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OPTIONS PRICING MODELS

AN ADDITIVE APPROACH

Working Paper No 40

Introduction

Options have existed for a long time, although their use as a financial instrument was limited until the creation of the Chicago Board Options Exchange in 1973. This date marked the beginning of a constant growth in the volume of trading of options. At the present time options are traded on almost every stock exchange and are expanded into most financial instruments. Besides, this incomparable success has been accompanied by a rapid development of the theory, whose corner stone is the option pricing model of Black and Scholes in 1973. Since that time the financial literature has abounded in new developments in option theory, and it also seems that as much from the theoretical point of view as from the practical point of view, options play an increasing role in the financial markets.

In this paper I present a simple model for valuing options on securities whose returns are generated by a binomial process. My approach is quite similar to that followed by Cox, Ross and Rubinstein (1979), as well as by Rendleman and Bartter (1979), although the assumptions are different. Both of them assumed that the returns are generated by a multiplicative binomial process, while I based my research on an additive process. In the multiplicative case, if one considers a large number of periods or stages, this means that the return of the security is log normal, and the multiplicative binomial option pricing model contains the Black and Scholes model as a special limiting case. As for the additive case, I will show that although a simple general formulation is difficult to derive in the discrete version, its limiting case, which assume that the returns are normally distributed, is simple and attractive. Besides its usefulness to value options on assets whose returns are additive, the additive option pricing model is particularly relevant to price options on portfolios which are additive by construction.

In the first section of this paper, I present the terminology used in option theory, as well as some basic relationships between the option and its underlying asset. There exist a large number of types of option, but since the two most common are the put and the call option on a security, I will limit my presentation to them. Furthermore, since both the call and the put options represent a claim on the same asset, there exists between them a put-call parity relationship, reviewed in the section, which enables me to use the same methodology for both the call and the put. Section two reviews the two main option pricing models, that is, the multiplicative option pricing formula and the Black and Scholes option pricing formula. Following Cox, Ross and Rubinstein, I will derive the discrete time model using arbitrage methods and I will present the Black and Scholes model as a limiting case. Then, using the same arbitrage methods, I will develop and discuss in the third section another option valuation formula, assuming the return of the security price follows an additive binomial process. I will show in the fourth section that this additive binomial option pricing model, although it has not been easily derived, reduces to a more satisfactory option pricing model in continuous time. The last section deals with the multivariate case, considering more specifically options on portfolios.

1 Definitions and Basic Relationships

1.1 Definitions

An option is a contract between two persons which gives one, the owner, the right, not the obligation, to buy or to sell a designed security from or to another one, the writer, at a specified price, either before or on a determined date.

The characteristics of an option are :

- its nature : the two main options are the call and the put options. A call option gives the right to the owner to buy a number of shares of the security, and the put option gives the owner the right to sell. As one can always buy or sell an option, four states are possible, one person can be:
    - buyer of a call option,
    - seller of a call option,
    - buyer of a put option,
    - seller of a put option.

  The buyer is generally called the owner of the option, and the seller the writer of the option.

- its value : the price the owner has to pay to the writer to obtain the option. This value belongs to the writer whatever the owner may decide.
- its maturity : the period during which or at the end of which the right can be exercised. In this regard a distinction must be made between the European option and the American option. A European option can only be exercised at the end of the period, while the American option can be exercised at any date before the expiration date. However, as it can be proved that it never pays to exercise an American option before the expiration date, such options can always be considered as European options. In fact, to sell an American option before the expiration date is more profitable than to exercise it.
- its exercise price : the price the owner pays (for a call) or receives (for a put) if he decides to exercise his option.

From what precedes it appears that the price of an option C is determined by the price S of the underlying security, its maturity T and the exercise price K.

1.2 Basic relationships

Before discussing the option pricing models, some fundamental constraints on option prices must be defined, because they must be met by any option pricing model. So, it has been demonstrated (Smith, 1976) that some option pricing models, that proved incorrect, did not take account of all of these relationships.

- Of a European call option and an American call option on the same underlying security and with the same maturity and the same exercise price, the American call option must have the higher value. This is obvious if one considers the fact that an American call option can be exercised before the expiration date. This gives the owner an additional opportunity to make profits.[1]
- If one considers call options on the same security and with the same exercise price, the longer the maturity of the option, the more valuable this

[1]This is true only for a dividend protected call.

option is. Again, this is obvious if one considers that the longer the period is, the higher the probability that the security price exceeds the exercise price.[1]

- The value of a call option decreases as the exercise price increases. Indeed, of two call options on the same security and with the same maturity but with different exercise price, the lowest exercise price option must be the most valuable.
- The price C of a European call option cannot be negative and at its maturity date it is equal to the maximum of either the difference between the price S of the security and the exercise price K , or zero.

$$C = \max \{0, S-K\}$$

- As an American call is always more valuable than a European call and can be exercised at any time before the expiration date, the relationship, which is valid for any date, becomes for an American call option:

$$C \geq \max\{0, S-K\}$$

1.3 The Put-Call parity

Assuming a perfect market, no dividend and equivalent borrowing and lending rate, the following relationship between the values of the European call and put options of a security, called put-call parity, holds:

$$C = S + P - \frac{K}{r}$$

where

- C = value of the call option,
- P = value of the put option,
- S = price of the security,
- K = exercise price for both the call and the put option

[1]This is true only if the variance of the security price constantly increases.

- r = one plus the risk free rate

According to this relationship, the value of a European call is equal to the value of a portfolio composed of the underlying security, one put on this security and an borrowed amount equal to K/r. This result can be derived if we compare the payoffs at the expiration date of the call option and the portfolio for different values S_1 of the security.

	$S_1 < K$	$S_1 \geq K$
call	0	$S_1 - K$
portfolio		
security	S_1	S_1
put	$K - S_1$	0
borrowing	$-K$	$-K$
total	0	$S_1 - K$

The payoffs of the call option and the portfolio are equivalent whatever the price of the security may be in respect to the exercise price K. Consequently the values of these two investments must also be identical, otherwise arbitraging is possible, and the value of a put option can always be derived from the value of its corresponding call. Therefore, any option analysis and option pricing model have only to deal with either call or put options.

2 Multiplicative Option Pricing Models

The two main Option Pricing Models generally used are the Binomial Option Pricing formula (BOP) and the Black and Scholes option Pricing formula (BS). Prior to these models other equilibrium models were presented, but as pointed out before, they appeared to be inconsistent with one or more basic relationships I presented in the preceding section (Smith, 1976). The two valuation pricing formulas, the BOP and the BS formulas, are fundamentally consistent. Cox, Ross and Rubinstein (1976) showed that the BS model is in fact a special limiting case of the BOP model. Their difference resides in the assumption made concerning the process of the changes in the security price. The BOP formula assumes that the security price follows a discrete process, the multiplicative binomial process, while the BS formula assumes a log-normal process. So when the discrete interval used in the BOP formula is infinitely divided in more subintervals, the BOP model converges to the BS model.

2.1 The Binomial Option Pricing Formula

The Binomial Option Pricing Formula assumes that the price of the underlying security of a European call option follows a multiplicative binomial process. At each period, the security price S can either increase to the amount uS with probability q or decrease to the amount dS with probability (1-q), u and d being constant over time. In both cases, u-1 and d-1 can be interpreted as rate of return of the security.

Let us describe the process for one period, assuming a perfect market and no dividend, and defining:

- S as the initial security price,
- K as the exercise price of the option,
- u as one plus the percentage change in the security price if it increases,
- d as one plus the percentage change in the security price if it decreases,
- C as the price of the call option at the beginning of the period,
- C_u as the price of the call option at the end of the period if the security price increases,
- C_d as the price of the call option at the end of the period if the security price decreases,
- r as one plus the riskless interest rate of the period.

Since the value of the option at the end of the period is a linear function of the security price, it is possible to construct a portfolio, called hedge portfolio, combining one call option and a proportion Δ of the security, which has a certain payoff, no matter the price of the security at the end of the period.

$$- C_u + \Delta uS = - C_d + \Delta dS$$

Then Δ, or the number of shares of the security which must be purchased per unit of call option, is:

$$\Delta = \frac{C_u - C_d}{uS - dS} \tag{2.2}$$

Since the payoff of this portfolio is independent of the price of the stock, it can be considered as a riskless investment. Therefore, to preclude arbitrage, this payoff must be equal to the payoff of an investment B of the same value in the riskless asset. More formally, the equality between the payoffs and the investments are :

$$rB = C_u - \Delta uS = C_d - \Delta dS \tag{2.3}$$

$$B = C - \Delta S \tag{2.4}$$

Substituting the value of Δ of (2.2) in (2.3) yields the value of B

$$B = \frac{1}{r}\left(\frac{uC_d - dC_u}{u - d} \right) \tag{2.5}$$

Hence the value of the call option C of (2.4) is equal to :

$$C = \left(\frac{C_u - C_d}{uS - dS}\right)S + \frac{1}{r}\left(\frac{uC_d - dC_u}{u - d}\right)$$

or, rearranging

$$C = \frac{1}{r}\left(\frac{r-d}{u-d} C_u + \frac{u-r}{u-d} C_d \right) \tag{2.6}$$

Let us now define

$$p = \frac{r-d}{u-d}$$

$$1 - p = \frac{u-d - (r-d)}{u-d} = \frac{u-r}{u-d}$$

then (2.6) becomes

$$C = \frac{1}{r}\left(pC_u + (1-p)C_d\right) \tag{2.7}$$

Besides, I note that the relationship $u > r > d$ must hold, otherwise portfolio composed of the security and bonds could generate riskless profit, hence $0 < p < 1$.

For that reason, p and (1-p) can be considered as the pseudo probability[1] values of a trial whose outcomes are C_u and C_d, and the price of the option is equal to the expected value of the option prices at the end of the period, discounted by the risk free rate. Recalling now that at the end of the period the price of the option is equal to the maximum of zero and the difference between the security price and the exercise price, that is,

$$C_u = \max(0, uS - K)$$

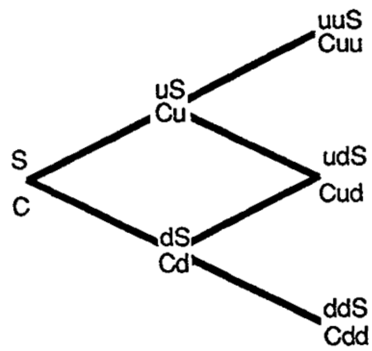
$$C_d = \max(0, dS - K)$$

then (2.7) can be rewritten

$$C = \frac{1}{r}\left(p \max(0, uS - K) + (1-p) \max(0, dS - K)\right) \qquad (2.8)$$

Now, if it is assumed that u,d and r are constant over time, the preceding relationship is easily extended to a multiperiod case. Considering such case as a tree diagram where the successive trials generate sequences of ups and downs in the security price, the initial value of a call is determined recursively by applying iteratively the relationship (2.8) to the nodes of each trial or period, starting with the last period and working backward to the initial period.

Let us take, for example, the two period case. The tree diagram of the values the security and the call option could have is :



[1]Rendleman and Bartter used the term pseudo probability for p and (1-p) in order to distinguish them from the true probabilities q and (1-q) and Harrison and Kreps call them equivalent martingale measure.

The relationship (2.8) is used first to determine the values of the option C_u and C_d at the beginning of the second period,

$$C_u = \frac{1}{r}\left(p \max(0,u^2S - K) + (1-p) \max(0,udS - K)\right)$$

$$C_d = \frac{1}{r}\left(p \max(0,udS - K) + (1-p) \max(0,d^2S - K)\right)$$

afterwards, it is used to calculate the initial value of the option, substituting their values for C_u and C_d.

$$C = \frac{1}{r}\left(p\, C_u + (1-p)\, C_d\right)$$

$$C = \frac{1}{r^2}\left(p^2\max(0,u^2S-K)+2p(1-p)\max(0,udS-K)+(1-p)^2\max(0,d^2S-K)\right) \quad (2.9)$$

Generalizing for n periods, (2.9) becomes

$$C = \frac{1}{r^n}\left(\sum_{x=0}^{n} \frac{n!}{x!(n-x)!}\, p^x(1-p)^{n-x} \max(0,u^x d^{n-x}S - K)\right) \qquad (2.10)$$

Since in the tree diagram there are some paths or sequences of ups and downs which have a number of downward movements such that the security price is lower than the exercise price, $u^x d^{n-x}S - K < 0$, then the value of the option for these sequences is:

$$\max(0,u^x d^{n-x}S - K) = 0$$

This means that for these sequences, the call option will never be exercised. Therefore such sequences can be eliminated from the valuation formula. So, defining by 'a' the minimum number of upward moves such that the value of the option is positive for a sequence, (2.10) is rewritten:

$$C = \frac{1}{r^n}\left(\sum_{x=a}^{n} \frac{n!}{x!(n-x)!}\, p^x(1-p)^{n-x}\, (u^x d^{n-x}S - K)\right)$$

hence, splitting the last member and rearranging,

$$C = S\sum_{x=a}^{n}\frac{n!}{x!(n-x)!}\left(\frac{pu}{r}\right)^{x}\left(\frac{(1-p)d}{r}\right)^{n-x} - \frac{K}{r^{n}}\sum_{x=a}^{n}\frac{n!}{x!(n-x)!}p^{x}(1-p)^{n-x} \quad (2.11)$$

Let us set p'=pu/r and (1-p')=(1-p)d/r. They can also be interpreted as pseudo probabilities since they are positive and their sum is equal to 1[1]

It appears that (2.11) is the sum of two binomial expansions, therefore the general formulation of the Binomial Option Pricing formula is:

$$\boxed{C = S\,B[a;n;p'] - \frac{K}{r^{n}}B[a;n;p]} \quad (2.11)$$

where

- $p = \frac{r-d}{u-d}$
- $(1 - p) = \frac{u-r}{u-d}$
- $p' = \frac{up}{r}$
- $(1 - p') = \frac{d(1-p)}{r}$
- a is the smallest positive integer such that $u^{a}d^{n-a}\,S - K > 0$
- B[a;n;p] and B[a;n,p'] are the cumulative probabilities of the complementary binomial distribution of obtaining a number of upward moves equal to or lower than a, for n trials and with a probability of occurrence of one upward move in the security price for a trial equal to p and p'.

Interestingly, q and (1-q), the true probabilities of occurrence of the upward and downward moves do not appear in the formula of the option pricing. This is not surprising if we remember that this model is based on the hedging, which guarantees to the hedge portfolio a certain payoff however upward or downward the security price

[1]As p,u,d and r are positive, p' and (1-p') are also positive, and their sum

$$p'+(1-p')=\frac{pu}{r}+\frac{(1-p)d}{r}$$

$$p'+(1-p') = \frac{1}{r}[pu + d - pd]$$

$$p'+(1-p') = \frac{1}{r}[p(u-d) + d]$$

Substituting the value of p

$$p'+(1-p') = \frac{1}{r}[(r-d) + d] = 1$$

may go. As a consequence the probability of occurrence of the changes in the price does not intervene in the hedging nor in the calculation of the call option. Even if two investors have different subjective probabilities they attribute an identical value to the same option. The only requirement concerns their agreement on the two future prices of the underlying security.

2.2 The Black and Scholes Option Pricing Formula

According to Cox, Ross and Rubinstein, the BS formula can be considered as a limiting case of the BOP formula. Their idea is the following: as trading takes place continuously, they divided the time to expiration t of an option into n equal subperiods of lenght h.

$$h \approx \frac{t}{n}$$

When n goes to infinity, they showed that the BOP formula converges to the BS formula by setting

$$u = e^{\sigma\sqrt{t/n}}$$

$$d = e^{-\sigma\sqrt{t/n}}$$

$$r = r^{t/n}$$

$$q = \frac{1}{2} + \frac{\mu}{2\sigma}\sqrt{t/n}$$

where

- σ is the standard deviation of the continuously compounded rate of return of the security price,
- r is one plus the continuously compounded riskless rate of interest,
- μ is the mean of the continuously compounded rate of return of the security price,
- q is the subjective probability to have an upward move.

In that case, the multiplicative binomial probability distribution of the security price converges to a log-normal distribution, and the BOP model (2.11) converges to the BS model.

$$\boxed{C = SN(x) - Kr^{-t} N(x - \sigma\sqrt{t}\,)} \qquad (2.12)$$

where

$$x = \frac{\log(S/Kr^{-t})}{\sigma\sqrt{t}} + \frac{1}{2}\sigma\sqrt{t}$$

N(.) is the cumulative normal distribution.

In this regard I think that the method followed by Rendleman and Bartter is more rigorous. In order to determine the limits of u and d, they hold the logarithmic mean and variance of the security rate of returns constant over the life of the option. The values of u and d they obtained are:

$$u = \exp\left[\frac{\mu t}{n} + \sigma\sqrt{\frac{t}{n}}\sqrt{\frac{1-q}{q}}\,\right]$$

$$d = \exp\left[\frac{\mu t}{n} + \sigma\sqrt{\frac{t}{n}}\sqrt{\frac{q}{1-q}}\,\right]$$

Unlike the discrete case the probabilities q and (1-q), as well as μ enter the definition of u and d. This suggests that the option valuation depends on the subjective probabilities and the preferences of the investors. But as Rendleman and Bartter explained, this results from the fact that the binomial distribution is only an approximation of the continuous distribution. The value of u and d which are implicit in the latter may be reflected in the discrete model. When n goes to infinity, the two distributions are identical, and μ and q disappear.

It can be observed from (2.12) that the price of an option is function of five variables: the security price, the standard deviation of the security continuously compounded rate of return, the exercise price, the time to maturity of the option and the continuously compounded riskless interest rate. As I noted before, the expected rate of return as well as any measure of utility does not appear in the formula, and among the five variables, the standard deviation is the only one which is unknown and must be estimated using past prices.

3 The Additive Binomial Option Pricing Formula

The Binomial Option Pricing Formula derived in the preceding section assumes that the security price follows a multiplicative binomial distribution, or in the limit a lognormal distribution. This implies that the security price of the upward sequence is continuously increasing , it follows a geometric progression of ratio u and consequently could have no limit. With regard to the downward sequences, the situation is identical if $d > 1$, but if $d < 1$ the lower limit of the price is zero. As far as I am concerned, I think that this assumption of lognormality in the returns is restrictive and not always justified. In this regard, let us remark that the assumption of normal distribution is often preferred in equilibrium asset pricing models. Therefore I oriented my research to the derivation of option pricing models under the assumption of constant and additive changes in the security price. Under this assumption, at each period the security price S can only either increase to the amount $S + u^*$ or decrease to the amount $S + d^*$.

Let us consider a one period case and define

S = price of the security at the beginning of the period

K = the exercise price,

u^* = the change in the security price if it increases,

d^* = the change in the security price if it decreases,

C^* = the price of the call option at the beginning of the period,

C_u^* = the price of the call option at the end of the period if the security price increases,

C_d^* = the price of the call option at the end of the period if the security price decreases,

$r = 1 + R$ = one plus the riskless interest rate for the period.

To obtain the value of C^*, the hedge ratio which sets equal the payoffs at the end of the period is calculated in a way such that the payoff of the hedge portfolio is independent of the value of the security.

$$- C_u^* + \Delta(S + u^*) = - C_d^* + \Delta(S + d^*) \tag{3.1}$$

hence

$$\Delta = \frac{C_u^* - C_d^*}{S + u^* - S - d^*} = \frac{C_u^* - C_d^*}{u^* - d^*} \tag{3.2}$$

Again this riskless investment in the hedge portfolio must have the same return as an investment B^* in bonds at the risk free rate,

$$B^* = \frac{1}{r}[C_u^* - \Delta (S + u^*)]$$

then replacing Δ by its value

$$B^* = \frac{1}{r}\left(C_u^* - \frac{C_u^* - C_d^*}{u^* - d^*}(S + u^*)\right)$$

$$B^* = \frac{1}{r}\left(\frac{C_d^*(S + u^*) - C_u^* (S + d^*)}{(u^* - d^*)}\right) \tag{3.3}$$

and

$$C^* - \Delta S - B^* = 0$$

hence the value of the call is

$$C^* = \left(\frac{C_u^* - C_d^*}{u^* - d^*}\right)S + \frac{1}{r}\left(\frac{C_d^*(S + u^*) - C_u^* (S + d^*)}{u^* - d^*}\right)$$

$$C^* = \frac{1}{r}\left(\frac{Sr\, C_u^* - Sr\, C_d^* + S\, C_d^* + u^* C_d^* - S\, C_u^* - d^* C_u^*}{u^* - d^*}\right)$$

Since r=1+R, S(r-1) is equal to the profit obtained if S was invested at the risk free rate, that is, SR, then

$$C^* = \frac{1}{r}\left(\frac{C_u^*(SR - d^*) + C_d^*(u^* - SR)}{u^* - d^*}\right) \tag{3.4}$$

Defining $\frac{SR - d^*}{u^* - d^*}$ and $\frac{u^* - SR}{u^* - d^*}$ by p^* and $(1-p^*)$, I obtain the following one period option pricing formula,

$$C^* = \frac{1}{r}\left(p^* C_u^* + (1-p^*) C_d^*\right) \tag{3.5}$$

This relationship is similar to that obtained for the multiplicative model (2.7). Similarly, the no arbitrage condition implies that $d^* < SR < u^*$. Therefore p^* and $(1-p^*)$ can also be interpreted as pseudo probabilities values of a trial whose outcomes are C_u^* and C_d^* . And the value of the call option is equal to the expected value of its price at the end of the period, discounted by the risk free rate. Likewise the multiplicative case, the values of the call option at the end of the period are:

$$C_u^* = \max(0, S + u^* - K)$$

$$C_d^* = \max(0, S + d^* - K)$$

then (3.5) becomes

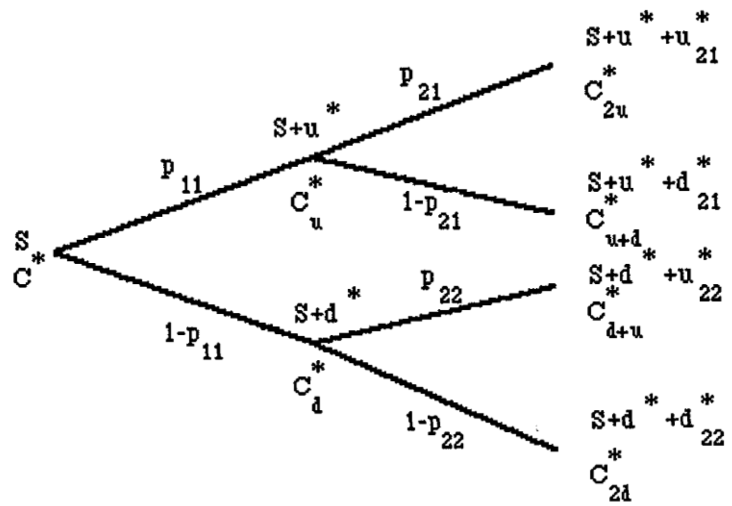
$$C^* = \frac{1}{r}\left(p^* \max(0, S + u^* - K) + (1-p^*) \max(0, S + d^* - K)\right) \tag{3.6}$$

Now, I will extend this one period option pricing formula to a multiperiod framework. In this respect, different sets of assumptions can be made. At this stage however, it seems better to start with a general formulation without assumptions. So let us consider a two period case, referring to

- u^* as the upward movement for the first period,
- d^* as the downward movement for the first period,
- p^*, $(1-p^*)$ as the pseudo probabilities of the upward and downward movements for the first period,
- r_1 as one plus the riskless interest rate (R_1) for the first period
- r_2 as one plus the riskless interest rate (R_2) for the second period.

- u^*_{21} as the upward movement for the second period, after a upward movement during the first period,
- u^*_{22} as the upward movement for the second period, after a downward movement during the first period,
- d^*_{21} as the downward movement for the second period after a upward movement during the first period,
- d^*_{22} as the downward movement for the second period after a downward movement during the first period,
- p_{21} , $(1-p_{21})$ as the pseudo probabilities of the upward and downward movements for the second period after an upward movement during the first period,
- p_{22} , $(1-p_{22})$ as the pseudo probabilities of the upward and downward movements for the second period after a downward movement during the first period,
- C^*_u , C^*_d , C^*_{2u} , C^*_{u+d}, C^*_{d+u} and C^*_{2d} the value of the call option after respectively one upward move, one downward move, two upward moves, one upward move followed by one downward move, one downward move followed by one upward move and two downward moves.

The tree diagram of the different values for the security and the call option is



The procedure I used to determine the call value for the one period case can be applied, starting recursively at the end of the tree and working backward to the initial period.Applying (3.4), the call values of the second and the first periods are:

$$C_u^* = \frac{1}{r_2}\left(\frac{C_{2u}^*\left(R_2(S+u^*)-d_{21}^*\right) + C_{u+d}^*\left(u_{21}^*- R_2(S+u^*)\right)}{u_{21}^* - d_{21}^*}\right) \tag{3.7}$$

$$C_d^* = \frac{1}{r_2}\left(\frac{C_{u+d}^*\left(R_2(S+d^*)-d_{22}^*\right) + C_{2d}^*\left(u_{22}^*- R_2(S+d^*)\right)}{u_{22}^* - d_{22}^*}\right) \tag{3.8}$$

$$C^* = \frac{1}{r_1}\left(\frac{C_u^*\left(R_1S - d^*\right) + C_d^*\left(u^*- R_1S\right)}{u^* - d^*}\right) \tag{3.9}$$

From the observation of (3.7), (3.8) and (3.9), it can be seen that one cannot assume both u^* and d^*, and p^* constant for the two periods, since in this case we should have

$$p_{11} = \frac{R_2(S+u^*) - d^*}{u^* - d^*} = \frac{R_2(S+d^*) - d^*}{u^* - d^*} = \frac{R_1S}{u^* - d^*}$$

These relationships could be true only if the two riskless interest rates R_1 and R_2 are both assumed equal to zero, which is unreal. Therefore as it is p^* and $(1-p^*)$ that are determined by the value of the movements and the riskless interest rate, I will first examine the case assuming u^* , d^* and R constant over the periods. In such case the value of $p^*$of one period is not constant, and it depends on the moves of the previous periods. Another approach consists in letting u^* and d^* vary over the periods, although assuming a certain relationship from one period to another. So, for example , I will show that if both u^* and d^* vary from one period to another, in such a way that the changes in percentage of both upward and downward moves in the security price are constant, the additive process reduces to the multiplicative process.

3.1 Case assuming that u^*, d^* and R are constant

Starting from (3.7), (3.8) and (3.9) and assuming u^*, d^* and R constant, the initial value of the call becomes

$$C^* = \frac{1}{r^2}\Big(p_{11}p_{21}C_{2u}^* + p_{11}(1-p_{21})C_{u+d}^* + (1-p_{11})p_{22}C_{d+u}^* + (1-p_{11})(1-p_{22})C_{2d}^* \Big) \tag{3.10}$$

where

$$p_{11} = \frac{SR - d^*}{u^* - d^*}$$

$$p_{21} = \frac{(S+u^*)R - d^*}{u^* - d^*}$$

$$p_{22} = \frac{(S+d^*)R - d^*}{u^* - d^*}$$

The probabilities p_{21} and p_{22} can also be rewritten:

$$p_{21} = p_{11} + \frac{u^*R}{u^* - d^*}$$

$$p_{22} = p_{11} + \frac{d^*R}{u^* - d^*}$$

To simplify the notations, let us call p_{11} by p^*, and let us replace $\frac{u^*R}{u^* - d^*}$ and $\frac{d^*R}{u^* - d^*}$ by u' and d'. Substituting these changes in (3.10) and rearranging yields:

$$C = \frac{1}{r^2}\left\{p^{*2}C^*_{2u} + p^*(1-p^*)C^*_{u+d} + p^*(1-p^*)C^*_{d+u} + (1-p^*)^2C^*_{2d}\right\} + \frac{1}{r^2}\left\{p^*u'\left(C^*_{2u} - C^*_{u+d}\right) + (1-p^*)d'\left(C^*_{d+u} - C^*_{2d}\right)\right\} \quad (3.11)$$

Therefore, recalling that

$$C^*_{2u} = \max(0, S+2u^* - K)$$

$$C^*_{u+d} = C^*_{d+u} = \max(0, S+u^* + d^* - K)$$

$$C^*_{2d} = \max(0, S+2d^* - K)$$

then (3.11) becomes

$$\begin{aligned} C = \frac{1}{r^2}\{&p^{*2}\max(0,S+2u^*-K) + 2p^*(1-p^*)\max(0,S+u^*+d^*-K) \\ &+ (1-p^*)^2\max(0,S+2d^*-K)\} \\ +\frac{1}{r^2}\{&p^*u'\left(\max(0,S+2u^*-K) - \max(0,S+u^*+d^*-K)\right) \\ +\ &(1-p^*)d'\left(\max(0,S+u^*+d^*-K) - \max(0,S+2d^*-K)\right)\} \end{aligned} \quad (3.12)$$

Let us call the first part in bracket of the equation (3.12) T_1 and the second part T_2, and let us examine their form for n periods. The coefficients of T_1 have the form of a binomial expression, therefore the relationship for n periods is:

$$T_1 = \frac{1}{r^n}\left(\sum_{x=0}^{n} \frac{n!}{x!(n-x)!} p^{*x} (1-p)^{*(n-x)} \max(0;S+xu^*+(n-x)d^*-K) \right) \quad (3.13)$$

Let us eliminate now the sequences which give a zero value at the call option, that is sequences whose number of ups is such that

$$S + xu^* + (n-x)d^* - K < 0$$

If the minimum number of upward moves giving a positive value to the call option is defined by 'a', (3.13) becomes:

$$T_1 = \frac{1}{r^n}\left(\sum_{x=a}^{n} \frac{n!}{x!(n-x)!} p^{*x} (1-p)^{*(n-x)} \left(S + xu^* + (n-x)d^* - K\right) \right) \quad (3.14)$$

As for the second part T_2 of (3.12), when the number of periods increases, it becomes rapidly very complex, and it can be shown that it is an expression of n degree terms in u', d',p^* and ($1-p^*$), such as for example

$$p^{*n-1}u',\ p^{*n-2}u'^2,\ \dots,\ p^*u'^{n-1},$$

$$p^{*n-2}u'd',\ \dots,\ p^*u'^{n-2}d',\ \dots .$$

Obviously, it is not easy to derive a simple general expression for T_2, as well as to manipulate it even for small number of periods, since its number of terms becomes rapidly large. Let us remark that if the riskless interest rate is assumed to be equal to zero, the call valuation formula reduces to T_1 .

However, let us note that given the definitions of u' and d', the terms whose degree in u' and d' is greater than one are insignificant in comparison with the terms of first degree in u' and d', and can be eliminated from the expression. In this case, it can be shown that the general formulation T_2 is:

$$T_2 \cong \frac{1}{r^n}\left(\sum_{j=0}^{n-2}\left(\sum_{i=1}^{n-1} i\right)\frac{(n-2)!}{(n-j-2)!j!}\ u'p^{*\,n-j-1}(1-p^*)^j\right.$$

$$\Big(\max\big(0;S+(n-j)u^*+jd^*-K\big) - \max\big(0;S+(n-j-1)u^*+(j+1)d^*-K\big)\Big)$$

$$+\sum_{j=1}^{n-1}\left(\sum_{i=1}^{n-1} i\right)\frac{(n-2)!}{(n-j-1)!(j-1)!}\ d'p^{*\,n-j-1}(1-p^*)^j$$

$$\left.\Big(\max\big(0;S+(n-j)u^*+jd^*-K\big) - \max\big(0;S+(n-j-1)u^*+(j+1)d^*-K\big)\Big)\right) \quad (3.15)$$

Considering again only the sequences of price which give a positive value to the option, ie sequences whose minimum number of upward movements is equal to a, T_2 is rewritten:[1]

$$T_2 \cong \frac{1}{r^n}[u^* - d^*]\left(\sum_{i=1}^{n-1} i\right)\left(\sum_{j=0}^{n-a-1}\frac{(n-2)!}{(n-j-2)!j!}\ u'p^{*\,n-j-1}(1-p^*)^j\right.$$

$$\left.+\sum_{j=1}^{n-a-1}\frac{(n-2)!}{(n-j-1)!(j-1)!}\,d'p^{*n-j-1}(1-p^*)^j\right)$$

$$+\frac{1}{r^n}\left(\left(\sum_{i=1}^{n-1} i\right)p^{*a-1}(1-p^*)^{n-a}\max\big(0;S+au^*+(n-a)d^*-K\big)\right)$$

$$\left(\frac{(n-2)!}{(n-a)!(a-2)!}\,u' + \frac{(n-2)!}{(n-a-1)!(a-1)!}\,d'\right) \quad (3.16)$$

Hence recalling $u' = \dfrac{u^*R}{u^* - d^*}$ and $d' = \dfrac{d^*R}{u^* - d^*}$,

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1
If the option values for respectively (n-j) and (n-j-1) upward movements are positive, that is,

$$\max\big(0;S+(n-j)u^*+jd^*-K\big) = S+(n-j)u^*+jd^*-K$$

$$\max\big(0;S+(n-j-1)u^*+(j+1)d^*-K\big) = S+(n-j-1)u^*+(j+1)d^*-K)$$

then their difference is equal to $(u^* - d^*)$.

$$T_2 \cong \frac{1}{r^n}\left(\sum_{i=1}^{n-1} i\right)\left(\sum_{j=0}^{n-a-1} \frac{(n-2)!}{(n-j-2)!j!}\ u^* R p^{*n-j-1}(1-p^*)\right.$$

$$\left. + \sum_{j=1}^{n-a-1} \frac{(n-2)!}{(n-j-1)!(j-1)!} d^* R p^{*n-j-1}(1-p^*)^j \right)$$

$$+ \frac{1}{r^n}\left(\sum_{i=1}^{n-1} i\right)\left(p^{*a-1}(1-p^*)^{n-a} \max\left(0; S+au^*+(n-a)d^*-K\right)\right)$$

$$\left(\frac{R}{u^* - d^*}\right)\left(\frac{(n-2)!}{(n-a)!(a-2)!}\, u^* + \frac{(n-2)!}{(n-a-1)!(a-1)!}\, d^*\right) \qquad (3.17)$$

What about the value of T_2 in comparison with the value of T_1 ? In this regard, it can be observed from (3.14) and (3.17) that the value K of the exercise price is determinant. Considering T_1 , K determines the number of positive terms in the valuation formula, as well as their magnitude; the smaller K is, the more positive terms there are and the greater they are. As for the expression of T_2 , K only determines its number of positive terms. I conclude that, other things being equal, the magnitude of T_2 is less important in the valuation formula for the *in the money option* than for the *out of the money option.* Although it could be more justified in some cases where the returns of an asset are normally distributed, the discrete additive valuation formula I derived in this section is not easily applicable, especially for large number of periods.

3.2 Derivation of the multiplicative OPM

Let us now suppose that u^* and d^* vary, assuming however that the percentage of the upward and downward movements of the security price are constant and equal to u" and d". The riskless interest rate is also assumed constant.

In that case, the movements in the tree diagram are

$u^*_{21} = (S+ u^*)u''$
$u^*_{22} = (S+ d^*)u''$
$d^*_{21} = (S+ u^*)d''$
$d^*_{22} = (S+ d^*)d''$
$u^* = Su''$
$d^* = Sd''$

Replacing these values in (3.7), (3.8) and (3.9) and rearranging I obtain,

$$C_u^* = \frac{1}{r}\left(\frac{C_{2u}^*(R\text{-}d'') + C_{u+d}^*(u''\text{-}R)}{u''\text{-}d''}\right) \qquad (3.18)$$

$$C_d^* = \frac{1}{r}\left(\frac{C_{u+d}^*(R\text{-}d'') + C_{2d}^*(u''\text{-}R)}{u''\text{-}d''}\right) \qquad (3.19)$$

$$C = \frac{1}{r}\left(\frac{C_u^*(R\text{-}d'') + C_d^*(u''\text{-}R)}{u''\text{-}d''}\right) \qquad (3.20)$$

If I substitute again the preceding values of C_u^* and C_d^* in (3.20) and if I refer to p'' and (1-p'') as $\frac{R\text{-}d''}{u''\text{-}d''}$ and $\frac{u''\text{-}R}{u''\text{-}d''}$, the price C of the call option becomes:

$$C = \frac{1}{r_2}\left(C_{2u}^*\ p''^2 + 2\ C_{u+d}^*\ p''(1\text{-}p'') + C_{2d}^*\ (1\text{-}p'')^2\right) \qquad (3.21)$$

This equation is equivalent to the two period case expression of the multiplicative binomial formula (2.9). To be convinced, one must consider that p'' and (1-p'') are respectively equal to p and (1-p),[1] and that C_{2u}^* , C_{u+d}^* and C_{2d}^* are respectively equal to C_{uu}, C_{ud} and C_{dd}.[2]

4 Continuous Time Version of the Additive BOPM

In this section I derive the continuous time version of the additive BOPM, letting the number of periods become infinite. To do so, I cannot simply use a larger number of periods, month, week or day, such as it was the case for the general binomial formula, but I must divide this calendar time in infinitesimally small

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1

$$p'' = \frac{R\text{-}d''}{u''\text{-}d''} = \frac{R+1\text{-}1\text{-}d''}{1+u''\text{-}1\text{-}d''} = \frac{r\text{-}d}{u\text{-}d}$$

$$(1\text{-}p'') = \frac{u''\text{-}R}{u''\text{-}d''} = \frac{1+u''\text{-}1\text{-}R}{1+u''\text{-}1\text{-}d''} = \frac{u\text{-}r}{u\text{-}d}$$

2

$C_{2u}^* = S + u^* + u_{21}^* - K$

$C_{2u}^* = S + Su'' + (S+Su'')u'' - K$

$C_{2u}^* = S(1 + 2u'' + u''^2) = S(1+u'')^2 = Su^2 = C_{uu}$

Similarly

$C_{u+d}^* = S(1+u'')(1+d'') = Sud = C_{ud}$

$C_{2d}^* = S(1+d'')^2 = Sd^2 = C_{dd}$

subperiods, assuming that the option is continuously traded. The reason is threefold. First, because options are really continuously traded. Secondly because considering a large number of period, such as month or week, is unreal given the limited life of an option. And thirdly because the constant additive assumptions made imply that the rate of return of the security is decreasing in time, their limit being equal to zero, while the riskless interest rate is kept constant.

Considering infinitesimally small subperiods or intervals, the values of u^* and d^* must be adapted so that they become sufficiently small to impede large change in the security price for small time intervals. One way to do that is to keep the mean and the variance of the security price equivalent for the discrete and the continuous models for t periods. As for the riskless interest rate R_n for an infinitesimal part of the t periods, it is equal to

$$R_n = r^{t/n} - 1 \approx t\frac{r}{n} - 1 = \frac{tR}{n}$$

As I have two parts T_1 and T_2 in the discrete valuation formula (3.10), I will examine each of them separately. Of course, as I suggested in the preceding section, the first part prevails on the second one, I will therefore focus more particularly on the first one in order to determine the values of u^* and d^* . Afterwards, I will examine the impact of these values for the second term T_2. Let us refer to u_n^* and d_n^* as the u^* and d^* for an infinitesimal subperiod. In order to determine the values of u_n^* and d_n^* , I will first calculate the mean and the variance of the change in the security price for the t periods. The expected change in the security price after t periods is equal to the sum of the expected number of upward and downward moves in the price, that is,

$$E[S-S_t] = E[x]u^* + (t-E[x])d^*$$

and its variance,

$$\mathrm{var}(S-S_t) = \mathrm{var}\left(xu^* + (t-x)d^*\right)$$

$$\mathrm{var}(S-S_t) = \mathrm{var}\left(x(u^* - d^*) + td^*\right) = (u^* - d^*)^2 \,\mathrm{var}(x)$$

As the sequence of ups and downs follows a binomial process whose probabilities are q and (1-q), then the expected number of up moves and their variance are

$$E[x] = tq$$

$$\mathrm{var}(x) = tq(1-q)$$

Substituting these two values in the expected value and the variance of the change in the security price yields

$$E(S-S_t) = tqu^* + (t-tq)d^* = t\left(qu^* + (1-q)d^*\right) = t\mu$$

$$\mathrm{var}(S-S_t) = (u^* - d^*)^2\, tq(1-q) = t\sigma^2$$

where μ and σ^2 are respectively the mean and variance of the change in the security price for one period.[1]

Afterwards, in order to keep the same mean and variance for the t periods when these periods are divided into n infinitesimally subperiods,the values of u_n^* and d_n^* must be such that

$$n[qu_n^* + (1-q)d_n^*] = E(S-S_t) = t\mu \tag{4.1}$$

$$nq(1-q)(u_n^* - d_n^*)^2 = \mathrm{var}(S-S_t) = t\sigma^2 \tag{4.2}$$

[1]For one period, the mean of the change in the security price is

$$\mu = qu^* + (1-q)d^*$$

and its variance

$$\sigma^2 = q(u^* - \mu)^2 + (1-q)(d^* - \mu)^2$$

$$\sigma^2 = q\left(u^* - qu^* - (1-q)d^*\right)^2 + (1-q)\left(d^* - qu^* - (1-q)d^*\right)^2$$

$$\sigma^2 = q\left((u^* - d^*)(1-q)\right)^2 + (1-q)\left((u^* - d^*)q\right)^2$$

$$\sigma^2 = q(1-q)(u^* - d^*)^2$$

The value of u_n^* and d_n^*, which are a solution to this two equation system with two unknowns, are[1]

$$u_n^* = \frac{\mu t}{n} + \sigma\sqrt{\frac{t}{n}}\sqrt{\frac{1-q}{q}}$$

$$d_n^* = \frac{\mu t}{n} + \sigma\sqrt{\frac{t}{n}}\sqrt{\frac{q}{1-q}}$$

Likewise in the multiplicative case q and (1-q), the true probabilities and μ, the mean of the return, intervene in the definition of u_n^* and d_n^*. This suggests that the option valuation depends on both the subjective probabilities and the preference of the investor. Again this is due to the approximation of the continuous distribution by a binomial distribution, and I will show further that both q and μ will not appear in the continuous time valuation relationship.

Now the question is to know how the binomial distribution with probabilities p^* and $(1-p^*)$ converges. In this regard I must determine the limit values of p^* and $(1-p^*)$. To do that, let us replace u^*, d^* and R in the determination of p^* and $(1-p^*)$ by u_n^*, d_n^* and $\frac{tR}{n}$.

$$p^* = \frac{SR - d^*}{u^* - d^*} = \frac{\frac{StR}{n} - d_n^*}{u_n^* - d_n^*}$$

$$p^* = \frac{\frac{StR}{n} - \left(\mu\frac{t}{n} - \sigma\sqrt{\frac{t}{n}}\sqrt{\frac{q}{1-q}}\right)}{\frac{\mu t}{n} + \sigma\sqrt{\frac{t}{n}}\sqrt{\frac{1-q}{q}} - \frac{\mu t}{n} + \sigma\sqrt{\frac{t}{n}}\sqrt{\frac{q}{1-q}}}$$

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[1]From the equation (4.1), the expression of u_n^*, function of d_n^*, is

$$u_n^* = \frac{t\mu - n(1-q)d_n^*}{nq}$$

The substitution of this result in (4.2) yields the second degree polynome

$$n^2 d_n^{*2} - 2\mu t n d_n^* + t^2\mu^2 - t\sigma^2\frac{nq}{1-q}$$

whose roots such that $u_n^* > d_n^*$ are

$$u_n^* = \frac{\mu t}{n} + \sigma\sqrt{\frac{t}{n}}\sqrt{\frac{1-q}{q}}$$

$$d_n^* = \frac{\mu t}{n} - \sigma\sqrt{\frac{t}{n}}\sqrt{\frac{q}{1-q}}$$

$$p^* = \frac{\frac{StR}{n} - \mu \frac{t}{n} + \sigma \sqrt{\frac{t}{n}}\sqrt{\frac{q}{1-q}}}{\sigma \sqrt{\frac{t}{n}}\sqrt{\frac{1}{q(1-q)}}}$$

$$p^* = \frac{(SR - \mu)\sqrt{tq(1-q)}}{\sigma\sqrt{n}} + q \tag{4.3}$$

Similarly

$$(1-p^*) = \frac{(SR - \mu)\sqrt{tq(1-q)}}{\sigma\sqrt{n}} + (1-q) \tag{4.4}$$

So, as n becomes large, the first term of the expression of p^* and $(1-p^*)$ reduces to zero and p^* and $(1-p^*)$ converge respectively to q and (1-q). This is illustrated by an example in figure 1 with p=.35 and q=.5.

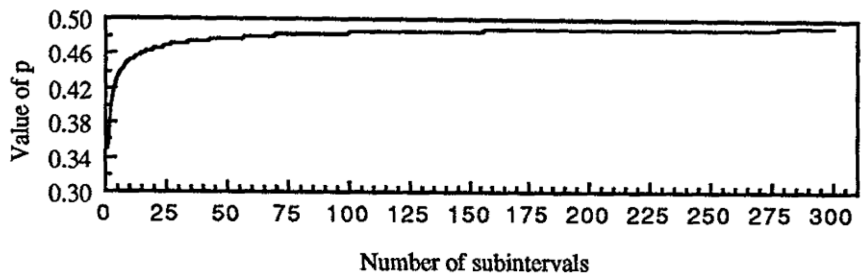

Figure 1

The central limit theorem says that the normal distribution provides the best accurate approximation to the binomial distribution when n is large and q is close to $\frac{1}{2}$, and a fairly good approximation when the probability is different from $\frac{1}{2}$ and the number of observations is very large.[1]

Furthermore when n goes to infinity the approximation will be exact. Therefore the cumulative binomial distribution with probability p^* can be approximated by a cumulative normal distribution. Since every normal distribution is characterized by a combination of mean and standard deviation, it remains to evaluate the mean μ_p and the standard deviation σ_p of the normal distribution of the returns.

[1]The rule of thumb is that $np > 5$ and $n(1-p) > 5$.

$$\mu_p = n\left(p^* u_n^* + (1-p^*) d_n^*\right)$$

$$\mu_p = n \frac{\frac{StR}{n} u^* - d_n^* u_n^* + u_n^* d_n^* - \frac{StR}{n} d_n^*}{u_n^* - d_n^*}$$

$$\mu_p = StR \qquad (4.5)$$

$$\sigma_p^2 = (u_n^* - d_n^*)^2 np^*(1-p^*)$$

$$\sigma_p^2 = \frac{(u_n^* - d_n^*)^2 \; n \left(\frac{StR}{n} - d_n^*\right)\left(u_n^* - \frac{StR}{n}\right)}{(u_n^* - d_n^*)\;(u_n^* - d_n^*)}$$

$$\sigma_p^2 = n\left(\frac{StR}{n} - d_n^*\right)\left(u_n^* - \frac{StR}{n}\right)$$

Substituting u_n^* and d_n^* by their value I obtain,

$$\sigma_p^2 = \sigma^2 t - \frac{1}{n}(\mu\, t\text{-}StR)^2 - (\mu\, t\text{-}StR)\sigma\sqrt{\frac{t}{n}}\left(\sqrt{\frac{q}{1-q}} - \sqrt{\frac{1-q}{q}}\right)$$

Since n becomes very large, the second and the third terms of the right member of the equation vanish, and

$$\sigma_p^2 = \sigma^2 t \qquad (4.6)$$

This is illustrated in figure 2. The estimated variance however converges more quickly to the true variance when the number of subperiods is increased than does p.

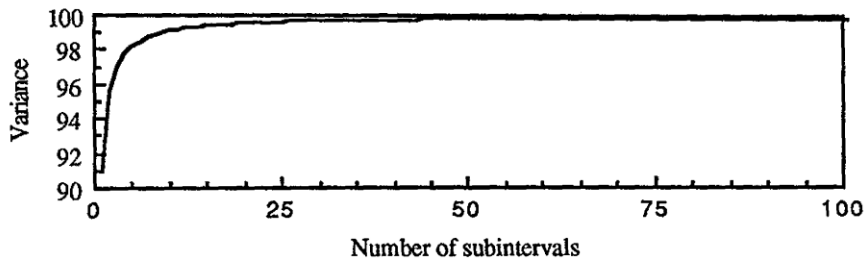

Figure 2

As tables of normal probability values are generally based on the standard normal distribution, it remains to convert the values of the normal variable, ie the change in the security price, into standard normal value. That is, I have to evaluate the lower limit zl of the cumulative standard normal distribution.

$$zl = \frac{Y_a - \mu_p}{\sigma_p}$$

where Y_p is the minimum change in the security price involving enough up moves so that the price is at least equal to the exercise price. Let us recall that according to the additive BOPM, the number of upward moves is

$$S + au_n^* + (n\text{-}a)d_n^* - K \geq 0$$

hence the minimum change in the security price

$$Y_a = au_n^* + (n\text{-}a)d_n^* \geq K - S$$

Since n becomes infinite, the difference in the preceding inequality becomes so small that the inequality can be considered as an equality, hence

$$Y_a = K - S$$

Replacing Y_a , μ_p and s_p by their value, the standard normal value zl becomes

$$zl = \frac{(K\text{-}S) - StR}{\sigma\sqrt{t}}$$

$$zl = \frac{K - StR - S}{\sigma\sqrt{t}} \approx \frac{K - S(1+R)^t}{\sigma\sqrt{t}} = \frac{K - Sr^t}{\sigma\sqrt{t}} \tag{4.7}$$

Let us first derive the continuous version of T_1 , denoting the security return by the normal variable Y,

$$T_1 = \frac{1}{r^t} \int_{Y_a}^{\infty} (S + Y - K) \frac{1}{\sigma_p\sqrt{2\pi}} e^{\left(-\frac{(Y - \mu_p)^2}{2\sigma_p^2}\right)} dY \tag{4.8}$$

Let us now convert the normal variable Y into standard normal variable Z,

$$\frac{Y - \mu_p}{\sigma_p} = \frac{Y - StR}{\sigma\sqrt{t}} = Z$$

then

$$Y = Z\sigma\sqrt{t} + StR$$

Then splitting the integral of (4.8) into two parts and replacing the normal variable Y by the standard normal variable Z yields:

$$T_1 = \frac{1}{r^t}\,(S + StR - K) \int_{Zl}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{\left(\frac{-Z^2}{2}\right)} dZ + \frac{1}{r^t}\sigma\sqrt{t} \int_{Zl}^{\infty} Z\,\frac{1}{\sqrt{2\pi}}\, e^{\left(\frac{-Z^2}{2}\right)} dZ$$

$$T_1 = \frac{1}{r^t}\,(Sr^t - K) \int_{Zl}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{\left(\frac{-Z^2}{2}\right)} dZ + \frac{1}{r^t}\sigma\sqrt{t} \int_{Zl}^{\infty} Z\,\frac{1}{\sqrt{2\pi}}\, e^{\left(\frac{-Z^2}{2}\right)} dZ$$

Furthermore, as I need the probability of the complementary normal distribution, that is, 1-Nc(zl), and given the symmetry property of the normal distribution which states 1-Nc(zl)=Nc(-zl), the standard normal value zl for the first term of the right member of the equation is :

$$zl = \frac{Sr^t - K}{\sigma\sqrt{t}}$$

Hence the expression of T_1 in continuous time is:[1]

$$T_1 = (S - \frac{K}{r^t})\, Nc\left[\frac{Sr^t - K}{\sigma\sqrt{t}}\right] + \frac{1}{r^t}\sigma\sqrt{t}\, N\left[\frac{K - Sr^t}{\sigma\sqrt{t}}\right] \qquad (4.9)$$

where Nc[.] is the cumulative standard normal distribution and N[.] is the

[1]See Winkler R.L., Roodman G.M. and Britney R.R., 'The Determination of Partial Moments', *Management Science*, 19, 1972, 290-296.

standard normal distribution.

It remains to derive the corresponding expression of T_2 when u^*, d^* and R are replaced by u_n^* ,d_n^* and $\frac{tR}{n}$. Since all terms of T_2 are function of u' and d', let us have a look at the values of u' and d':

$$u' = \frac{u_n^* \frac{tR}{n}}{u_n^* - d_n^*} = \frac{\mu R t^{3/2}}{2\sigma n^{3/2}} + \frac{tR}{2n}$$

$$d' = \frac{d_n^* \frac{tR}{n}}{u_n^* - d_n^*} = \frac{\mu R t^{3/2}}{2\sigma n^{3/2}} + \frac{tR}{2n}$$

Clearly, the different terms of T_2 converge to zero when n goes to infinity, therefore the resulting additive call valuation formula in continuous time is:

$$\boxed{C = (S - \frac{K}{r^t})\, Nc\left[\frac{Sr^t - K}{\sigma\sqrt{t}}\right] + \frac{1}{r^t}\sigma\sqrt{t}\, N\left[\frac{K - Sr^t}{\sigma\sqrt{t}}\right]} \qquad (4.10)$$

where

- S is the initial security price,
- K is the exercise price,
- r is one plus the annual riskless interest rate R,
- σ is the standard deviation of the security prices,
- t is the number of periods in years,
- Nc[.] is the cumulative standard normal distribution,
- N[.] is the standard normal distribution.

The call valuation expressed in equation (4.10) is consistent with the single period call option formula Brennan (1979) derived, although the methods are different. While I used arbitrage methods, Brennan used utility functions. He showed that under the assumption of normality distribution of the prices, it is possible to derive a call option pricing relationship, also called risk neutral valuation relationship, for options if the utility function of the investor exhibits constant absolute risk aversion. This is of course consistent with the assumptions I made here. In this regard, I note that I never assumed $q=(1-q)=\frac{1}{2}$ since the central limit theorem guarantees that the normal distribution is a good approximation of the binomial distribution, even if q is different from $\frac{1}{2}$, ie the investor is risk averse.

Unlike the discrete call valuation formula described in the preceding section, the continuous additive option pricing formula is quite interesting. Similarly to the BS model, it is determined by five variables: the security price, the standard deviation of the security returns, the exercise price of the option, the time to maturity and the riskless interest rate. This formula can also be easily tested, and in order to use it, one only needs to estimate the standard deviation of the asset prices and to manipulate tables of the standard normal distribution.

The impact of changes in these five variables on the call value satisfies the basic relationships specified in the first section.[1]

a) If the security price increases, the call value will increase,

$$\frac{\delta C}{\delta S} = Nc\left(\frac{Sr^t - K}{\sigma\sqrt{t}}\right) > 0$$

This partial derivative gives in fact the value of the hedge ratio or proportion of the security in the hedge portfolio.

b) If the exercise price increases, the call value will decrease,

$$\frac{\delta C}{\delta K} = -\, r^t\, Nc\left(\frac{Sr^t - K}{\sigma\sqrt{t}}\right) < 0$$

c) If the volatility of the security increases, the call value will increase,

$$\frac{\delta C}{\delta \sigma} = \sqrt{t}\, r^t\, N\left(\frac{S - Kr^{-t}}{\sigma\sqrt{t}}\right) > 0$$

d) If the time to expiration increases, the call value will increase,

$$\frac{\delta C}{\delta t} = Kr^{-t}R\; Nc\left(\frac{Sr^t - K}{\sigma\sqrt{t}}\right) + \frac{\sigma r^{-t}}{2\sqrt{t}}(1\text{-}2Rt)\; N\left(\frac{K - Sr^t}{\sigma\sqrt{t}}\right) > 0$$

e) If the riskless interest rate increases, the call value will increase,

[1] See appendix 1 for the derivation of the partial derivatives.

$$\frac{\delta C}{\delta r} = Kt\, r^{-t}\, Nc\left(\frac{Sr^t - K}{\sigma\sqrt{t}}\right) - \frac{\sigma}{r^t\sqrt{t}} N\left(\frac{K - Sr^t}{\sigma\sqrt{t}}\right) > 0$$

The call value of the additive model is bounded between the two straight lines $C = K$ and $C = S - Kr^{-t}$. The value of a call on the one hand cannot be greater than the security price, and on the other hand it must be greater than the difference between the security price and the discounted value of the exercise price. The lower bound is approached when S becomes very large, as is illustrated in figure 3.

In order to compare the B-S and the additive option pricing models, I calculated the call values using both models for the same example with K=100, R=.12 and t=180, letting the security price vary. A part of the two curves is represented in figure 4. The value of the call of the additive model is slightly greater than the B-S model for *out of the money* call options and the reverse for *in the money* call options. For *at the money* call options there is practically no difference between the call values for both models. This is not surprising: the B-S model assumes that the security returns follow a lognormal distribution. It consequently gives a higher(lower) probability of occurrence to the large(low) security prices than does the normal distribution.

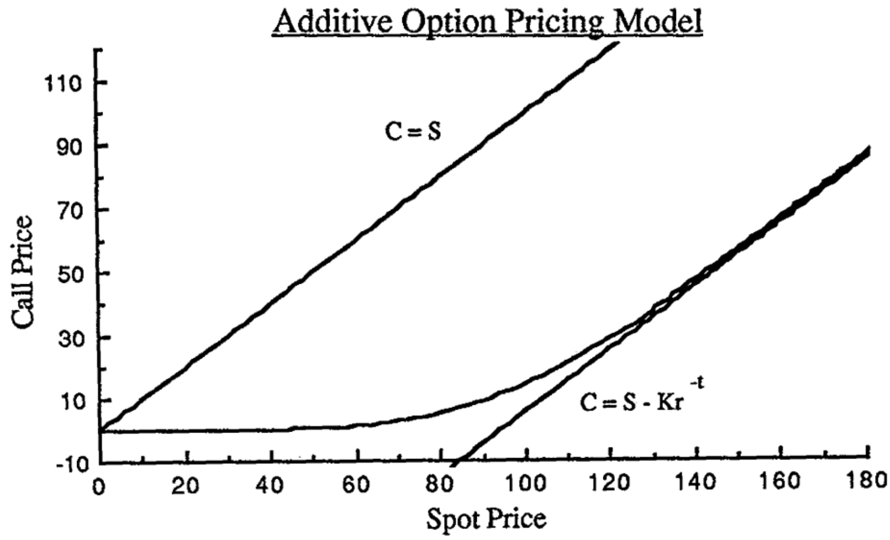

Figure No.3

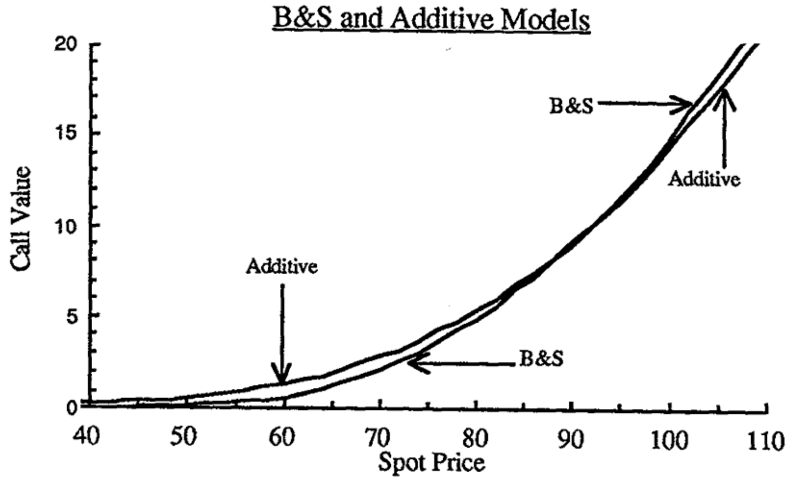

Figure No.4

5 Option Pricing Models on Portfolios

Since the payoffs of a lot of contingent claims depend on several random variables, it is interesting to extend the previous option valuation models to the multivariate case, considering more particularly the case of a portfolio. If the additive option pricing model is relevant to each of the asset of a portfolio, then the additive model is also relevant to valuate calls on the portfolio. Indeed, as the additive option pricing model assumes that the returns of an asset are normally distributed and as the return of a portfolio is by construction a linear combination of the asset returns, any linear combination of the asset returns is also normally distributed. The additive option valuation model can therefore be used to value options on both the individual assets and the portfolio. As for the B-S model it assumes that the returns are lognormally distributed. Therefore since a linear combination of lognormal distributions is not lognormal, there is an inconsistency in using the B-S model for both the individual assets and the portfolio.

By definition the additive option pricing model is relevant to valuate call options on portfolios when it is relevant to the individual assets of the portfolio. So let us, for example, consider a portfolio composed of two securities S_1 and S_2. The individual tree diagrams of these securities for one period are represented in figure 5 as well as a possible tree diagram of their portfolio. The portfolio tree diagram can be represented as a two step tree. The first step shows the different states of the security S_1 and the second step shows the states of the security S_2 conditional on the states of S_1. The probability p_2' at the second step is the conditional probability to obtain the S_2 states given a certain state of S_1. The diagram tree of figure 5 can be replaced by a binomial tree whose values of u^* and d^* are calculated so that the mean and the variance of the changes in the portfolio prices are the same. Keeping the same mean and variance guarantees that at the limit, that is when the period is subdivised into an infinity of subperiods, the two tree diagrams will be identical.

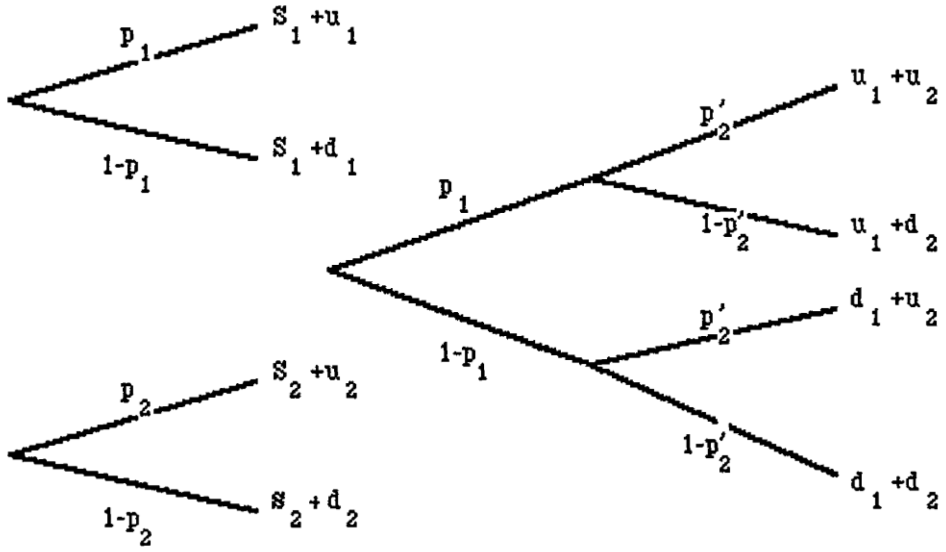

Figure 5

If I note by x_i the proportion of security i in the portfolio, the expected return and the variance of a portfolio composed of m assets are for one period:

$$\mu_p = \sum_{i=1}^{m} x_i \, \Delta \, S_i$$

$$\sigma_p^2 = \sum_{i=1}^{m} \sum_{j=1}^{m} x_i \, x_j \, \sigma_{ij}$$

where ΔS_i and σ_{ij} are respectively the return of security i and the covariance between security i and j. The ups and downs of the binomial tree must be such that the mean and the variance of the binomial tree are equal to μ_p and σ_p^2. The value of the probability q can arbitrarily be set to $\frac{1}{2}$. That is:

$$q\, u_p + (1-q)\, d_p = \mu_p$$

$$q(1-q)\left(u_p - d_p\right)^2 = \sigma_p^2$$

The values of u_p and d_p which solve these two equations are :

$$u_p = \sum_{i=1}^{m} x_i \Delta S_i + \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{m} x_i x_j \sigma_{ij}}$$

$$d_p = \sum_{i=1}^{m} x_i \Delta S_i - \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{m} x_i x_j \sigma_{ij}}$$

Generalizing for t periods, the continuous value u_n and d_n of u_p and d_p of the portfolio tree diagram are :

$$u_n = \sum_{i=1}^{m} x_i \Delta S_i \frac{t}{n} + \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{m} x_i x_j \sigma_{ij}} \sqrt{\frac{t}{n}} \sqrt{\frac{1-q}{q}} \qquad (5.1)$$

$$d_n = \sum_{i=1}^{m} x_i \Delta S_i \frac{t}{n} - \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{m} x_i x_j \sigma_{ij}} \sqrt{\frac{t}{n}} \sqrt{\frac{q}{1-q}} \qquad (5.2)$$

Appendix 1

$$C = (S - \frac{K}{r^t})\, Nc\left[\frac{Sr^t - K}{\sigma\sqrt{t}}\right] + \frac{1}{r^t}\sigma\sqrt{t}\, N\left[\frac{K - Sr^t}{\sigma\sqrt{t}}\right]$$

Let us note $\frac{K - Sr^t}{\sigma\sqrt{t}}$ by x

a) $$\frac{\delta C}{\delta S} = Nc[-x] + (S - Kr^{-t})\frac{\delta Nc[-x]}{\delta S} + \sigma\sqrt{t}\, r^{-t}\frac{\delta N(x)}{\delta S}$$

$$\frac{\delta C}{\delta S} = Nc[-x] + (S - Kr^{-t})\frac{\delta Nc[-x]}{\delta x}\frac{\delta x}{\delta S} + \sigma\sqrt{t}\, r^{-t}\frac{\delta N(x)}{\delta x}\frac{\delta x}{\delta S}$$

$$\frac{\delta C}{\delta S} = Nc[-x] + (S - Kr^{-t})\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\frac{\delta x}{\delta S} + \sigma\sqrt{t}\, r^{-t}\, x\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\frac{\delta x}{\delta S}$$

$$\frac{\delta C}{\delta S} = Nc[-x] + \frac{\delta x}{\delta S}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\left(S - Kr^{-t} + \sigma\sqrt{t}\, r^{-t}\frac{K - Sr^t}{\sigma\sqrt{t}}\right)$$

$$\frac{\delta C}{\delta S} = Nc\left[\frac{Sr^t - K}{\sigma\sqrt{t}}\right] > 0$$

b) $$\frac{\delta C}{\delta K} = -\, r^{-t}\, Nc[-x] + (S - Kr^{-t})\frac{\delta Nc[-x]}{\delta K} + \sigma\sqrt{t}\, r^{-t}\frac{\delta N(x)}{\delta K}$$

$$\frac{\delta C}{\delta K} = -\, r^{-t}\, Nc[-x] + (S - Kr^{-t})\frac{\delta Nc[-x]}{\delta x}\frac{\delta x}{\delta K} + \sigma\sqrt{t}\, r^{-t}\frac{\delta N(x)}{\delta x}\frac{\delta x}{\delta K}$$

As in case a the second and third terms disappear, and

$$\frac{\delta C}{\delta K} = -\, r^{-t}\, Nc\left[\frac{Sr^t - K}{\sigma\sqrt{t}}\right] < 0$$

c) $$\frac{\delta C}{\delta \sigma} = (S - Kr^{-t})\frac{\delta Nc[-x]}{\delta \sigma} + \sigma\sqrt{t}\, r^{-t}\frac{\delta N(x)}{\delta \sigma} + \sqrt{t}\, r^{-t}\, N[x]$$

$$\frac{\delta C}{\delta \sigma} = (S - Kr^{-t})\frac{\delta Nc[-x]}{\delta x}\frac{\delta x}{\delta \sigma} + \sigma\sqrt{t}\, r^{-t}\frac{\delta N(x)}{\delta x}\frac{\delta x}{\delta \sigma} + \sqrt{t}\, r^{-t}\, N[x]$$

As in case a the first and second terms disappear, and

$$\frac{\delta C}{\delta \sigma} = \sqrt{t}\, r^{-t}\, N\left[\frac{K - Sr^t}{\sigma\sqrt{t}}\right] > 0$$

d) $$\frac{\delta C}{\delta t} = (S - Kr^{-t})\frac{\delta Nc[-x]}{\delta t} + Nc(-x)\frac{\delta(-Kr^{-t})}{\delta t} + \sigma\sqrt{t}\, r^{-t}\frac{\delta N(x)}{\delta t}$$
$$+ \sigma\sqrt{t}\, N(x)\frac{\delta(-r^{-t})}{\delta t} + \sigma\, N(x)\, r^{-t}\frac{\delta\sqrt{t}}{\delta t}$$

As in case a the first and third terms disappear, and

$$\frac{\delta C}{\delta t} = Nc(-x)\,\frac{\delta(-Kr^{-t})}{\delta t} + \sigma\sqrt{t}\,N(x)\,\frac{\delta(-r^{-t})}{\delta t} + \sigma\, N(x)\, r^{-t}\,\frac{\delta\sqrt{t}}{\delta t}$$

$$\frac{\delta C}{\delta t} = Kr^{-t}\,Nc(-x)\,\log(r) - \sigma\sqrt{t}\,N(x)r^{-t}\log(r) + \sigma\, N(x)\, r^{-t}\,\frac{1}{2\sqrt{t}}$$

As $\log(r) = R$

$$\frac{\delta C}{\delta t} = KRr^{-t}\,Nc\left[\frac{Sr^{t} - K}{\sigma\sqrt{t}}\right] + N\left[\frac{K - Sr^{t}}{\sigma\sqrt{t}}\right]\frac{\sigma\, r^{-t}}{2\sqrt{t}}(1 - 2tR) > 0$$

Since when t is large N(d) reduces to zero.

e) $$\frac{\delta C}{\delta r} = (S - Kr^{-t})\,\frac{\delta Nc[-x]}{\delta r} + Nc(-x)\,\frac{\delta(-Kr^{-t})}{\delta r} + \sigma\sqrt{t}\,r^{-t}\,\frac{\delta N(x)}{\delta r} + \sigma\sqrt{t}\,N(x)\,\frac{\delta(r^{-t})}{\delta r}$$

As in case a the first and third terms disappear, and

$$\frac{\delta C}{\delta r} = Kt\, r^{-t-1}\,Nc[-x] - t\,\sigma\sqrt{t}\,r^{-t-1}\,N[x]$$

$$\frac{\delta C}{\delta r} = Kt\, r^{-t-1}\,Nc\left[\frac{Sr^{t} - K}{\sigma\sqrt{t}}\right] - \frac{\sigma}{\sqrt{t}}r^{-t-1}\,N\left[\frac{K - Sr^{t}}{\sigma\sqrt{t}}\right] > 0$$

Since $Kt > \sigma/\sqrt{t}$ and $Nc[-x] > N[x]$ for any x.

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