A DESCRIPTION OF n-ARY SEMIGROUPS POLYNOMIAL-DERIVED FROM INTEGRAL DOMAINS

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ABSTRACT. We provide a complete classification of the n-ary semigroup structures defined by polynomial functions over infinite commutative integral domains with identity, thus generalizing Głazek and Gleichgewicht's classification of ternary semigroups.

1. INTRODUCTION

Let R be an infinite commutative integral domain with identity and let $n \geqslant 2$ be an integer. In this note we provide a complete description of all the n-ary semigroup structures defined by polynomial functions over R (i.e., the n-ary semigroup structures polynomial-derived from R).

For any integer $k \geqslant 1$, let $[k] = \{1,\ldots,k\}$. Recall that a function $f\colon R^n \to R$ is said to be *associative* if it solves the following system of $n-1$ functional equations:

$$\begin{aligned} & f(x_1\ldots,f(x_i,\ldots,x_{i+n-1}),\ldots,x_{2n-1}) \\ = \;& f(x_1\ldots,f(x_{i+1},\ldots,x_{i+n}),\ldots,x_{2n-1}), \qquad i\in[n-1]. \end{aligned} \tag{1}$$

In this case, the pair (R,f) is called an *n-ary semigroup*.

The introduction of n-ary semigroups goes back to Dörnte [1] and led to the generalization of groups to n-ary groups (polyadic groups). The next extensive study on polyadic groups was due to Post [10] and was followed by several contributions towards the description of n-ary groups and similar structures. To mention a few, see [2–6, 8, 9, 11].

We now state our main result, which provides a description of the possible associative polynomial functions from R^n to R. Let $\mathrm{Frac}(R)$ denote the fraction field of R and let $\mathbf{x} = (x_1,\ldots,x_n)$.

Main Theorem. *A polynomial function $p\colon R^n \to R$ is associative if and only if it is one of the following functions:*

- *(i)* $p(\mathbf{x}) = c$, *where* $c \in R$,
- *(ii)* $p(\mathbf{x}) = x_1$,
- *(iii)* $p(\mathbf{x}) = x_n$,
- *(iv)* $p(\mathbf{x}) = c + \sum_{i=1}^n x_i$, *where* $c \in R$,
- *(v)* $p(\mathbf{x}) = \sum_{i=1}^n \omega^{i-1} x_i$ *(if $n \geqslant 3$), where* $\omega \in R \setminus \{1\}$ *satisfies* $\omega^{n-1} = 1$,
- *(vi)* $p(\mathbf{x}) = -b + a\prod_{i=1}^n (x_i + b)$, *where* $a \in R\setminus\{0\}$ *and* $b \in \mathrm{Frac}(R)$ *satisfy* $ab^n - b \in R$ *and* $ab^k \in R$ *for every* $k \in [n-1]$.

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The Main Theorem shows that the associative polynomial functions of degree greater than 1 are symmetric, i.e., invariant under any permutation of the variables.

Example 1. The third-degree polynomial function $p\colon\mathbb{Z}^3\to\mathbb{Z}$ defined on the ring $\mathbb{Z}$ of integers by

$$p(x_1,x_2,x_3)=9\,x_1x_2x_3+3\,(x_1x_2+x_2x_3+x_3x_1)+x_1+x_2+x_3$$

is associative since it is the restriction to $\mathbb{Z}$ of the associative polynomial function $q\colon\mathbb{Q}^3\to\mathbb{Q}$ defined on the field $\mathbb{Q}$ of rationals by

$$q(x_1,x_2,x_3)=-\frac{1}{3}+9\prod_{i=1}^{3}\Big(x_i+\frac{1}{3}\Big).$$

The classification given in the Main Theorem was already obtained for ternary semigroups (i.e., when $n=3$) by Głazek and Gleichgewicht [7]. Surprisingly, the classification for arbitrary n remains essentially the same except for certain solutions of type (v) (already mentioned in [1]), whose existence is subordinate to that of nontrivial roots of unity. Note that, when n is odd, (v) always provides the solution

$$p(\mathbf{x})=\sum_{i=1}^{n}(-1)^{i-1}\,x_i\,,$$

which was the unique solution of type (v) found in [7] for $n=3$.

In Section 2 we give the proof of the Main Theorem. In Section 3 we analyze some properties of these n-ary semigroup structures: we show that they are medial, determine the n-ary groups defined by polynomial functions, and discuss irreducibility issues for these n-ary semigroups.

2. Technicalities and proof of the Main Theorem

Throughout this section, with every function $f\colon R^n\to R$ we associate n functions $f_i\colon R^{2n-1}\to R$, $i\in[n]$, defined by

$$f_i(x_1,\ldots,x_{2n-1})=f(x_1\ldots,f(x_i,\ldots,x_{i+n-1}),\ldots,x_{2n-1}). \tag{2}$$

It follows that f is associative if and only if $f_1=\cdots=f_n$.

It is clear that the definition of R enables us to identify the ring $R[x_1,\ldots,x_n]$ of polynomials of n indeterminates over R with the ring of polynomial functions of n variables from R^n to R. Recall that, for any integer $d\geqslant 0$, a polynomial function $p\colon R^n\to R$ of degree $\leqslant d$ can be written as

$$p(\mathbf{x})=\sum_{j_1+\cdots+j_n\leqslant d}c_{j_1,\ldots,j_n}\,x_1^{j_1}\cdots x_n^{j_n}\,,\qquad c_{j_1,\ldots,j_n}\in R.$$

For every $i\in[n]$, we denote the degree of p in x_i by $\deg(p,x_i)$. We also denote the degree of p by $\deg(p)$.

Proposition 2. *For every associative polynomial function $p\colon R^n\to R$, we have $\deg(p,x_i)\leqslant 1$ for every $i\in[n]$. Moreover, if $\deg(p,x_n)=0$ (resp. $\deg(p,x_1)=0$), then p is either a constant or the projection on the first (resp. the last) coordinate.*

Proof. Let $p\colon R^n\to R$ be an associative polynomial function and let $p_1,\ldots,p_n$ be the polynomial functions associated with p as defined in (2). For every $i\in[n]$, we let $d_i=\deg(p,x_i)$. By associativity, we have

$$\begin{aligned} d_1&=\deg(p_i,x_1)=\deg(p_1,x_1)=d_1^2\,, && i\in[n]\smallsetminus\{1\},\\ d_n&=\deg(p_i,x_{2n-1})=\deg(p_n,x_{2n-1})=d_n^2\,, && i\in[n]\smallsetminus\{n\},\end{aligned}$$

which shows that $d_1 \leqslant 1$ and $d_n \leqslant 1$.

Again by associativity, we have

$$\begin{array}{ll} d_i d_{n-i+1} = \deg(p_i, x_n) = \deg(p_1, x_n) = d_1 d_n\,, & i \in [n],\\ d_i = \deg(p_{i+1}, x_i) = \deg(p_i, x_i) = d_1 d_i\,, & i \in [n-1],\\ d_i = \deg(p_{i-1}, x_{n+i-1}) = \deg(p_i, x_{n+i-1}) = d_i d_n\,, & i \in [n] \smallsetminus \{1\}. \end{array}$$

If $d_1 = d_n = 1$, then the first set of equations shows that $d_i = 1$ for every $i \in [n]$. If $d_n = 0$, then the third set of equations shows that p is of the form $p(\mathbf{x}) = c_1 x_1 + c_0$ and hence we can conclude immediately. We proceed similarly if $d_1 = 0$. □

By Proposition 2 an associative polynomial function $p\colon R^n \to R$ can always be written in the form

$$p(\mathbf{x}) = \sum_{j_1,\ldots,j_n \in \{0,1\}} c_{j_1,\ldots,j_n}\, x_1^{j_1} \cdots x_n^{j_n}\,, \qquad c_{j_1,\ldots,j_n} \in R.$$

Using subsets of $[n]$ instead of Boolean indexes, we obtain

$$p(\mathbf{x}) = \sum_{J\subseteq[n]} c_J \prod_{j\in J} x_j, \qquad c_J \in R. \tag{3}$$

In order to prove the Main Theorem, we only need to describe the class of associative polynomial functions of the form (3).

To avoid cumbersome notation, for every subset $S = \{j_1, \ldots, j_k\}$ of integers and every integer m, we set $S + m = \{j_1 + m, \ldots, j_k + m\}$. Also, for every $i \in [n]$, we let

$$\begin{array}{lclcl} A_i & = & \{1, \ldots, i-1\} & = & [i-1],\\ B_i & = & \{i, \ldots, i+n-1\} & = & [n] + i - 1,\\ C_i & = & \{i+n, \ldots, 2n-1\} & = & [n-i] + n + i - 1, \end{array}$$

with the convention that $A_1 = C_n = \varnothing$.

Lemma 3. *If $p\colon R^n \to R$ is of the form (3), then for every $i \in [n]$ the associated function $p_i\colon R^{2n-1} \to R$ is of the form*

$$p_i(x_1, \ldots, x_{2n-1}) = \sum_{S\subseteq[2n-1]} r_S^i \prod_{j\in S} x_j \tag{4}$$

and its coefficients are given in terms of those of p by

$$r_S^i = \begin{cases} c_{J_S^i \cup\{i\}}\, c_{K_S^i}\,, & \text{if } S \cap B_i \neq \varnothing,\\ c_{J_S^i \cup\{i\}}\, c_{\varnothing} + c_{J_S^i}\,, & \text{otherwise,} \end{cases}$$

where $J_S^i = (S \cap A_i) \cup ((S \cap C_i) - n + 1)$ and $K_S^i = (S \cap B_i) - i + 1$.

Proof. We first note that

$$p(x_i, \ldots, x_{n+i-1}) = \sum_{K\subseteq[n]} c_K \prod_{k\in K} x_{k+i-1} = \sum_{K\subseteq[n]} c_K \prod_{k\in K+i-1} x_k.$$

Then, partitioning $J \subseteq [n]$ into $J \cap A_i$, $J \cap \{i\}$, and $J \cap (C_i - n + 1)$, we obtain

$$\begin{aligned} p_i(x_1, \ldots, x_{2n-1}) &= \sum_{J\subseteq[n]} c_J \prod_{j\in J\cap A_i} x_j \prod_{j\in(J+n-1)\cap C_i} x_j \prod_{j\in J\cap\{i\}} \Bigg(\sum_{K\subseteq[n]} c_K \prod_{k\in K+i-1} x_k \Bigg)\\ &= \sum_{J\subseteq[n], J\ni i}\ \sum_{K\subseteq[n]} c_J\, c_K \prod_{j\in J\cap A_i} x_j \prod_{j\in(J+n-1)\cap C_i} x_j \prod_{k\in K+i-1} x_k\\ &\quad + \sum_{J\subseteq[n], J\not\ni i} c_J \prod_{j\in J\cap A_i} x_j \prod_{j\in(J+n-1)\cap C_i} x_j. \end{aligned}$$

The result is then obtained by reading the coefficient of $\prod_{i\in S} x_i$ in the latter expression. □

Proposition 4. *Let* $p\colon R^n \to R$ *be an associative polynomial function of the form (3). If* $c_{[n]} = 0$*, then* $\deg(p) \leqslant 1$.

Proof. We assume that $c_{[n]} = 0$ and prove by induction that $c_J = 0$ for every $J \subseteq [n]$ such that $|J| \geqslant 2$. Suppose that $c_J = 0$ for every $J \subseteq [n]$ such that $|J| \geqslant k$ for some $k \geqslant 3$. Fix $J_0 \subseteq [n]$ such that $|J_0| = k-1$. We only need to show that $c_{J_0} = 0$.

Assume first that $\ell = \min(J_0) \leqslant (n+1)/2$.

(*i*) Case $\ell = 1$. Let

$$S = J_0 \cup \big((J_0 + n - 1) \smallsetminus \{n\}\big) \subseteq [2n-1].$$

We have $S \cap A_1 = \varnothing$, $S \cap B_1 = J_0$, and $(S \cap C_1) - n + 1 = J_0 \smallsetminus \{1\}$. By Lemma 3, we have $r_S^1 = c_{J_0}^2$. Setting $m = \min([n] \smallsetminus J_0)$,[1] we also have $|S \cap A_m| = |A_m| = m-1$ and $|(S \cap C_m) - n + 1| = |J_0| - (m-1)$. Moreover, $S \cap B_m \neq \varnothing$ for otherwise we would have $J_0 = \{1\}$, which contradicts $|J_0| \geqslant 2$. Thus, using Lemma 3, associativity, and the induction hypothesis, we have $r_S^m = 0$ and therefore

$$c_{J_0}^2 = r_S^1 = r_S^m = 0.$$

(*ii*) Case $1 < \ell \leqslant (n+1)/2$. Let

$$S = (J_0 + \ell - 1) \cup \big((J_0 + n - 1) \smallsetminus \{n + \ell - 1\}\big) \subseteq [2n-1].$$

We proceed as above to obtain $r_S^\ell = c_{J_0}^2$. By associativity it is then sufficient to show that $r_S^{2\ell-1} = 0$. Using the notation of Lemma 3, we can readily see that $|K_S^{2\ell-1}| \geqslant |J_0|$. Hence by Lemma 3 we only need to show that $c_{K_S^{2\ell-1}} = 0$. If $|K_S^{2\ell-1}| > |J_0|$, then we conclude by using the induction hypothesis. If $|K_S^{2\ell-1}| = |J_0|$, then we can apply case (*i*) since $\min(K_S^{2\ell-1}) = 1$.

If $\ell > (n+1)/2$, we proceed symmetrically by setting $\ell' = \max(J)$ and considering the cases $\ell' = n$ and $(n+1)/2 \leqslant \ell' < n$ separately. □

Proposition 5. *A polynomial function* $p\colon R^n \to R$ *of the form (3) with* $c_{[n]} = 0$ *is associative if and only if it is one of the functions of types* (*i*)–(*v*).

Proof. It is straightforward to see that the functions of types (*i*)–(*v*) are associative polynomial functions.

Now, by Proposition 4 the polynomial function p has the form

$$p(\mathbf{x}) = c_0 + \sum_{i=1}^{n} c_i x_i\,, \qquad c_0, \dots, c_n \in R.$$

Comparing the coefficients of x_1 in p_1 and p_2, we obtain the equation $c_1^2 = c_1$. Similarly, we show that $c_n^2 = c_n$. If $c_1 = 0$ or $c_n = 0$, we conclude by Proposition 2. Thus we can assume that $c_1 = c_n = 1$. Comparing the coefficients of x_i in p_i and p_{i-1} for $2 \leqslant i \leqslant n$, we obtain the equations $c_1 c_i = c_2 c_{i-1}$, or equivalently, $c_i = c_2^{i-1}$ and $c_2^{n-1} = 1$. Finally, since the constant term in p_i is $c_0 + c_i c_0$, we must have $c_0 = 0$ unless $c_1 = \cdots = c_n$. □

[1] In fact, $m = \mathrm{mex}_{[n]}(J_0)$, where 'mex' stands for the *minimal excluded number*, well known in combinatorial game theory.

Lemma 6. *Let* $p\colon R^n \to R$ *be an associative polynomial function of the form (3). If* $c_{[n]} \neq 0$*, then* p *is a symmetric function.*

Proof. Let us first prove that $c_J = c_{J'}$ for every $J, J' \in [n]$ such that $|J| = |J'| = n-1$. Setting $S = [2n-1] \smallsetminus \{n\}$, we see by Lemma 3 that $r_S^i = c_{[n]}c_{[n]\smallsetminus\{n-i+1\}}$ for $i \in [n]$ and we conclude by associativity.

We now proceed by induction. Suppose that $c_J = c_{J'}$ for every $J, J' \in [n]$ such that $|J| = |J'| \geqslant k$ for some $2 \leqslant k \leqslant n-1$ and set $c_{|J|} = c_J$ for every $J \subseteq [n]$ such that $|J| \geqslant k$. Fix J_0 such that $|J_0| = k-1$ and set $S = J_0 \cup C_1$ and $m = \min([n] \smallsetminus J_0) \leqslant n-1$. By Lemma 3 and associativity we have $c_{[n]}c_{J_0} = r_S^1 = r_S^{m+1} = c_{n-1}c_{|J_0|+1}$. □

The interest of Lemma 6 is shown by the following obvious result.

Lemma 7. *A symmetric function* $f\colon R^n \to R$ *is associative if and only if the associated functions* $f_1, \ldots, f_n$ *satisfy the condition* $f_1 = f_2$.

Recall that the n-variable *elementary symmetric polynomial functions* of degree $k \leqslant n$ are defined by

$$P_k(\mathbf{x}) = \sum_{K\subseteq[n],\,|K|=k} \prod_{i\in K} x_i.$$

Proposition 8. *A polynomial function* $p\colon R^n \to R$ *such that* $\deg(p) > 1$ *is associative if and only if it is of the form*

$$p(\mathbf{x}) = \sum_{k=0}^{n} c_k\, P_k(\mathbf{x}), \tag{5}$$

where the coefficients $c_k \in R$ *satisfy the conditions*

$$c_{j+1}c_k + c_j\delta_{k,0} = c_jc_{k+1}\,, \qquad j \in [n-1],\ k \in [n]-1. \tag{6}$$

Proof. By Proposition 5 and Lemma 6, any associative polynomial function $p\colon R^n \to R$ such that $\deg(p) > 1$ is of the form (5). By Lemma 7, such a polynomial function is associative if and only if $p_1 = p_2$, that is, with the notation of Lemma 3, $r_S^1 = r_S^2$ for every $S \subseteq [2n-1]$.

Set $j = |S \cap C_1|$, $k = |S \cap B_1|$, $j' = |S \cap A_2| + |S \cap C_2|$, and $k' = |S \cap B_2|$. We have either $j' = j-1$ and $k' = k+1$, or $j' = j+1$ and $k' = k-1$, or $j' = j$ and $k' = k$. Therefore we get the equations

$$\begin{aligned} c_{j+1}c_k + c_j\delta_{k,0} &= c_jc_{k+1}\,, && j \in [n-1],\ k \in [n]-1,\\ c_{j+2}c_{k-1} + c_{j+1}\delta_{k-1,0} &= c_{j+1}c_k\,, && j \in [n-1]-1,\ k \in [n]. \end{aligned}$$

We conclude by observing that both sets of conditions are equivalent. □

Let us now consider the special case where R is a field.

Proposition 9. *Assume that* R *is a field. The associative polynomial functions from* R^n *to* R *of degree* > 1 *are of the form*

$$p_{a,b}(\mathbf{x}) = -b + a\prod_{i=1}^{n}(x_i + b), \tag{7}$$

where $a \in R \smallsetminus \{0\}$ *and* $b \in R$.

Remark 1. The functions $p_{a,b}$ defined in (7) can be written in several equivalent forms. It is easy to see that they are associative since so are $p_{a,0}$ and $p_{a,b} = \varphi \circ p_{a,0} \circ (\varphi^{-1}, \ldots, \varphi^{-1})$ where $\varphi(x) = x - b$.

Proof. Since the coefficient c_n in (5) is nonzero by Proposition 5, we can set $a = c_n$ and $b = c_{n-1}/a$. Using equation (6) for $j = n-1$ and $k \geqslant 1$, we obtain $c_k = b\,c_{k+1}$, that is, $c_k = ab^{n-k}$. Using again equation (6) for $j = n-1$ and $k = 0$, we obtain $c_0 = -b + ab^n$. We conclude by observing that the function $p_{a,b}$ is associative (see remark above). □

We see from the proof of Proposition 9 that the system of equations (6) has a unique solution in $\mathrm{Frac}(R)$. Therefore we can characterize the associative polynomial functions of degree > 1 as the restrictions to R of a nonzero multiple of the product function, up to an affine transformation in $\mathrm{Frac}(R)$.

Proposition 10. *Any associative polynomial function* $p\colon R^n \to R$ *such that* $\deg(p) > 1$ *is of type* (vi).

3. Further properties

We now investigate a few properties of the semigroup structures that we have determined.

3.1. **n-ary groups.** After classifying the ternary semigroups defined by polynomial functions, Głazek and Gleichgewicht [7] determined the corresponding ternary groups. Using the Main Theorem, we can also derive a description of the n-ary groups defined by polynomial functions. Recall that an *n-ary quasigroup* is given by a nonempty set G and an n-ary operation $f\colon G^n \to G$ such that for every $a_1, \ldots, a_n, b \in G$ and every $i \in [n]$ the equation

$$f(a_1, \ldots, a_{i-1}, z, a_{i+1}, \ldots, a_n) = b, \tag{8}$$

has a unique solution in G. An *n-ary group* is then an n-ary quasigroup (G, f) that is also an n-ary semigroup. Recall also that in an n-ary group, with any element x is associated the element $\overline{x}$, called *skew to x*, defined by the equation $f(x, \ldots, x, \overline{x}) = x$.

Proposition 11. *The n-ary groups (R, p) defined by polynomial functions $p\colon R^n \to R$ of degree $\leqslant 1$ are of type (iv) with $\overline{x} = (2-n)x - c$ and type (v) with $\overline{x} = x$.*

Proof. We immediately see that the polynomials of types (i)–(iii) do not define n-ary groups. It is well known that the polynomials of types (iv) and (v) define n-ary groups.[2] □

In general, the n-ary semigroups (R, p) defined by type (vi) are not n-ary groups. In the special case where R is a field, we have the following immediate result.

Proposition 12. *If R is a field, the n-ary semigroup $(R \setminus \{-b\}, p_{a,b})$, where $p_{a,b}$ is defined in (7), is an n-ary group. It is isomorphic to $(R \setminus \{0\}, p_{a,0})$.*

3.2. **Medial n-ary semigroup structures.** We observe that all the n-ary semigroup structures given in the Main Theorem are medial. This is a general fact for functions of degree $\leqslant 1$ on a commutative ring. This is also immediate for the function $p_{a,b}$ defined in (7) because it is the restriction to R of an n-ary operation that is isomorphic to a nonzero multiple of the n-ary product operation on $\mathrm{Frac}(R)$. From this observation it follows that, for the n-ary groups given in Proposition 11, the map $x \mapsto \overline{x}$ is an endomorphism.

[2]Polynomial functions of type (v) were already considered by Dörnte [1, p. 5] in the special case of complex numbers.

3.3. **(Ir)reducibility of n-ary semigroup structures.** Recall that if $(G,\circ)$ is a semigroup, then there is an obvious way to define an n-ary semigroup by $f(x_1,\ldots,x_n) = x_1\circ\cdots\circ x_n$. In this case, the n-ary semigroup (G,f) is said to be *derived* from $(G,\circ)$ or *reducible* to $(G,\circ)$, otherwise it is said to be *irreducible*. It is clear that the n-ary semigroups defined in types (i)–(iii) are derived from the corresponding semigroups. However, the n-ary semigroups defined in type (v) are not reducible. Indeed, otherwise we would have $y = y\circ 0\circ\cdots\circ 0$ for all $y\in R$, and therefore

$$x\circ y = x\circ(y\circ 0\circ\cdots\circ 0) = x\circ(y\circ 0)\circ 0\circ\cdots\circ 0 = x+\omega(y\circ 0),$$

for $x,y\in R$, which leads to $x\circ y = x+\omega y+c$, where $c=\omega^2(0\circ 0)$. We know from the Main Theorem that this function does not define a semigroup. We can prove similarly that the n-ary semigroups defined in type (iv) are reducible if and only if $c=(n-1)c_0$ for $c_0\in R$ and, when R is a field, that the semigroup $(R\smallsetminus\{0\},p_{a,0})$ is derived from a semigroup if and only if $a=a_0^{n-1}$ for $a_0\in R$.

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