmatrix} = \begin{Bmatrix} q_i(t_1) \\ \vdots \\ \ddot{q}_i(t_n) \end{Bmatrix} \tag{9}$$

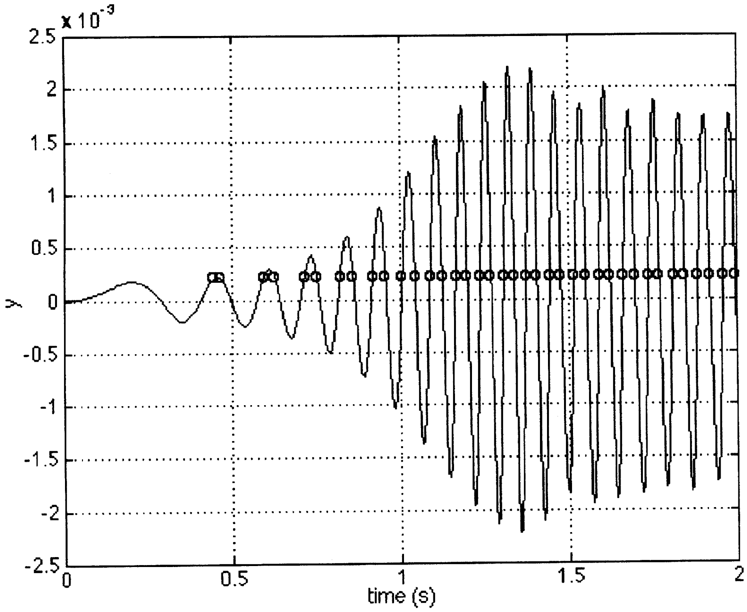

Fig. 1 Constant level response points used in the identification process

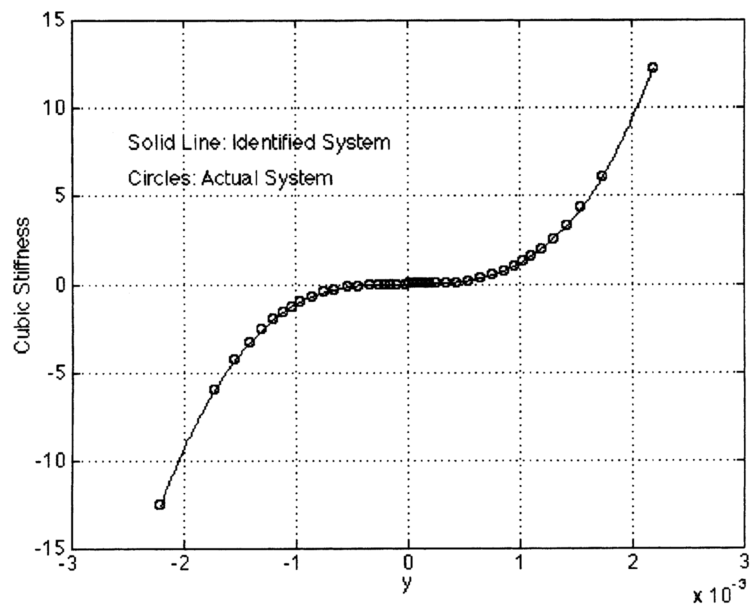

Fig. 2 True and estimated cubic stiffness

for $i = 1, \ldots, m$, where m is the number of modes. $\hat{K}_{ii}$, $\hat{C}_{ii}$, $\hat{F}_i$ are the various elements of the matrices $\hat{\mathbf{K}} = \mathbf{M}^{-1}\mathbf{K}$, $\hat{\mathbf{C}} = \mathbf{M}^{-1}\mathbf{C}$ and $\hat{\boldsymbol{F}} = \mathbf{M}^{-1}\boldsymbol{F}$. In this example $\boldsymbol{N}$ is a vector containing all the non-linearities in the system. Any non-linear terms from the damping or stiffness matrices are moved to $\boldsymbol{N}$, together with their associated linear terms (as in the previous case of $N = k_1 y + \hat{N}$), so that all the elements in the matrices are linear or zero. Equations (9) also demonstrate an additional advantage of multiplying throughout by the inverse of the mass matrix, namely that the number of unknowns is reduced, speeding up the computation and also improving the accuracy of the fit.

In equation (5), y was not included at all to avoid rendering the equations singular. This should also be the case in equations (9). However, since the position of the non-linearity and the variable it depends on are not known, it is impossible to predetermine which of $q_1, \ldots, q_m, \dot{q}_1, \ldots, \dot{q}_m$ should be kept constant and excluded. Even the number of modes, m, is unknown for a real system. However, preliminary analysis would give an indication via frequency response function (FRF) plots.

The number and types of non-linearities present in each mode are not known; hence the procedure becomes speculative at this point. It is first assumed that there is a non-linearity depending on, say, q_1 (e.g. a stiffness non-linearity or Coulombic friction). Then points where q_1 has a constant value are identified in the output. Assuming that enough such points have been identified in the output of the system, each of equations (9) is solved using a least-squares process, each yielding the elements of one line of the mass, structural damping and structural stiffness matrices as well as the value of the non-linear term. When all the sets of equations are solved, the equivalent of equation (6) is

$$\boldsymbol{N} = -\ddot{\boldsymbol{q}} - \hat{\mathbf{C}}\dot{\boldsymbol{q}} - \hat{\mathbf{K}}\boldsymbol{q} + \hat{\boldsymbol{F}} \qquad (10)$$

(where the column associated with q_1 in $\hat{\mathbf{K}}$ is made up of zeros), which is employed to calculate the values of the non-linear terms for every instant in time, since all the other matrices are now considered to be known. Since the equations of motion have been multiplied throughout by the inverse of the mass matrix there will be non-linear terms in each of the m equations (9), even if there is only one non-linearity in one mode. The non-linear terms, N_i, are then plotted against q_1. If the plots are single-valued functions of q_1, then the non-linearity is assumed to depend on the correct variable and the mode has been identified correctly. If the curve has a phase-plot-type shape, then this means that the non-linearity depends on some other variable and, hence, the procedure needs to be repeated from the beginning, keeping another one of $q_2, \ldots, q_m, \dot{q}_1, \ldots, \dot{q}_m$ constant until a successful identification is obtained. Finally, after the non-linear terms have been evaluated for all instants in time, they can be curve-fitted to yield continuous functions.

In order for the identification process to succeed, the input and output data need to be interpolated to obtain a set of instances in time where the desired variable has exactly the same value. This value needs to be near the equilibrium level so that enough such points can be obtained. Cubic interpolation has been found to be quite adequate, yielding sets of points that are almost exact solutions to the equations of motion. The excitation force also needs to be such that it excites all the important features of the systems,

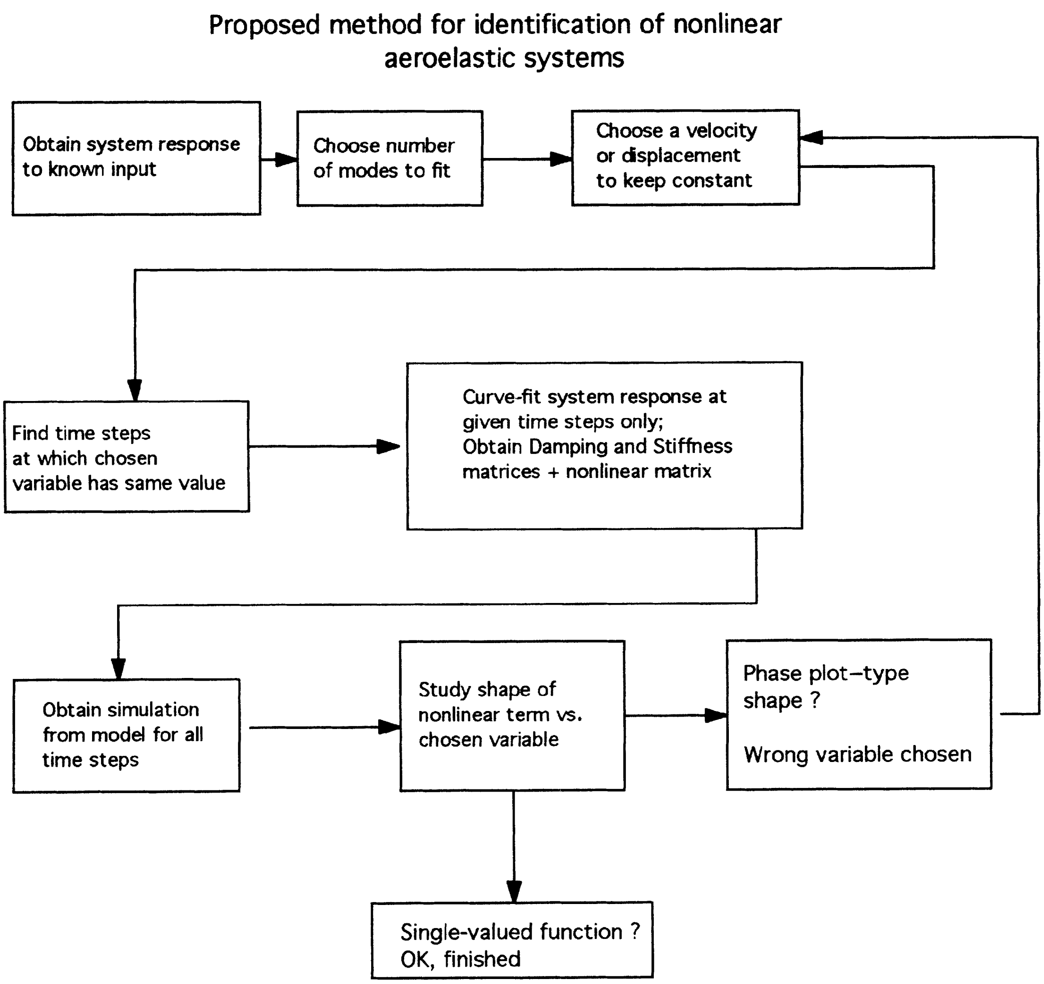

Fig. 3 System identification algorithm

including the non-linearity. Sine-sweep or banded random excitations are suitable since they allow several frequencies of excitation to be applied to the system in one test.

To make the application of the method clearer, the algorithm is presented below in pseudocode and in graphical form in Fig. 3:

1. Choose the number of modes by which to represent the system.
2. Choose a suitable input and measure the response of the chosen modes to the input.
3. Assume the variable on which the non-linear term depends, e.g. $N_1 = f(q_1)$.
4. Find the set of time steps, t_c, where q_1 =constant (using interpolation), from the measured data.
5. For $i = 1$ to number of modes, solve the ith equation of (5) at t_c to evaluate the ith line of $\hat{\mathbf{C}}$ and $\hat{\mathbf{K}}$ as well as $N_i[q_1(t_c)]$.
6. Solve equations (10) at all time steps to evaluate $\boldsymbol{N}$.
7. For $i = 1$ to number of modes:
   (a) Plot N_i against the variable on which it was assumed to vary, e.g. q_1.
   (b) If the plot has the shape of a phase-plane plot, the wrong assumption was made. Go back to step 3 and make a different assumption.
   (c) If the plot is a single-valued function, the correct assumption was made and the mode has been evaluated correctly. Curve-fit the plot to obtain the type of non-linearity.
   (d) End loop.
8. To validate the model solve at all time steps and compare with measured data.

An example of the application of the method is given in the next section. It should be noted that the non-linearities identified by the method need not be single-valued functions. Hysteresis-type non-linearities can be identified but, if such a non-linearity is expected, then the procedure needs to be applied on constant values of the variable with only positive or only negative derivatives, as demonstrated in a later section.

A further consideration regarding the proposed method concerns the effect of performing the identification procedure at various levels and not just the one. This was tried for a few very simple test cases but was not found to improve the accuracy of the resulting system estimates. However, in the case where a significant amount of noise is present in the response data, it is suggested that using a large number of levels would have a beneficial effect, since it would average out the noise contribution.

It should be noted that the method will only identify systems that contain non-linearities dependent upon one variable. For instance, it will identify a wing with friction and freeplay in the wing-root pitch degree of freedom; however, it will not identify a system with freeplay both in the wing-root pitch and the wing-root heave degree of freedom. This limitation comes from the fact that the equations of motion are identified as if they had been multiplied throughout by the inverse of the mass matrix. Hence, every non-linearity that exists in the system appears in the equation for every mode. Since the method works on the assumption that it is possible to keep the non-linear term in each equation constant, two or more non-linear terms that depend on two different variables will cause it to fail.

Finally, the method can be applied at a single level, but it can also be applied simultaneously at a number of levels, thus increasing the accuracy of the prediction in the presence of experimental noise in the measured input and output signals.

5 A MORE COMPLEX APPLICATION

The method is here demonstrated by applying it to a multi-degree-of-freedom simulated aeroelastic system. The system is a rigid, rectangular, flat-plate wing with three degrees of freedom, one in the wing pitch, one in the wing heave and one in the control surface pitch. The control surface pitch spring is bilinear [**8**]. The equation of motion for the system is

$$\begin{pmatrix} I_\gamma & I_{\gamma\theta} & I_{\gamma\beta} \\ I_{\gamma\theta} & I_\theta & I_{\theta\beta} \\ I_{\gamma\beta} & I_{\theta\beta} & I_\beta \end{pmatrix} \begin{Bmatrix} \ddot{\gamma} \\ \ddot{\theta} \\ \ddot{\beta} \end{Bmatrix} + \tfrac{1}{2}\rho V sc \begin{pmatrix} \dfrac{s^2 a_1}{3} & 0 & 0 \\ -\dfrac{scb_1}{2} & -c^2 M_{\dot{\theta}} & 0 \\ -\dfrac{scc_1}{2} & 0 & -c^2 M_{\dot{\beta}} \end{pmatrix} \begin{Bmatrix} \dot{\gamma} \\ \dot{\theta} \\ \dot{\beta} \end{Bmatrix} + \begin{pmatrix} K_\gamma & 0 & 0 \\ 0 & K_\theta & 0 \\ 0 & 0 & K_\beta \end{pmatrix} \begin{Bmatrix} \gamma \\ \theta \\ \beta \end{Bmatrix} + \tfrac{1}{2}\rho V^2 sc \begin{pmatrix} 0 & \dfrac{sa_1}{2} & \dfrac{sa_2}{2} \\ 0 & -\dfrac{cb_1}{2} & -\dfrac{cb_2}{2} \\ 0 & -\dfrac{cc_1}{2} & -\dfrac{cc_2}{2} \end{pmatrix} \begin{Bmatrix} \gamma \\ \theta \\ \beta \end{Bmatrix} = \begin{Bmatrix} F y_{\mathrm{ff}} \\ F(x_{\mathrm{ff}} - x_{\mathrm{f}}) \\ 0 \end{Bmatrix} \quad (11)$$

where γ, θ and β are the three degrees of freedom, I_γ, $I_{\gamma\theta}$, etc., are the second moments of area, c is the chord, s is the span, t is the thickness of the wing, ρ is the density of air, V is the free-stream velocity, a_1, a_2, b_1, etc., are aerodynamic lift and moment curve slopes, $M_{\dot{\theta}}$ and $M_{\dot{\beta}}$ are unsteady aerodynamic derivatives, x_{ff}, y_{ff} is the point of application of the excitation, x_{f} is the x coordinate of the flexural axis of the wing, K_γ, K_θ, K_β are the stiffnesses of the three springs and F is the excitation function. K_β in this case is substituted by a bilinear function of β.

The fact that the system is numerical implies that there is no measurement error. However, numerical errors are introduced by the process of differentiation of the response (introduction of higher derivatives). In order to apply the identification routine usefully, the time step needs to be sufficiently small. The input used here was a sine sweep. The first step is to choose the number of modes. Because the model is simple, wing heave γ, wing pitch θ and control surface pitch β are the three modes required for a successful identification.

According to the procedure outlined in the previous section, the next step is to assume that there is a non-linearity which appears in every equation. Therefore, it is assumed that the wing heave equation contains a non-linearity depending on the wing heave. Equations (9) are solved and using the estimates for $\hat{\mathbf{C}}$ and $\hat{\mathbf{K}}$, the three non-linear terms are calculated for each time step. As it happens, because the single non-linearity in this model depends on β, the identification process fails and needs to be repeated for the case where the non-linearity depends on the control surface pitch. For a particular case of successful identification, the resulting damping matrix was

$$\hat{\mathbf{C}}_{\text{id}} = \begin{bmatrix} 0.0381 & -0.0700 & 0.0100 \\ -0.1665 & 0.1293 & -0.0387 \\ 0.2000 & -0.2763 & 0.1159 \end{bmatrix}$$

The actual damping matrix was

$$\hat{\mathbf{C}} = \begin{bmatrix} 0.0381 & -0.0700 & 0.0100 \\ -0.1665 & 0.1292 & -0.0387 \\ 0.2000 & -0.2763 & 0.1158 \end{bmatrix}$$

The identified stiffness matrix was

$$\hat{\mathbf{K}}_{\text{id}} = \begin{bmatrix} 16.7993 & -12.2321 & 0.0000 \\ -74.6637 & 225.4430 & 0.0000 \\ 74.6637 & -481.9899 & 0.0000 \end{bmatrix}$$

The actual stiffness matrix was

$$\hat{\mathbf{K}} = \begin{bmatrix} 16.7994 & -12.2321 & 0.0259 \\ -74.6637 & 225.4430 & 0.0738 \\ 74.6637 & -481.9898 & -0.3048 \end{bmatrix}$$

The two sets of matrices are virtually identical apart from the last column of $\hat{\mathbf{K}}$, which is zero in the identified case. This is due to the fact that the non-linearity appears in all three elements of that column and the identification process merges the linear and non-linear parts of these elements, as in the earlier example where $N = k_1 y + k_2 y^3$. However, unlike the case of cubic stiffness, the bilinear function is discontinuous and, therefore, cannot be differentiated or curve-fitted. Additionally, since it is linear in parts, N_i cannot be fitted by least squares as the sum of a linear and a bilinear function. Hence, separating the linear and non-linear parts of N_i is not as straightforward as in the previous example. The problem can only be partly solved by considering the fact that the linear part of N_i is made up of a structural and an aerodynamic term. Aerodynamic stiffness terms depend on the square of the free-stream velocity [**9**, **10**]. Hence

$$N_i(V, \beta) = (k_\text{s} + k_\text{a} V^2)\beta + \hat{N}_i(\beta) \tag{12}$$

where V is the free-stream velocity, k_s is the structural contribution and k_a is the aerodynamic contribution. Since the purely non-linear term is structural, $\hat{N}_i$ does not depend on V. By performing identifications at two different airspeeds, k_a can be evaluated; however, the linear and non-linear structural terms will remain merged in a new non-linear function equal to $k_\text{s}\beta + \hat{N}_i(\beta)$. Consequently, it is possible to isolate the aerodynamic contribution to the linear part of the system's stiffness but not the structural one.

The best test of the accuracy of the method is to use the new matrices, together with the non-linear terms obtained to solve the identified model, and compare its response to that of the actual system. The non-linear terms are handled as lookup tables since their discontinuities prohibit the use of interpolation or curve-fitting.

Figure 4 shows the percentage error. The large peaks occur at points where the real system's response is very close to zero. The comparison between the actual non-linear term in wing heave and that produced by the identification method can be seen in Fig. 5. Figure 6 shows the non-linear surface for the same degree of freedom, i.e. an equivalent of the restoring force surface given by the restoring force method. This plot should be compared to graphs in reference [**11**].

The identification method was tried on a system with freeplay non-linearity, again with satisfactory results. Figure 7 shows the comparison between actual and identified non-linear terms in wing pitch and Fig. 8 shows the freeplay non-linear surface.

Several special cases of the simple wing model have been identified using the present method, especially where the response is chaotic, with satisfactory results.

6 IDENTIFYING HYSTERESIS-TYPE NON-LINEARITIES

Hysteresis is characterized by the fact that the response lies on one path while increasing and on another one while decreasing [**12**]. Hence, hysteresis-type non-linearities can be easily identified by the proposed method with a slight modification. When isolating response levels, only points in the response that lie on the level but also have a first

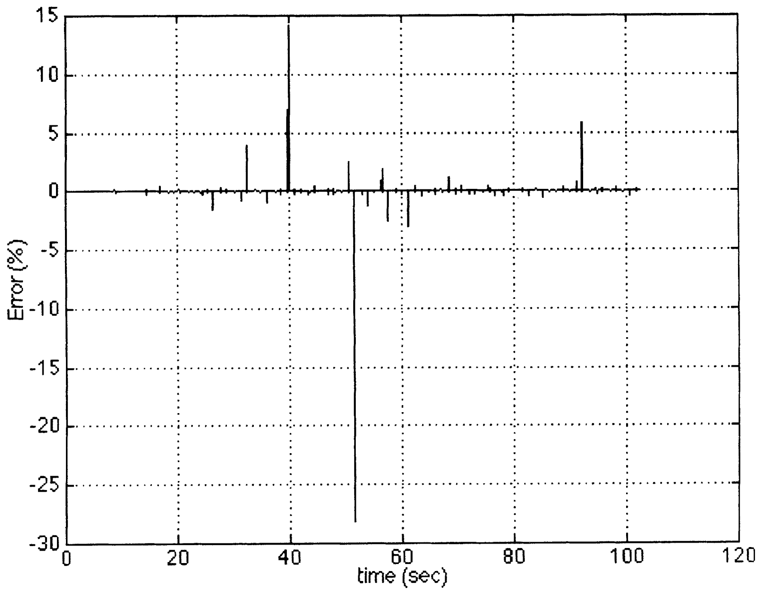

Fig. 4 Percentage error in the identified control surface pitch response

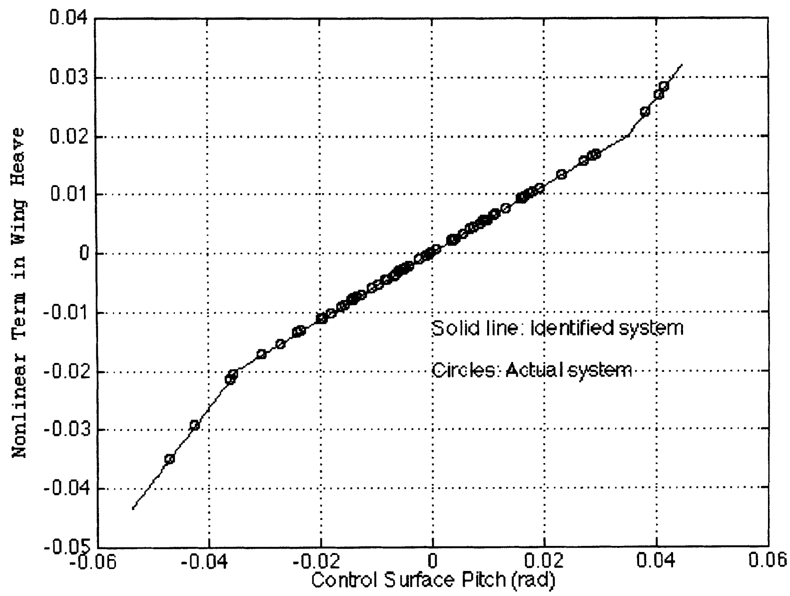

Fig. 5 True and identified bilinear term in the wing heave

derivative with respect to time of the same sign can be used. This is demonstrated in Fig. 9.

The simple single degree-of-freedom system presented earlier but with hysteretic stiffness was identified successfully using the proposed method. For a particular test case, the values of the mass and damping coefficients were $m = 1.2$, $c = 18.9$. The identified values were $m = 1.200\,000\,83$ and $c = 18.900\,001\,4$. Figure 10 shows the percentage error in the identification of the hysteretic stiffness variation with y.

7 IDENTIFICATION OF LARGE SYSTEMS

A real system will contain a large number of modes and the identification of the entire system will be difficult to perform accurately. It is therefore of interest to determine whether the proposed method could deliver acceptable results when less modes are used in the identification procedure than there are in the real system.

A second mathematical model of a wing was developed, this time without a control surface but with a multi-mode

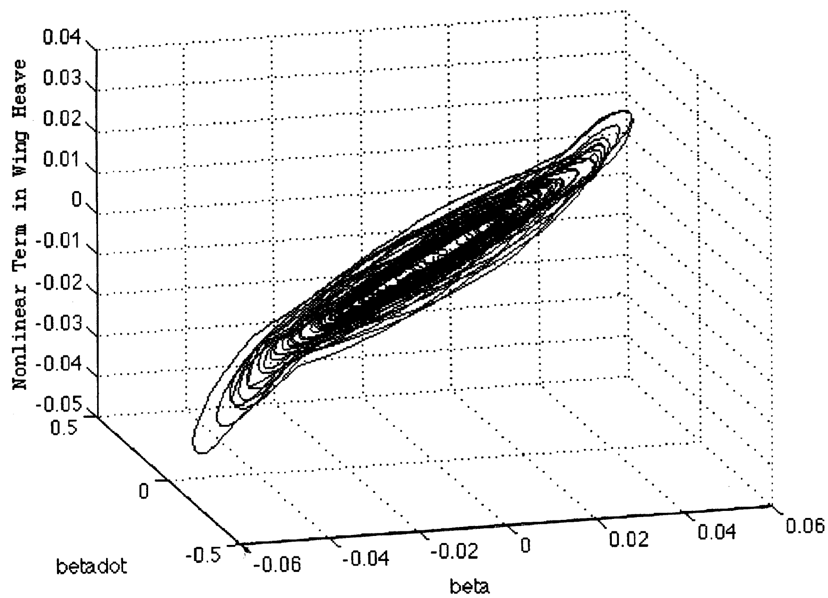

Fig. 6 Wing heave bilinear surface

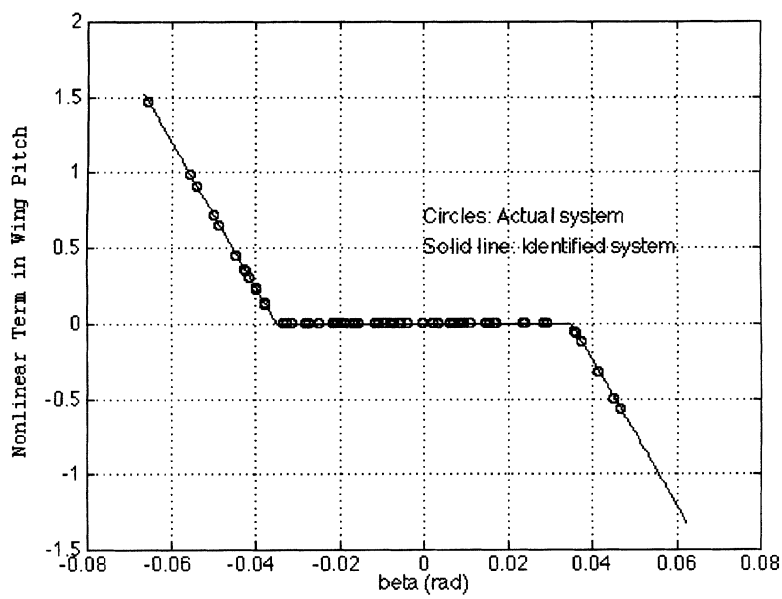

Fig. 7 True and identified non-linear term in the wing pitch for freeplay

Rayleigh–Ritz [**13**] series approximating the flexibility of the wing. The two wing-root rigid modes (wing-root pitch and heave) were retained as a mechanism of introducing non-linearities. Since in order to identify a non-linear system the modes that contain the non-linearities need to be identified, the two rigid modes always need to appear in the identification process.

The results presented below are for a 5 DOF rectangular wing with bilinear stiffness in the wing-root pitch. The Rayleigh–Ritz series contains two bending and one torsional mode. The system was identified using models of increasing complexity from 2 DOF to 5. The wing's damping and stiffness matrices were

$$\hat{\mathbf{C}} = \begin{bmatrix} 5.2101 & -0.0392 & -12.2802 & -7.7218 & 0.2345 \\ -41.5151 & 1.7436 & 89.6726 & 58.1212 & 1.8682 \\ 3.3185 & 0.000 & 1.6930 & -1.6144 & -0.1776 \\ -9.4815 & 0.000 & 7.3802 & 8.8887 & 0.5074 \\ -68.1641 & 0.000 & 164.0038 & 102.5024 & 4.8612 \end{bmatrix}$$

$$\hat{\mathbf{K}} = 10^4 \begin{bmatrix} 0.4293 & 0.0121 & 0.0000 & 0.0000 & 0.0156 \\ -1.5331 & -0.1121 & 0.0000 & 0.0000 & -0.1245 \\ 0.0000 & 0.0136 & 2.3893 & 0.4540 & -0.0405 \\ 0.0000 & -0.0309 & -3.9099 & -0.7300 & 0.1158 \\ 0.0000 & -0.1518 & -0.4661 & -0.0194 & 3.0015 \end{bmatrix}$$

The results obtained by the 5 DOF identification were

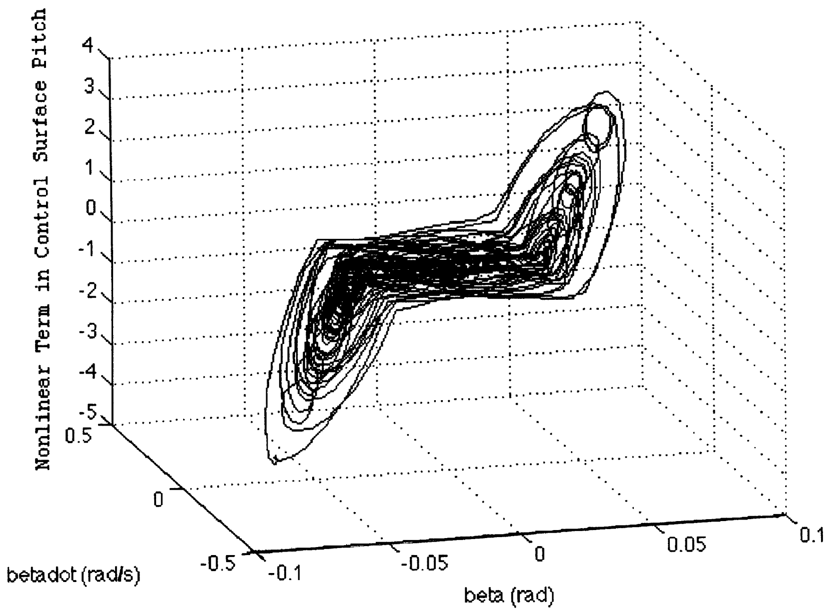

Fig. 8 Wing pitch freeplay surface

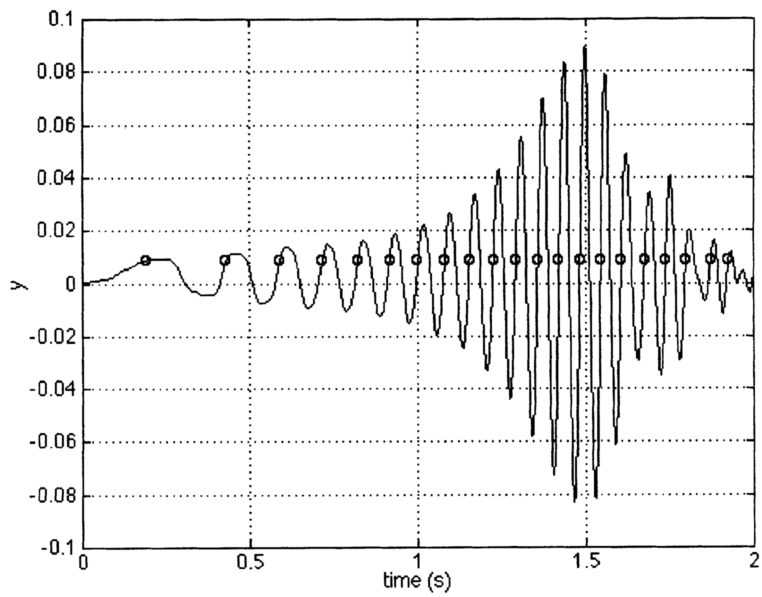

Fig. 9 Constant level response points used in the identification of hysteresis

$$\hat{\mathbf{C}}_{\mathrm{id}} = \begin{bmatrix} 5.2164 & -0.0398 & -12.2586 & -7.7137 & -0.2334 \\ -41.5106 & 1.7484 & 89.5107 & 58.0764 & 1.8697 \\ 3.3629 & -0.0007 & 1.7712 & -1.5794 & -0.1864 \\ -9.5281 & 0.0015 & 7.2587 & 8.8269 & 0.5166 \\ -67.5750 & 0.0069 & 162.8999 & 101.7764 & 4.7698 \end{bmatrix}$$

$$\hat{\mathbf{K}}_{\mathrm{id}} = 10^4 \begin{bmatrix} 0.4289 & 0.0000 & -0.0000 & 0.0000 & 0.0155 \\ -1.5331 & 0.0000 & 0.0001 & -0.0001 & -0.1231 \\ 0.0004 & 0.0000 & 2.3793 & 0.4522 & -0.0404 \\ -0.0006 & 0.0000 & -3.8933 & -0.7270 & 0.1151 \\ -0.0006 & 0.0000 & -0.4601 & -0.0188 & 2.9788 \end{bmatrix}$$

The results obtained by the 4 DOF identification were

$$\hat{\mathbf{C}}_{\mathrm{id}} = \begin{bmatrix} 5.6880 & -0.0564 & -11.3243 & -7.1511 \\ -45.3634 & 1.8817 & 82.0752 & 53.5962 \\ 3.9890 & -0.0012 & 1.9721 & -2.0110 \\ -9.9812 & 0.0101 & 8.0456 & 8.2341 \end{bmatrix}$$

$$\hat{\mathbf{K}}_{\mathrm{id}} = 10^4 \begin{bmatrix} 0.4991 & 0.0000 & -0.0000 & 0.0000 \\ -1.1298 & 0.0000 & 0.0000 & -0.0002 \\ 0.0019 & 0.0000 & 2.8748 & 0.6753 \\ -0.0002 & 0.0000 & -4.7892 & -0.5499 \end{bmatrix}$$

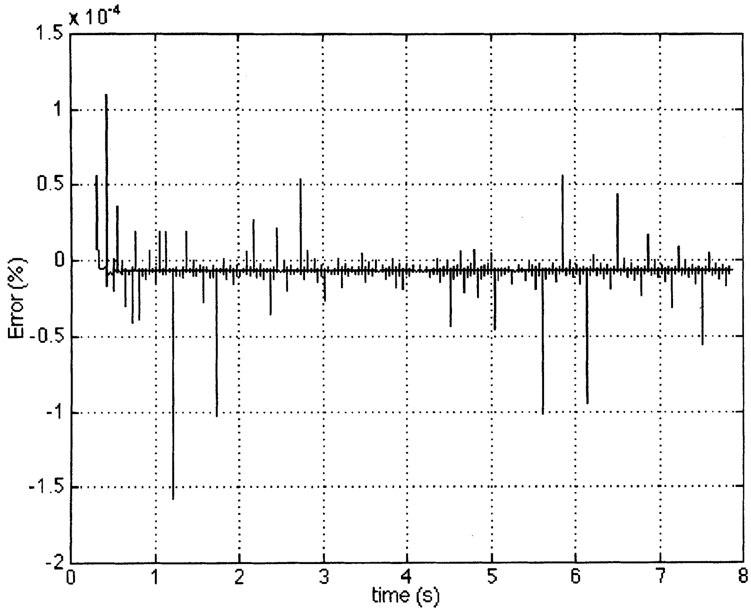

Fig. 10 Percentage error in the identification of the hysteretic non-linear term

The results obtained by the 2 DOF identification were

$$\hat{\mathbf{C}}_{\mathrm{id}} = \begin{bmatrix} 6.3704 & -0.0944 \\ -53.4671 & 2.1364 \end{bmatrix}$$

$$\hat{\mathbf{K}}_{\mathrm{id}} = 10^4 \begin{bmatrix} 0.3879 & 0.0000 \\ -1.2271 & 0.0000 \end{bmatrix}$$

The agreement between the comparable parts of the actual and identified sets of matrices deteriorates with decreasing identification model order. Again, the second column of the identified stiffness matrices is zero because its elements were absorbed in the non-linear terms. However, despite the drop in accuracy, the system has been identified properly, even in the two-mode case, as can be seen by the comparison of the non-linear terms presented in Fig. 11. In other words, the type and location of the non-linearity have been identified accurately.

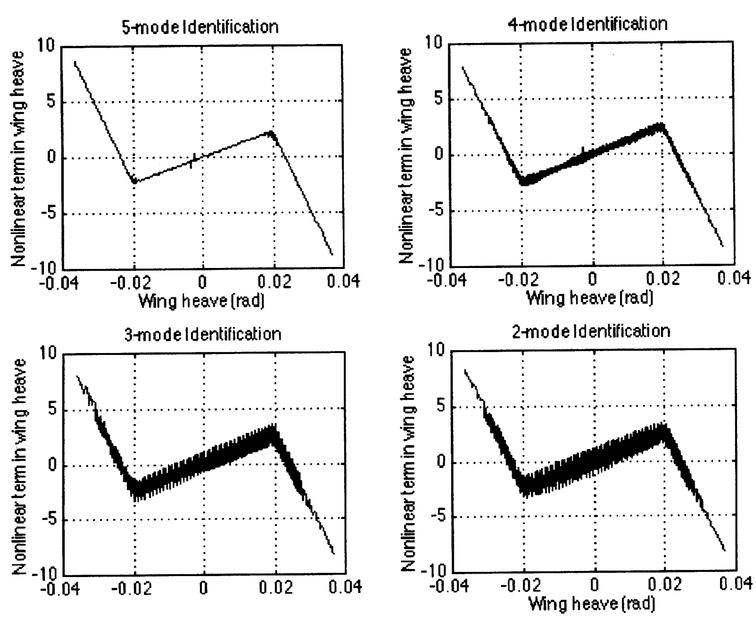

Fig. 11 Identification of a 5 DOF system with five-, four-, three- and two-mode models

8 CONCLUSIONS

A new method for identification of non-linear aeroelastic systems is proposed, based on the restoring force method. The main thrust of the technique consists of curve-fitting the system at time points where the response of a mode and, hence, the non-linearity dependent on it has a constant value. The method was demonstrated on a simple single degree-of-freedom system and then applied to a multi-degree-of-freedom system representing a rigid wing, yielding in both cases models whose response was in very good agreement with that of the actual systems. The method was found capable of identifying a wide range of non-linearities, including discontinuous and hysteresis-type non-linearities.

The proposed method was also applied to a rigid, flexible wing system using less flexible modes in the model than in the system. The agreement between the response of the model and that of the system was noticeably worse than in the earlier examples. However, both the type and the position of the non-linearity were identified correctly. The quality of the identification was found to deteriorate with increasing disparity in the number of modes between the model and the system.

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