Natural and projectively equivariant quantizations

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Introduction

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$$qp \to \frac{1}{2}(QP + PQ)$$

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One finds then :

$$Q_M(\nabla, S)(f) = {p^*}^{-1}\left(\sum_{l=0}^k C_{k,l} \langle Div^{\omega^l} p^*S, \nabla^{\omega^{k-l}}_s p^*f\rangle\right),$$

with $C_{k,l} = \frac{(\lambda+\frac{k-1}{m+1})\cdots(\lambda+\frac{k-l}{m+1})}{\gamma_{2k-1}\cdots\gamma_{2k-l}} \begin{pmatrix} k \\ l \end{pmatrix}, \forall l \geq 1, \quad C_{k,0} = 1$

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Application γ :	Application γ :
$\mathcal{L}_{X^h} Q_{Aff}(S)(f) =$ $Q_{Aff}((L_{X^h} + \gamma(h))S)(f)$	$\mathcal{L}_{h^*} Q_\omega(S)(f) =$ $Q_\omega((L_{h^*} + \gamma(h))S)(f),\ h \in g_1$

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Casimirs C and $\mathcal{C}$ | "Casimirs" C^ω and $\mathcal{C}^\omega$

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•Conclusion : "Flat" case

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