EQUIVARIANT QUANTIZATIONS AND CARTAN CONNECTIONS

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1. Introduction

In the framework of geometric quantization, it is common to define a quantization procedure as a linear bijection from the space of classical observables to a space of differential operators acting on wave functions (see [31]).

In our setting, the space of observables (also called the space of *Symbols*) is the space of smooth functions on the cotangent bundle T^*M of a manifold M, that are polynomial along the fibres. The space of differential operators $\mathcal{D}_{\frac{1}{2}}(M)$ is made of differential operators acting on half-densities.

It is known that there is no natural quantization procedure : the spaces of symbols and of differential operators are not isomorphic as representations of $\mathrm{Diff}(M)$.

The idea of G-equivariant quantization is to reduce the set of (local) diffeomorphisms under consideration. If a Lie group G acts on M by local diffeomorphisms, this action can be lifted to symbols and to differential operators. A G-equivariant quantization was defined by P. Lecomte and V. Ovsienko in [21] as a G-module isomorphism from symbols to differential operators.

They first considered the projective group $PGL(m+1, \mathbb{R})$ acting locally on the manifold $M = \mathbb{R}^m$ by linear fractional transformations and defined the notion of *projectively equivariant quantization.* They proved the existence of such a quantization and its uniqueness, up to some natural normalization condition.

In [11], the authors considered the group $SO(p+1, q+1)$ acting on the space $\mathbb{R}^{p+q}$ or on a manifold endowed with a flat conformal structure. There again, the result was the existence and uniqueness of a *conformally equivariant quantization.*

Over vector spaces, or manifolds endowed with flat structures, similar results were obtained for other type of differential operators (see [1]) or for other Lie groups. The first part of this presentation is a survey of these results. Unless otherwise stated, the results are based on a collaboration with F. Boniver (see [3] and [4])

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At that point, the results held over vector spaces or manifolds endowed with a flat structure. It was then remarked by S. Bouarroudj ([6, 7]) that the formula for the projectively equivariant quantization for symbols of degree 2 and 3 could be expressed using a torsion free linear connection, in such a way that it only depends on the projective class of the connection.

In [23], P. Lecomte gave an exact formulation of this extension of projectively equivariant quantization to arbitrary manifolds : can we find a quantization procedure that would take a torsion free linear connection as a parameter, would be natural *in all its arguments, including the connection* and would only depend on the projective class of the connection ?

He also set the problem in the conformal situation. There, the quantization should depend on a pseudo-Riemannian metric, be natural in all its arguments, and only depend on the *Conformal class* of the metric.

In the projective case, a positive answer to the question was given by M. Bordemann in [5], using the notion of Thomas-Whitehead connection. His construction was adapted by S. Hansoul in order to deal with differential operators acting on tensor fields ([14, 13]).

In the second part of this presentation, we will show that these results can be obtained using the theory of Cartan connections. We will derive an explicit formula for the quantization in terms of the Cartan connection associated to a projective class of connections. This formula is nothing but the formula for the flat case up to replacement of the partial derivatives by invariant differentiation with respect to the Cartan connection. It then provides a closer link between the quantization over vector spaces and the general problem of natural and projectively equivariant quantization. This part is based on a collaboration with F. Radoux ([24]).

2. The data

Here we recall the definitions of the basic objects, such as tensor densities, differential operators and their symbols. Unless otherwise stated, we denote by M a smooth, Hausdorff and second countable manifold of dimension m.

2.1. **Tensor densities.** The vector bundle of tensor densities $F_\lambda(M) \to M$ is a line bundle associated to the linear frame bundle :

$$F_\lambda(M) = P^1M \times_\rho \Delta^\lambda(R^m),$$

where the representation ρ of the group $GL(m,\mathbb{R})$ on the one-dimensional vector space $\Delta^\lambda(R^m)$ is given by

$$\rho(A)e = |detA|^{-\lambda}e, \quad \forall A \in GL(m,\mathbb{R}),\ \forall e \in \Delta^\lambda(\mathbb{R}^m).$$

We denote by $\mathcal{F}_\lambda(M)$ the space of smooth sections of this bundle. The action of local diffeomorphisms and of vector fields on $\mathcal{F}_\lambda(M)$ are induced by its definition : in local coordinates over M, any $\psi \in \mathcal{F}_\lambda(M)$ writes

$$\psi(x) = f(x)|dx^1 \wedge \cdots \wedge dx^m|^\lambda.$$

If φ is a local diffeomorphism, we have

$$(\varphi.\psi)(x) = f(\varphi^{-1}(x))|(\frac{\partial\varphi^{-1}(x)}{\partial x})|^{\lambda}|dx^1 \wedge \cdots \wedge dx^m|^{\lambda}$$

and the Lie derivative is given by

$$(L_X^{\lambda}\psi)(x) = (\sum_i X^i \frac{\partial}{\partial x^i} f + \lambda(\sum_i \frac{\partial X^i}{\partial x^i})f)|dx^1 \wedge \cdots \wedge dx^m|^{\lambda},$$

for every vector field X.

It will also be interesting for our purpose to note that $\mathcal{F}_\lambda(M)$ can be identified with the space $C^\infty(P^1M, \Delta^\lambda(R^m))_{GL(m,\mathbb{R})}$ of functions f such that

$$f(uA) = \rho(A^{-1})f(u) \quad \forall u \in P^1M,\ \forall A \in GL(m,\mathbb{R}).$$

Finally, let us remark that the space $\mathcal{F}_0(M)$ is exactly the space of real valued functions on M, as a module over $\mathrm{Diff}(M)$ and of $\mathrm{Vect}(M)$, while the action of $\mathrm{Diff}(M)$ on $\mathcal{F}_1(M)$ shows that its elements behave like integrands under coordinates changes.

2.2. **Differential operators.** We denote by $\mathcal{D}_{\lambda,\mu}(M)$ (or simply by $\mathcal{D}_{\lambda,\mu}$) the space of linear differential operators from $\mathcal{F}_\lambda(M)$ to $\mathcal{F}_\mu(M)$. The actions of $\mathrm{Vect}(M)$ and $\mathrm{Diff}(M)$ are induced by the actions on tensor densities : with the notation introduced above, one has

$$(\varphi \cdot D)(f) = \varphi \cdot (D(\varphi^{-1} \cdot f)), \quad \forall f \in \mathcal{F}_\lambda(M), D \in \mathcal{D}_{\lambda,\mu},$$

and

$$\mathcal{L}_X D = L_X^{\mu} \circ D - D \circ L_X^{\lambda}.$$

There is a filtration

$$\mathcal{D}_{\lambda,\mu} = \cup_{k=0}^{\infty} \mathcal{D}_{\lambda,\mu}^k$$

defined by the order of differential operators. It is well-known that this filtration is preserved by the action of local diffeomorphisms and of vector fields.

Let us give a simple example : If $D \in \mathcal{D}_{\lambda,\lambda}^2$ writes in coordinates

$$D = A_2^{ij}\partial_i\partial_j + A_1^i\partial_i + A_0,$$

then

$$\begin{array}{rcl}(\mathcal{L}_X D)f & = & [X^k\partial_k + \lambda\partial_k X^k, D]f \\ & = & (L_X A_2)^{ij}\partial_{ij}f \\ & + & ((L_X A_1)^l - A_2^{ij}\partial_{ij}X^l - 2\lambda A_2^{li}\partial_{ij}X^j)\partial_l f \\ & + & (L_X A_0 - \lambda(A_1^k\partial_k + A_2^{ij}\partial_{ij})\partial_l X^l)f\end{array}$$

where

$$\begin{array}{rcl}(L_X A_2)^{ij} & = & X^k\partial_k A_2^{ij} - A_2^{kj}\partial_k X^i - A_2^{ki}\partial_k X^j \\ (L_X A_1)^l & = & X^k\partial_k A_1^l - A_1^k\partial_k X^l \\ L_X A_0 & = & X^k\partial_k A_0.\end{array}$$

Remark 1. We see that the term of highest order (A_2^{ij}) behaves like a symmetric tensor field under the action of any vector field. This is also the case for terms of lower order if the components of the vector field X are affine functions of the coordinates.

2.3. **Symbols.** The space of *symbols* is the graded space associated to $\mathcal{D}_{\lambda,\mu}$. It turns out that this space only depends on the *shift value* $\delta = \mu - \lambda$. Actually, it can be identified with the space

$$\mathcal{S}_\delta(M) = \bigoplus_{l=0}^{\infty} \mathcal{S}^l_\delta(M)$$

of contravariant symmetric tensor fields with coefficients in δ-densities, endowed with the classical actions of $\mathrm{Diff}(M)$ and of $\mathrm{Vect}(M)$.

Indeed, the *principal symbol operator*

$$\sigma : \mathcal{D}^l_{\lambda,\mu}(M) \to \mathcal{S}^l_\delta(M),$$

which associates to every differential operator its highest order term, commutes with the action of diffeomorphisms and of vector fields (see remark 1). It is a bijection from the quotient space $\mathcal{D}^l_{\lambda,\mu}(M)/\mathcal{D}^{l-1}_{\lambda,\mu}(M)$ to the space $\mathcal{S}^l_\delta(M)$.

It will also be useful to view symbols as equivariant functions on the linear frame bundle : one has

$$\mathcal{S}^l_\delta(M) \cong C^\infty(P^1M, S^l_\delta(\mathbb{R}^m))_{GL(m,\mathbb{R})},$$

where $S^l_\delta(\mathbb{R}^m) = S^l\mathbb{R}^m \otimes \Delta^\delta(\mathbb{R}^m)$ endowed with the natural representation of $GL(m,\mathbb{R})$.

3. A survey of $\mathfrak{g}$-equivariant quantizations

Here we recall the results that were obtained in the framework of $\mathfrak{g}$-equivariant quantizations over vector spaces. Throughout this section, M will be a vector space of dimension m, and $\mathfrak{g}$ will be a subalgebra of $\mathrm{Vect}(M)$.

3.1. **Equivariant quantizations and symbol maps.** A *quantization* on M is a linear bijection Q_M from the space of symbols $\mathcal{S}_\delta(M)$ to the space of differential operators $\mathcal{D}_{\lambda,\mu}(M)$ such that

$$\sigma(Q_M(T)) = T, \quad \forall T \in \mathcal{S}^k_\delta(M),\ \forall k \in \mathbb{N}.$$

The inverse of such a map is called a *Symbol map*.

A $\mathfrak{g}$-equivariant quantization is a quantization $Q_\mathfrak{g}$ which is moreover a $\mathfrak{g}$-module isomorphism from $\mathcal{S}_\delta(M)$ to $\mathcal{D}_{\lambda,\mu}(M)$.

3.2. **Affinely equivariant quantization.** Let us begin with the most simple example of equivariant quantization.

The constant and linear vector fields generate the *affine* subalgebra *Aff* of $\mathrm{Vect}(\mathbb{R}^m)$. Now, it is easily seen, as stated in remark 1 that the *total symbol map*

$$\begin{array}{l}\sigma_{\mathit{Aff}}:\mathcal{D}_{\lambda,\mu}(\mathbb{R}^m)\to\mathcal{S}_\delta(\mathbb{R}^m):\\ \quad\sum_{|\alpha|\leq k}c_\alpha(\frac{\partial}{\partial x^1})^{\alpha_1}\cdots(\frac{\partial}{\partial x^n})^{\alpha_n}\mapsto\sum_{|\alpha|\leq k}c_\alpha(\frac{\partial}{\partial x^1})^{\alpha_1}\vee\cdots\vee(\frac{\partial}{\partial x^n})^{\alpha_n}\end{array}$$

is an isomorphism of *Aff*-representations. The inverse of this map, also known as *standard ordering* is then an affinely equivariant quantization Q_{Aff}.

Now, on the one hand, we know that there exists an affinely equivariant quantization. On the other hand it is also known that for the whole algebra of polynomial vector fields $\mathit{Vect}_*(\mathbb{R}^m)$, symbols and differential operators are not isomorphic. The next question is then the existence of the $\mathfrak{g}$-equivariant quantization for an algebra $\mathfrak{g}$ such that

$$\mathit{Aff}\subset\mathfrak{g}\subset\mathit{Vect}_*(\mathbb{R}^m),$$

if such an algebra exists.

3.3. **The projective algebra of vector fields.** It turns out that such an algebra exists : it is isomorphic to $sl(m+1,\mathbb{R})$ and is given by

$$sl_{m+1}=\langle\{\partial_k,x^j\,\partial_k,x^j\sum_{k=1}^m x^k\partial_k\}\rangle. \tag{1}$$

This algebra can be defined in geometric terms :
Consider the projective group $G=PGL(m+1,\mathbb{R})=GL(m+1,\mathbb{R})/\mathbb{R}_0\mathrm{Id}$. Its Lie algebra $\mathfrak{g}=gl(m+1,\mathbb{R})/\mathbb{R}\mathrm{Id}$ is isomorphic to $sl(m+1,\mathbb{R})$ and decomposes into a direct sum of subalgebras

$$\mathfrak{g}=\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1=\mathbb{R}^m\oplus gl(m,\mathbb{R})\oplus\mathbb{R}^{m*}.$$

The isomorphism is given explicitly by

$$\psi:gl(m+1,\mathbb{R})/\mathbb{R}\mathrm{Id}\to\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1:\left[\begin{pmatrix}A&v\\ \xi&a\end{pmatrix}\right]\mapsto(v,A-a\,Id,\xi). \tag{2}$$

The group $G=PGL(m+1,\mathbb{R})$ acts on $\mathbb{R}P^m$. Since $\mathbb{R}^m$ can be seen as the open set of $\mathbb{R}P^m$ of equation $x^{m+1}=1$, there is a local action of G on $\mathbb{R}^m$. The vector fields associated to this action are given by

$$\left\{\begin{array}{lll}X^h_x&=&-h\quad\text{if}\,h\in\mathfrak{g}_{-1}\\ X^h_x&=&-[h,x]\quad\text{if}\,h\in\mathfrak{g}_0\\ X^h_x&=&-\frac{1}{2}[[h,x],x]\quad\text{if}\,h\in\mathfrak{g}_1\end{array}\right.\;, \tag{3}$$

where $x\in\mathfrak{g}_{-1}\cong\mathbb{R}^m$. These vector fields span the algebra sl_{m+1}.

One might think that this algebra is rather small in the algebra $\mathit{Vect}_*(\mathbb{R}^m)$ of polynomial vector fields over $\mathbb{R}^m$, but actually, we have the following result :

Theorem 1. *The projective algebra of vector fields sl_{m+1} is a maximal proper subalgebra of $Vect_*(\mathbb{R}^m)$.*

Hence, if we impose the equivariance with respect to the projective algebra sl_{m+1}, we actually impose the strongest condition we can.

3.4. **Projectively equivariant quantizations.** The first result concerning sl_{m+1}-equivariant quantizations is the following :

Theorem 2 (Lecomte, Ovsienko [21]). *There exists a unique sl_{m+1}-equivariant quantization*

$$Q_p : \mathcal{S}_0(\mathbb{R}^m) \to \mathcal{D}_{\lambda,\lambda}(\mathbb{R}^m).$$

This result corresponds to the case where the shift $\delta = \mu - \lambda$ vanishes. It was generalized by the following theorem

Theorem 3 (Duval, Ovsienko [12]). *There exists a unique sl_{m+1}-equivariant quantization*

$$Q_p : \mathcal{S}_\delta(\mathbb{R}^m) \to \mathcal{D}_{\lambda,\mu}(\mathbb{R}^m)$$

if δ is not a critical value. The critical values are known.

Moreover an explicit formula for the quantization was given in [21] and generalized in [12]. It is the following :

$$Q_p(T) = Q_{Aff}\Big(\sum_{l=0}^{k} \frac{(\lambda + \frac{k-1}{n+1}) \cdots (\lambda + \frac{k-l}{n+1})}{\gamma_{2k-1} \cdots \gamma_{2k-l}} \begin{pmatrix} k \\ l \end{pmatrix} Div^l T\Big), \tag{4}$$

where

$$\gamma_r = \frac{n + r - (n+1)\delta}{n+1} \tag{5}$$

and

$$Div^l T(\eta^1, \cdots, \eta^{k-l}) = \sum_{i_1,\cdots,i_l} \partial_{x^{i_1}} \cdots \partial_{x^{i_l}} T(dx^{i_1}, \cdots, dx^{i_l}, \eta^1, \cdots, \eta^{k-l}).$$

Remark 2. The critical values correspond to the vanishing of the denominators in formula (4) :

Definition 1. A value of δ is critical (for the algebra sl_{m+1}) if there exists $k, l \in \mathbb{N}$ such that $1 \leq l \leq k$ and $\gamma_{2k-l} = 0$.

Remark 3. There are some special situations were the sl_{m+1}-equivariant quantization can exist even if δ is a critical value. Roughly speaking, these situations correspond to the vanishing of the numerators in formula (4). We refer the reader [22, 20] for a detailed discussion.

3.5. **Conformally equivariant quantizations.** In [11], C. Duval, P. Lecomte and V. Ovsienko analysed the equivariant quantizations with respect to another algebra, namely the conformal algebra $so_{p+1,q+1}$.

If g is the pseudo-Riemannnian metric over $\mathbb{R}^m$ given by

$$\mathrm{diag}(\overbrace{1,\ldots,1}^{p},\overbrace{-1,\ldots,-1}^{q}),\quad (p+q=m),$$

then

$$\begin{array}{l} so_{p+1,q+1} = \{\sum_i a_i\partial_i | a\in\mathbb{R}^m\}\oplus\mathbb{R}\mathcal{E}\oplus\{\sum_i (Ax)^i\frac{\partial}{\partial x^i}|A\in so(p,q,\mathbb{R})\} \\ \oplus\{\alpha(x)\mathcal{E}-\frac{1}{2}\|x\|^2\sum_i g^{ii}\alpha_i\frac{\partial}{\partial x^i}|\alpha\in\mathbb{R}^{m*}\}, \end{array}$$

where $\mathcal{E}$ is the Euler field defined by $\mathcal{E}_x = \sum_i x^i\frac{\partial}{\partial x^i}$.

The results concerning this algebra are the following :

Theorem 4 (Duval, Lecomte, Ovsienko, [11])**.** *There exists a unique $so_{p+1,q+1}$-equivariant quantization map*

$$Q_{\mathrm{so}} : \mathcal{S}_\delta(\mathbb{R}^m)\to\mathcal{D}_{\lambda,\mu}(\mathbb{R}^m)$$

if δ is not critical. The critical values are known, zero is not critical.

Moreover, the algebra $so_{p+1,q+1}$ has the following property

Theorem 5 (Boniver, Lecomte,[2])**.** *The algebra $so_{p+1,q+1}$ is a maximal proper subalgebra of polynomial vector fields if $(p,q)\neq(1,1)$.*

3.6. **IFFT algebras and IFFT equivariant quantizations.** The projective and conformal algebras share the properties of being graded, finite dimensional subalgebras of polynomial vector fields. They are moreover maximal proper subalgebras of polynomial vector fields. This section presents results concerning the classification of the algebras possessing these properties, and the equivariant quantizations with respect to these algebras. It is based on a joint work with F. Boniver in [3] and [4].

Here again M will denote a vector space of dimension m over the field $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$. We denote by $Vect_*(M)$ the algebra of polynomial vector fields (holomorphic polynomials if $\mathbb{K}=\mathbb{C}$). This algebra admits a decomposition

$$Vect_*(M) = \oplus_{i=-1}^{\infty} Vect_i(M),$$

where $Vect_i(M)$ is the set of vector fields whose components are homogeneous polynomials of degree $i+1$.

It is easy to check that this decomposition fulfils the relation

$$[Vect_i(M), Vect_j(M)]\subset Vect_{i+j}(M) :$$

the algebra of polynomial vector fields is a graded algebra.

A subalgebra $\mathfrak{g}$ of $Vect_*(M)$ is graded if it writes $\mathfrak{g} = \oplus_{i=-1}^{\infty}\mathfrak{g}_i$ with $\mathfrak{g}_i\subset Vect_i(M)$ for all $i\geq -1$.

We investigate the maximality property in the set of proper subalgebras of $Vect_*(M)$: a finite dimensional subalgebra $\mathfrak{g}$ of $Vect_*(M)$ is maximal if it is not properly contained in any proper subalgebra of $Vect_*(M)$.

The first main result of [3] is the following necessary condition for a subalgebra to be maximal :

Theorem 6. *If* $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \cdots \oplus \mathfrak{g}_k$ *is a maximal graded subalgebra of* $Vect_*(M)$*, then* $\mathfrak{g}$ *fulfils the following conditions*

- *one has* $\mathfrak{g}_{-1} = Vect_{-1}(M)$*;*
- *the representation* $(\mathfrak{g}_{-1}, ad)$ *of* $\mathfrak{g}_0$ *is irreducible;*
- *the space* $\mathfrak{g}_1$ *is not trivial;*
- *if* $\mathbb{K} = \mathbb{R}$*, the representation* $(\mathfrak{g}_{-1}, ad)$ *of* $\mathfrak{g}_0$ *does not admit a complex structure.*

It follows from this theorem that any maximal graded and finite-dimensional subalgebra of polynomial vector fields belongs to the class of *Irreducible Filtered Finite-dimensional Transitive* Lie algebras, listed by Kobayashi and Nagano in [17] (IFFT algebras for short).

The most important properties of these algebras are the following :

- They are simple.
- Their grading contains exactly three terms :
$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1.$$
- $\mathfrak{g}_0$ is reductive : one has
$$\mathfrak{g}_0 = \mathfrak{h}_0 \oplus \mathbb{K}\mathcal{E},$$
where $\mathfrak{h}_0$ is the semisimple part of $\mathfrak{g}_0$ and where the *Euler* element $\mathcal{E}$ spans a one-dimensional center (in $\mathfrak{g}_0$).
- $\mathfrak{g}_p$ is the eigenspace of eigenvalue p of $ad(\mathcal{E})$.

It is worth noticing that in [17], the authors listed simple matrix algebras together with their gradings. But in [19], they described a standard procedure to view these algebras as subalgebras of polynomial vector fields over the vector space $\mathfrak{g}_{-1}$, which corresponds exactly to equation (3).

In [3], we proved that the subalgebras of vector fields obtained in this way from IFFT algebras are maximal proper subalgebras, provided they meet the additional requirement (which corresponds to the fourth necessary condition of theorem 6):

- When the base field is $\mathbb{R}$, $\mathfrak{g}$ has no complex structure.

In other words, every IFFT algebra $\mathfrak{g}$ gives rise (using equation (3)) to a maximal subalgebra of polynomial vector fields. But, when the algebra $\mathfrak{g}$ admits a complex structure, the algebra obtained by equation (3) is a maximal subalgebra of the algebra of holomorphic polynomial vector fields over the complex vector space $\mathfrak{g}_{-1}$. It is not a maximal subalgebra of polynomial vector fields over $\mathfrak{g}_{-1}$ considered as a real vector space.

The natural next question is the existence of equivariant quantizations with respect to this class of subalgebras. In [4], we extended the methods of [11] in order to deal with this question. We were able to present two new examples.

3.7. **The symplectic and orthogonal algebras.** The orthogonal algebra algebra $so(n,n,\mathbb{K})$ can be written as

$$\mathfrak{O}^{(n)} = \mathfrak{O}^{(n)}_{-1} \oplus \mathfrak{O}^{(n)}_{0} \oplus \mathfrak{O}^{(n)}_{1},$$

where $\mathfrak{O}^{(n)}_{-1} = \wedge^2\mathbb{K}^n$, $\mathfrak{O}^{(n)}_{1} = \wedge^2\mathbb{K}^{n*}$ and $\mathfrak{O}^{(n)}_{0} = gl(n,\mathbb{K})$ (note that this grading is different from the one of the conformal case). For all $A \in \mathfrak{O}^{(n)}_{0}$ and $h \in \mathfrak{O}^{(n)}_{-1} \oplus \mathfrak{O}^{(n)}_{1}$,

$$[A,h] = \rho(A)h,$$

where ρ is the natural representation of $\mathfrak{O}^{(n)}_{0}$ on $\mathfrak{O}^{(n)}_{-1} \oplus \mathfrak{O}^{(n)}_{1}$.

Similarly, the symplectic algebra $sp(2n,\mathbb{K})$ is written

$$\mathfrak{S}^{(n)} = \mathfrak{S}^{(n)}_{-1} \oplus \mathfrak{S}^{(n)}_{0} \oplus \mathfrak{S}^{(n)}_{1},$$

where $\mathfrak{S}^{(n)}_{-1} = S^2\mathbb{K}^n$, $\mathfrak{S}^{(n)}_{1} = S^2\mathbb{K}^{n*}$ and $\mathfrak{S}^{(n)}_{0} = gl(n,\mathbb{K})$. The same statements about the bracket hold.

These algebras are IFFT algebras. The result concerning the equivariant quantizations are the following.

Theorem 7. *For both algebras, if $\delta = \mu - \lambda$ is not critical, then there exists an equivariant quantization*

$$Q_{\mathfrak{g}} : \mathcal{S}_\delta \to \mathcal{D}_{\lambda\mu}.$$

All critical shift values belong to the set

$$CV = \{\frac{n}{n+1} + \frac{\sum_{i=1}^{n}(k_i - l_i)(k_i - l_i + 2i)}{4(n-1)(k-l)} : \vec{k} > \vec{l}\}$$

in the orthogonal case and

$$CV = \{1 + \frac{\sum_{i=1}^{n}(k_i - l_i)(k_i - l_i + 2i)}{4(n+1)(k-l)} : \vec{k} > \vec{l}\}$$

in the symplectic case, where $\vec{k} = (k_1, \cdots, k_n)$ and $\vec{l} = (l_1, \cdots, l_n)$ belong to $\mathbb{N}^n$ and fulfil the properties

- $k_1 \geq \cdots \geq k_n \geq 0$, $l_1 \geq \cdots \geq l_n \geq 0$,
- *In the orthogonal case,* $k_{2i-1} = k_{2i}$, (resp. $l_{2i-1} = l_{2i}$) $\quad \forall i \in \{1, \ldots, \lfloor n/2 \rfloor\}$, *and $k_n = 0$ (resp. $l_n = 0$) if n is odd.*
- *In the symplectic case, k_i and l_i belong to $2\mathbb{N}$ for all $i \leq n$.*

In particular, they are greater than 0.

Theorem 8. *If the shift is not in the set CV of the previous theorem then the $\mathfrak{g}$-equivariant quantization is unique.*

4. Natural and projectively equivariant quantizations

From now on to the end of this presentation, we will show how to extend the problem of projectively equivariant quantizations to arbitrary manifolds. We let M denote a manifold of dimension m. The generalization of the problem involves the concept of projectively equivalent connections.

4.1. **Projectively equivalent connections.** We denote by $\mathcal{C}_M$ the space of torsion-free linear connections on M. Two such connections are *Projectively equivalent* if they define the same *paths*, that is, the same geodesics up to parametrization.

In algebraic terms, H. Weyl showed in [29] that two connections were projectively equivalent if and only if there exists a 1-form α such that their associated covariant derivatives ∇ and ∇' fulfil the relation

$$\nabla'_X Y = \nabla_X Y + \alpha(X)Y + \alpha(Y)X.$$

The notion of projective equivalence of connections was studied during the 1920's. One of the main problems that were addressed was to associate a single object to a projective class of connections on M. There are two main answers to this problem.

One of them is due to T.Y. Thomas [27], J.H.C. Whitehead [30] and O.Veblen [28]. There, the idea is to associate to a class $[\nabla]$ of torsion free connections a single torsion free linear connection $\widetilde{\nabla}$ on a manifold $\widetilde{M}$ of dimension $m+1$ (see also [15, 26, 25] for a modern formulation).

The second approach, due to E. Cartan [10], leads to the concept of Cartan projective connection, developed in a modern setting by S. Kobayashi and T. Nagano in [18, 16].

4.2. **Natural and equivariant quantizations.** Roughly speaking, a *natural quantization* is a quantization (in the sense of section 3.1) which depends on a torsion-free connection and commutes with the action of diffeomorphisms. More precisely, a natural quantization is a collection of maps (defined for every manifold M)

$$Q_M : \mathcal{C}_M \times \mathcal{S}_\delta(M) \to \mathcal{D}_{\lambda,\mu}(M)$$

such that

- For all ∇ in $\mathcal{C}_M$, $Q_M(\nabla)$ is a quantization,
- If ϕ is a local diffeomorphism from M to N, then one has

$$Q_M(\phi^*\nabla)(\phi^*T) = \phi^*(Q_N(\nabla)(T)), \quad \forall \nabla \in \mathcal{C}_N, \forall T \in \mathcal{S}_\delta(N).$$

A quantization Q_M is *projectively equivariant* if one has $Q_M(\nabla) = Q_M(\nabla')$ whenever ∇ and ∇' are projectively equivalent torsion-free linear connections on M.

This problem is indeed a generalization of sl_{m+1}-equivariant quantization over $\mathbb{R}^m$. A first direct link was given by P. Lecomte in [23] :

Theorem 9. *If Q_M is a natural projectively equivariant quantization and if we denote by ∇_0 the flat connection on $\mathbb{R}^m$, then $Q_{\mathbb{R}^m}(\nabla_0)$ is sl_{m+1}-equivariant.*

4.3. **An outline of M. Bordemann's existence theorem.** In [5], M. Bordemann proved that if δ is not a critical shift value for the projectively equivariant quantization (see definition 1), then there exists a natural and projectively equivariant quantization. His construction can be roughly summarized as follows :

- First associate to each projective class $[\nabla]$ of torsion-free linear connections on M a unique linear connection $\tilde{\nabla}$ on a principal line bundle $\tilde{M} \to M$ (the Thomas-Whitehead connection),
- Lift the symbols to a suitable space of tensors on $\tilde{M}$,
- Apply the *Standard ordering* to these tensors,
- Pull the so-defined differential operator back on M.

The first step ensures that the so defined quantization will be projectively equivariant, because it only depends on $\tilde{\nabla}$.

The delicate part lies in the second step. It is indeed not easy to find the suitable space of tensors on $\tilde{M}$ in order to establish a bijection with symbols on M. However, this construction has recently been generalized by S. Hansoul in her thesis ([13]).

5. Cartan connections and natural quantizations

We will show how to build a natural and projectively equivariant quantization, using projective Cartan connections instead of Thomas-Whitehead connections.

One of the advantages in using this formalism is that it might provide ideas in order to deal with the conformal situation. Indeed, the theories of projective Cartan connections and conformal Cartan connections are very similar (see for instance [16]).

The ingredients involved in the construction are similar to the one used by M. Bordemann : First associate a single Cartan connection on a manifold P to a projective class of torsion free linear connections, then lift the symbols and the arguments of differential operators to P, apply a kind of standard ordering, with respect to the Cartan connection. Finally pull the constructed operator back on M.

It turns out that the second step is obvious in our construction, but the fourth one is delicate.

5.1. **Step 1 : Cartan connections.** Let G be a Lie group and H a closed subgroup. Denote by $\mathfrak{g}$ and $\mathfrak{h}$ the corresponding Lie algebras. Let $P \to M$ be an H-principal bundle over M, such that $\dim M = \dim G/H$. A Cartan connection on P is a $\mathfrak{g}$-valued one-form ω on P such that

- If R_a denotes the right action of $a \in H$, then $R_a^*\omega = Ad(a^{-1})\omega$,
- If k^* is the vertical field associated to $k \in \mathfrak{h}$, then $\omega(k^*) = k$,
- $\forall u \in P, \omega_u : T_uP \to \mathfrak{g}$ is a linear bijection.

The most simple example of Cartan connection is the following. Suppose that G is a Lie group and H is a closed subgroup. Then $G \to G/H$ is an

H-principal bundle and it is easy to check that the Maurer-Cartan form ω of G is a Cartan connection.

In general, the curvature of a Cartan connection ω is the $\mathfrak{g}$-valued 2-form Ω on P given by

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega]. \tag{6}$$

The Maurer-Cartan equation shows that the curvature of the Maurer-Cartan form vanishes.

5.2. **Projective connections.** From now on, we let G denote the projective group $PGL(m+1,\mathbb{R})$ defined in section 3.3. We denote by H the subgroup associated to the subalgebra $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ and G_0 the subgroup associated to $\mathfrak{g}_0$. It is easy to see that H is the semidirect product of G_0 and $\mathbb{R}^{m*}$ and that G_0 is isomorphic to $GL(m,\mathbb{R})$.

Let us denote by G^2_m the group of 2-jets at the origin $0 \in \mathbb{R}^m$ of local diffeomorphisms defined on a neighborhood of 0 and that leave 0 fixed. The group H acts on $\mathbb{R}^m$ by linear fractional transformations that leave the origin fixed. This allows to view H as a subgroup of G^2_m.

Definition 2. A *Projective structure on M* is a reduction of the second order jet-bundle P^2M to the group H.

The following result ([16, Prop 7.2 p.147]) is the starting point of our method.

Proposition 10 (Kobayashi-Nagano)**.** *There is a natural one to one correspondence between the projective equivalence classes of torsion-free linear connections on M and the projective structures on M.*

The notion of *Normal* Cartan connection is defined by natural conditions imposed on the components of the curvature.

Now, the following result ([16, p. 135]) gives the relationship between projective structures and Cartan connections.

Proposition 11. *A unique normal Cartan connection with values in the algebra $sl(m+1,\mathbb{R})$ is associated to every projective structure P. This association is natural.*

The connection associated to a projective structure P is called the normal projective connection of the projective structure.

5.3. **Step 2 : Lift of equivariant functions.** We will establish a bijection between $GL(m,\mathbb{R})$-equivariant functions on P^1M and H-equivariant functions on P. The following results are quoted in [8, p. 47].

Definition 3. If (V,ρ) is a representation of $GL(m,\mathbb{R})$, then we define a representation (V,ρ') of H by

$$\rho' : H \to GL(V) : \left[\begin{pmatrix} A & 0 \\ \xi & a \end{pmatrix}\right] \mapsto \rho(\frac{A}{a})$$

for every $A \in GL(m,\mathbb{R}), \xi \in \mathbb{R}^{m*}, a \neq 0$.

Now, using the representation ρ', we can give the relationship between equivariant functions on P^1M and equivariant functions on P : If P is a projective structure on M, the natural projection $P^2M \to P^1M$ induces a projection $p : P \to P^1M$ and we have :

Proposition 12. *If (V, ρ) is a representation of $GL(m, \mathbb{R})$, then the map*

$$p^* : C^\infty(P^1M, V) \to C^\infty(P, V) : f \mapsto f \circ p$$

defines a bijection from $C^\infty(P^1M, V)_{\mathrm{GL}(m,\mathbb{R})}$ to $C^\infty(P, V)_H$.

This result is well-known and comes from the following facts

- (p, Id, π) is a morphism of principal bundles from P to P^1M
- the equivariant functions on P are constant on the orbits of the action of G_1 on P.

5.4. **Step 3 : Invariant differentiation.** Here we want to define an analog of the standard ordering, using the Cartan connection.

We will use the concept of invariant differentiation with respect to a Cartan connection developed in [8, 9]. Let P be a projective structure and let ω be the associated normal projective connection.

Definition 4. Let (V, ρ) be a representation of H. If $f \in C^\infty(P, V)$, then the invariant differential of f with respect to ω is the function $\nabla^\omega f \in C^\infty(P, \mathbb{R}^{m*} \otimes V)$ defined by

$$\nabla^\omega f(u)(X) = L_{\omega^{-1}(X)} f(u) \quad \forall u \in P, \quad \forall X \in \mathbb{R}^m.$$

We will also use an iterated and symmetrized version of the invariant differentiation

Definition 5. If $f \in C^\infty(P, V)$ then $(\nabla^\omega)^k f \in C^\infty(P, S^k\mathbb{R}^{m*} \otimes V)$ is defined by

$$(\nabla^\omega)^k f(u)(X_1, \ldots, X_k) = \frac{1}{k!} \sum_\nu L_{\omega^{-1}(X_{\nu_1})} \circ \ldots \circ L_{\omega^{-1}(X_{\nu_k})} f(u)$$

for $X_1, \ldots, X_k \in \mathbb{R}^m$.

Hence, the "standard ordering with respect to ω" is easy to define. It associates to $S \in C^\infty(P, S^k\mathbb{R}^m \otimes \Delta^\delta(\mathbb{R}^m))_H$ and $f \in C^\infty(P, \Delta^\lambda(\mathbb{R}^m))_H$ the function

$$\langle S, (\nabla^\omega)^k f \rangle.$$

5.5. **Step 4 : Pull-back to M.** Using steps 1 to 3, the basic idea is to define the natural and projectively equivariant quantization as follows :

- Take a symbol $T \in C^\infty(P^1M, S^k\mathbb{R}^m \otimes \Delta^\delta(\mathbb{R}^m))_{GL(m,\mathbb{R})}$ and lift it to $p^*T \in C^\infty(P, S^k\mathbb{R}^m \otimes \Delta^\delta(\mathbb{R}^m))_H$
- Apply the standard ordering with respect to ω to p^*T and p^*f for an f in $C^\infty(P^1M, \Delta^\lambda(\mathbb{R}^m))_{GL(m,\mathbb{R})}$
- Apply $(p^*)^{-1}$ to go down on M.

Doing this, we get the formula

$$Q_M(\nabla, T)(f) = (p^*)^{-1}(\langle p^*T, (\nabla^\omega)^k f\rangle).$$

Unfortunately, this simple construction does not work, because the function $\langle p^*T, (\nabla^\omega)^k f\rangle$ is not H-equivariant. The idea is then to add lower degree correcting terms to p^*T in order that this construction yields an H-equivariant function.

First, we can measure the default of equivariance of $(\nabla^\omega)^k f$:

Proposition 13. *If $f \in \mathcal{C}^\infty(P, \Delta^\lambda\mathbb{R}^m)_H$, then*

- $(\nabla^\omega)^k f$ *belongs to* $\mathcal{C}^\infty(P, S^k\mathbb{R}^{m*} \otimes \Delta^\lambda\mathbb{R}^m)_{G_0}$,
- *there holds*
$$L_{h^*}(\nabla^\omega)^k f = -k((m+1)\lambda + k - 1)((\nabla^\omega)^{k-1} f \vee h),$$
for every $h \in \mathbb{R}^{m*} \cong \mathfrak{g}_1$.

Next we build an analog of the divergence operator of the flat case in order to construct the correcting terms.

5.6. **The Divergence operator.** We fix a basis $(e_1, \ldots, e_m)$ of $\mathbb{R}^m$ and we denote by $(\epsilon^1, \ldots, \epsilon^m)$ the dual basis in $\mathbb{R}^{m*}$.

The *Divergence operator* with respect to the Cartan connection ω is then defined by

$$div^\omega : C^\infty(P, S^k_\delta(\mathbb{R}^m)) \to C^\infty(P, S^{k-1}_\delta(\mathbb{R}^m)) : S \mapsto \sum_{j=1}^m i(\epsilon^j)\nabla^\omega_{e_j} S,$$

where i denotes the inner product.

This operator is the curved generalization of the divergence operator used in [21]. The following proposition shows its most important properties.

Proposition 14. *For every $S \in C^\infty(P, S^k_\delta(\mathbb{R}^m))_H$,*

- *the function $(div^\omega)^l S$ belongs to $C^\infty(P, S^{k-l}_\delta(\mathbb{R}^m))_{G_0}$,*
- *there holds*
$$L_{h^*}(div^\omega)^l S = (m+1)l\gamma_{2k-l}\, i(h)(div^\omega)^{l-1} S,$$
for every $h \in \mathbb{R}^{m*} \cong \mathfrak{g}_1$.

5.7. **The formula.** Using propositions 13 and 14, we are now able to derive the formula for the quantization.

Theorem 15. *If δ is not critical, then the collection of maps $Q_M : \mathcal{C}_M \times \mathcal{S}_\delta(M) \to \mathcal{D}_{\lambda,\mu}(M)$ defined by*

$$Q_M(\nabla, S)(f) = {p^*}^{-1}\Big(\sum_{l=0}^k C_{k,l}\langle {div^\omega}^l p^* S, {\nabla^\omega_s}^{k-l} p^* f\rangle\Big), \forall S \in \mathcal{S}^k_\delta(M) \qquad (7)$$

defines a projectively invariant natural quantization if

$$C_{k,l} = \frac{(\lambda + \frac{k-1}{m+1}) \cdots (\lambda + \frac{k-l}{m+1})}{\gamma_{2k-1} \cdots \gamma_{2k-l}} \begin{pmatrix} k \\ l \end{pmatrix}, \forall l \geq 1, \quad C_{k,0} = 1.$$

Proof. The formula makes sense because the function

$$\sum_{l=0}^{k} C_{k,l}\langle (Div^{\omega})^{l} p^{*}S, (\nabla^{\omega})^{k-l} p^{*}f\rangle \tag{8}$$

is H-equivariant (the lower degree terms were added in order to obtain this property).

The principal symbol of $Q_M(\nabla, S)$ is exactly S, and formula (7) defines a quantization, that is projectively invariant, by the definition of ω. Next, the naturality of the quantization defined in this way follows from the naturality of all the tools used in order to define the formula. $\square$

Remarks :

- The formula coincides up to replacement of the partial derivatives by invariant differentiations to formula (4)
- One can show that formula (7) coincides for the case of third order differential operators with the formula provided by S. Bouarroudj in [6, 7].

Now, the proof of the previous theorem also allows us to analyse the existence problem when δ is a critical value : assume that there exist $k \in \mathbb{N}$ and $r \in \mathbb{N}$ such that $1 \leq r \leq k$ and $\gamma_{2k-r} = 0$. Then if there exists $i \in \{1, \ldots, r\}$ such that $\lambda = -\frac{k-i}{m+1}$, then one can replace the coefficients $C_{k,i}, \ldots, C_{k,k}$ by zero and the function (8) is still H-equivariant. Then the collection Q_M still defines a projectively equivariant and natural quantization. If λ does not belong to the set $\{-\frac{k-1}{m+1}, \ldots, -\frac{k-r}{m+1}\}$, then there is no solution since the sl_{m+1}- equivariant quantization in the sense of [21, 22] does not exist. To sum up, we have shown the following

Theorem 16. *There exists a natural and projectively equivariant quantization if and only if there exists an sl_{m+1}-equivariant quantization in the sense of* [21] *over $M = \mathbb{R}^m$.*

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