Control and Voting Power in Shareholding Networks

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Outline

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1. Shareholding networks and measurement of control
2. Simple games, Banzhaf indices and Boolean functions
3. Application to shareholding networks

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1. Shareholding networks and measurement of control
   - Shareholding networks
   - Measurement of control
2. Simple games, Banzhaf indices and Boolean functions
   - Simple games
   - Banzhaf index
3. Application to shareholding networks
   - From shareholders' control to Banzhaf indices
   - Dealing with real networks
4. Cycles and float

Shareholding networks

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- networks of entities (firms, banks, individual owners, pension funds,...) linked by shareholding relationships;
- their structure;
- notion and measurement of control in such networks.

Shareholding networks

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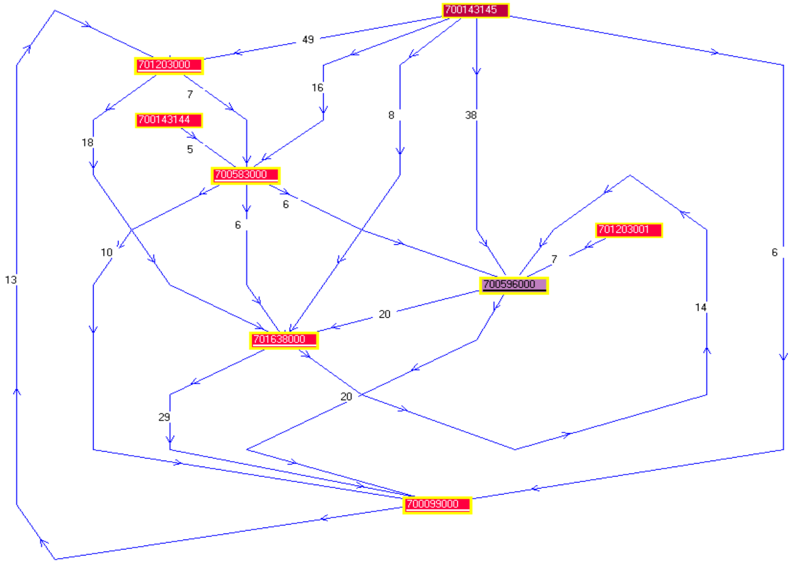

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Outsider vs. insider system

Outsider system:

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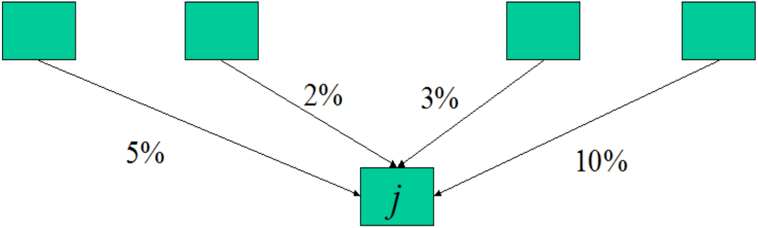
Outsider system:

- single layer of shareholders;
- dispersed ownership, high liquidity;
- transparent, open to takeovers;
- weak monitoring of management;
- typical of US and British stock markets.

Example: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Outsider vs. insider system

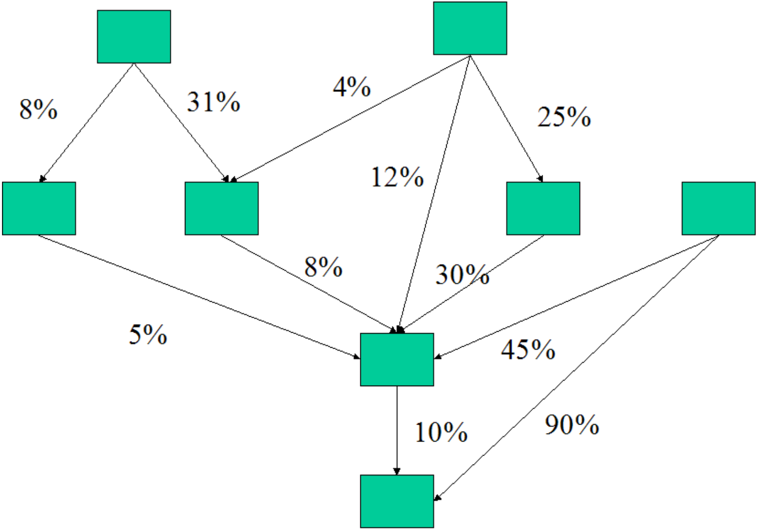
Insider system:

Outsider vs. insider system

Insider system:

- multiple layers of shareholders, possibly involving cycles
- concentrated ownership, low liquidity; controlling blocks
- strong monitoring of management
- typical of Continental Europe and Asia (Japan, South Korea, . . .)

Examples: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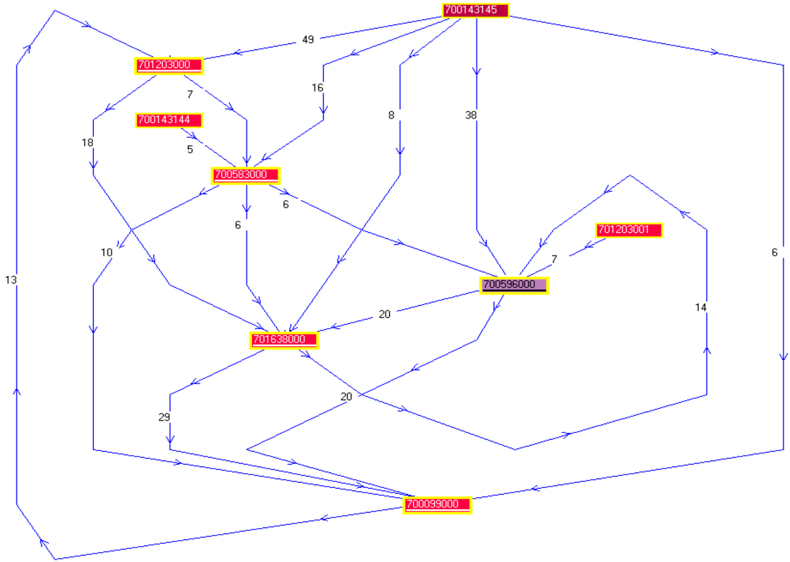

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Concentration patterns across countries

State	Ownership concentration[1]	Largest shareholder[2]
Belgium	63%	56%
France	52%	29%
Italy	68%	52%
UK	41%	15%
USA	30%	< 5%

Averages over large samples of quoted firms.
[1] Percentage of shares held by all disclosing shareholders (above 3-5%)
[2] Percentage of shares held by largest shareholder or block.

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- Not necessary to own more than 50% of the shares in order to control a firm.
- It has been argued that 20% to 30% are often sufficient.

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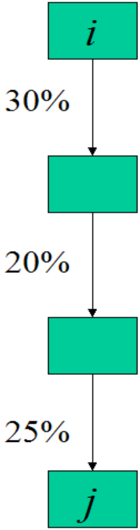
Example:

Control:
$x = 20\%$

i controls j



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Examples:

Compare the following networks.

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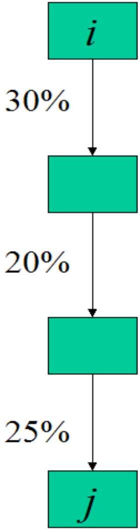

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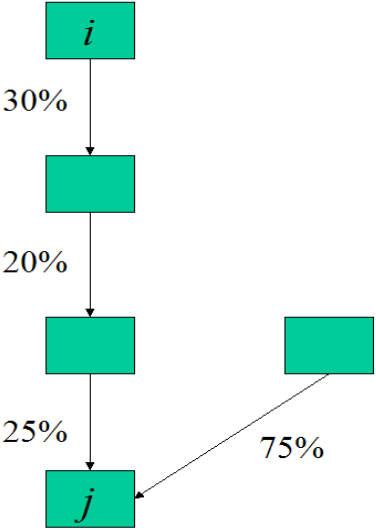

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Multiply the shareholdings along each path of indirect ownership; add up over all paths.

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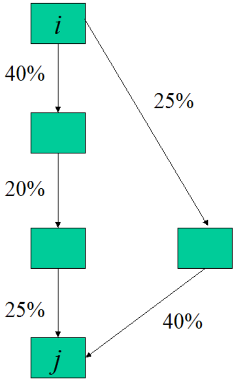
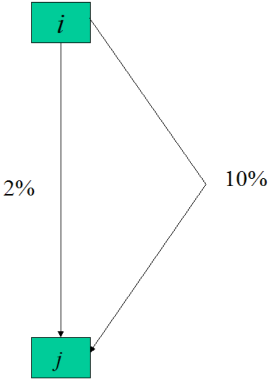
Example:

Direct ownership



Indirect ownership



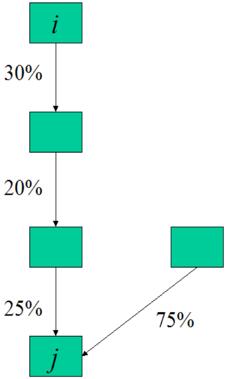
Measurement of control

Model 2:

Multiply the shareholdings along each path of indirect ownership; add up over all paths.

Weaknesses:

From the point of view of control, however, the following situations should be deemed equivalent: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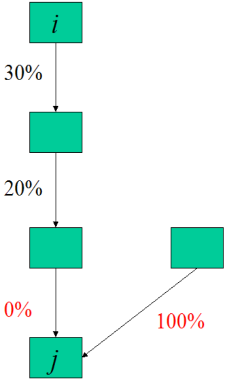
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Measurement of control

Model 3:

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- Look at the shareholders of firm j as playing a weighted majority game whenever a decision is to be made by firm j.

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Model 3:

- Look at the shareholders of firm j as playing a weighted majority game whenever a decision is to be made by firm j.
- Identify the level of control of player i over j with the Banzhaf power index of i in this game.

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Simple games

- A simple game is a monotonically increasing function $v : 2^N \rightarrow \{0, 1\}$, where $N = \{1, 2, \ldots, n\}$ is a finite set of players and 2^N is the collection of subsets of N.
- Interpretation: v describes the voting rule which is adopted by the players when a decision is to be made.
- If S is a subset of players, then $v(S)$ is the outcome of the voting process when all players in S say “Yes”.

A complex example

- Players are members of the Senate, members of the House of Representatives, and the President of the USA.
- In order to be adopted, a bill must receive
  (1) at least half of the votes in the House of Representatives and in the Senate, as well as the President's vote, or
  (2) at least two thirds of the votes in the House of Representatives and in the Senate.

For each subset S of players, the rules indicate whether $v(S) = 0$ or $v(S) = 1$ (bill is either rejected or adopted), as the outcome of the voting process when all players in S say "Yes".

Boolean digression

Link with Boolean functions:
a simple game can be viewed as a monotonically increasing Boolean function which associates a $\{0, 1\}$ output with every N-dimensional vector of $\{0, 1\}$ inputs (votes).
For instance, $v(0, 1, 1, 0, 1, 0) = 1$, etc.

Weighted majority games

- Each player i carries a voting weight w_i

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- Each player i carries a voting weight w_i
- There is a voting threshold t
- $v(S) = 1 \iff \sum_{i \in S} w_i > t$
- Typically, $t = \frac{1}{2} \sum_{i \in S} w_i$ or $t = \frac{2}{3} \sum_{i \in S} w_i$

Note that the previous example (USA) does not define a weighted majority game.

In Boolean jargon, a weighted majority game is called a threshold function.

Outline

Banzhaf index

- The Banzhaf index $Z(k)$ of player k is the probability that, for a random voting pattern (uniformly distributed), the outcome of the game changes from 0 to 1 when player k changes her vote from 0 to 1.
- Or: probability that player k is a swing player.

Example

- Member-states of the European Economic Community in 1958: Belgium, France, Germany, Italy, Luxembourg and The Netherlands.
- Council of Ministers: weighted majority decision rule.
- Voting weights: 4 for France, Germany and Italy, 2 for Belgium and The Netherlands, and 1 for Luxembourg.
- Needed: 12 votes to pass a resolution.

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- Needed: 12 votes to pass a resolution.

Observe that Luxembourg had no voting power at all.
The issue of each vote was always determined *regardless* of the vote by Luxembourg.

Banzhaf index

- The Banzhaf index provides a measure of the influence or power of player k in a voting game. (J. Banzhaf, Rutgers Law Review 1965.)
- The index is related to, but different from the Shapley-Shubik index.

Banzhaf index

- The Banzhaf index $Z(k)$ of player k is the probability that, for a random voting pattern (uniformly distributed), the outcome of the game changes from 0 to 1 when player k changes her vote from 0 to 1.
- Or: probability that player k is a swing player:

$$Z_v(k) = \frac{1}{2^{n-1}} \cdot \sum_{k \in T \subseteq N} (v(T) - v(T \setminus \{k\}))$$

Note: $v(T) - v(T \setminus \{k\}) = 1$ iff $v(T) = 1$ and $v(T \setminus \{k\}) = 0$ (that is: T wins, but $T \setminus \{k\}$ loses).

Boolean digression

Link with Boolean functions:
While attempting to characterize Boolean threshold functions, Chow (1961) has introduced $(n+1)$ parameters associated with a Boolean function $f(x_1, x_2, \ldots, x_n)$:

$$(\omega_1, \omega_2, \ldots, \omega_n, \omega)$$

where

- ω is the number of “ones” of f
- ω_k is the number of “ones” of f where $x_k = 1$.

Boolean digression

ω_k is the number of "ones" of f where $x_k = 1$

$\Longrightarrow \omega_k/2^{n-1}$ = probability that $f = 1$ when $x_k = 1$.

Can be interpreted as a measure of the *importance* or the *influence* of variable k over f.

Can be shown that Banzhaf indices are simple transformations of the Chow parameters:

$$Z(k) = (2\omega_k - \omega)/2^{n-1}.$$

Similar indices in reliability theory.

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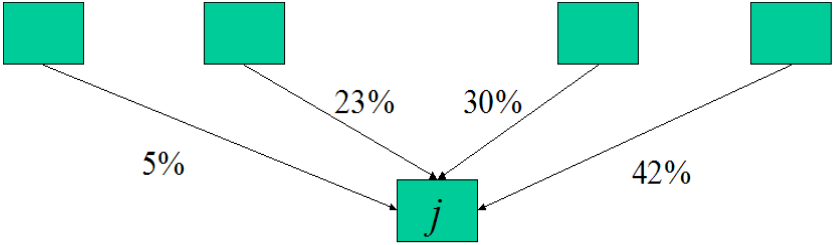
Shareholders' voting game

- Look at the shareholders of firm j as playing a weighted majority game (with quota 50%) whenever a decision is to be made by firm j
- In this model, the level of control of firm i over firm j can be measured by the Banzhaf index $Z(i,j)$ in the game
- Note: $Z(i,j)$ is equal to 1 if firm i owns more than 50% of the shares of j
- Note: More generally, $Z(i,j)$ is not proportional to the shareholdings $w(i,j)$.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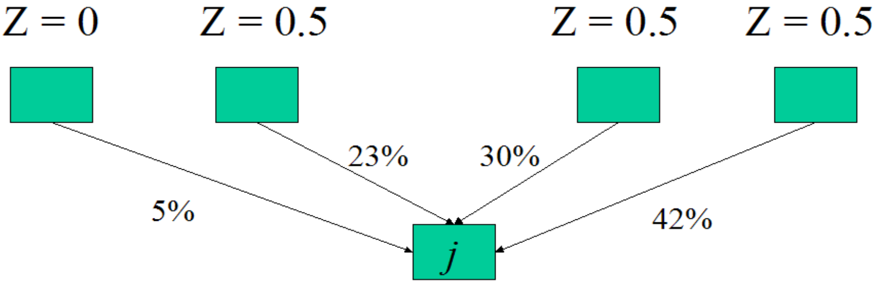
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Single layer of shareholders

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- In this case, computing Banzhaf indices is already NP-hard, but...
- Banzhaf indices can be "efficiently" computed by dynamic programming (pseudo-polynomial algo)

Additional complications

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- Ultimate relevant shareholders are not univoquely defined

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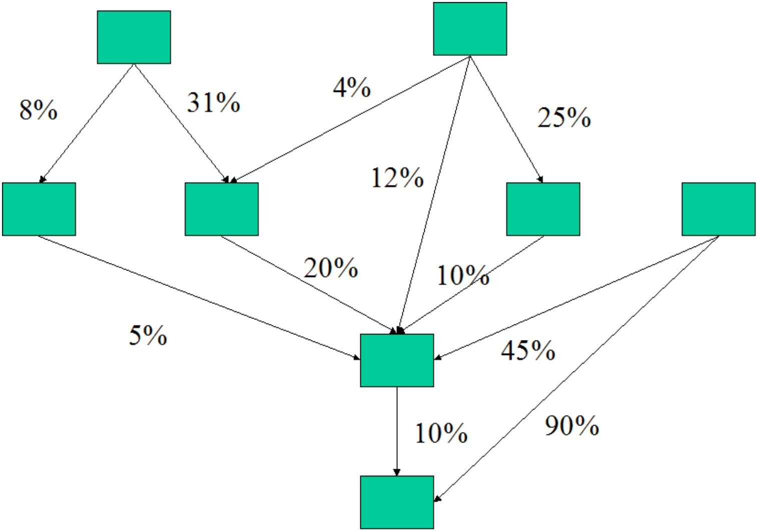
Multilayered networks

We extend previous studies in several ways:

- look at multilayered networks as defining compound games, i.e., compositions of weighted majority games.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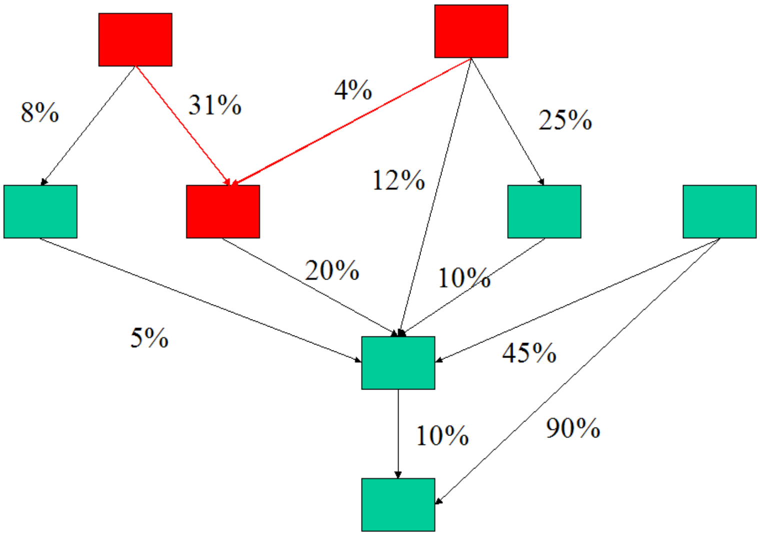
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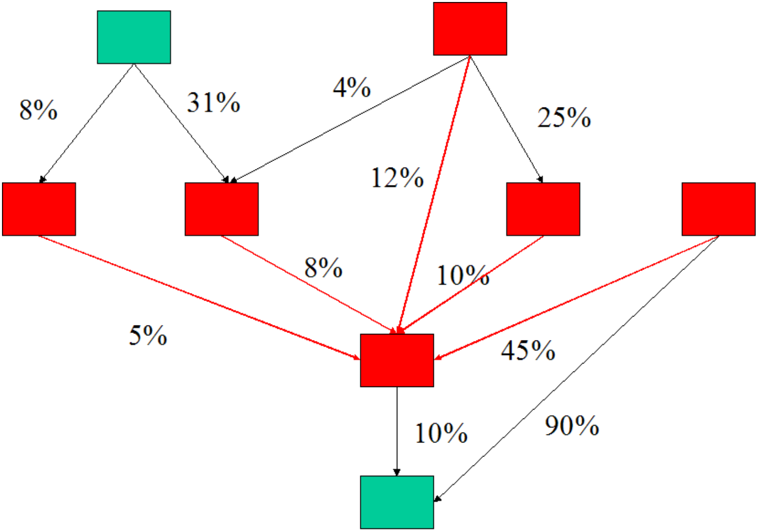
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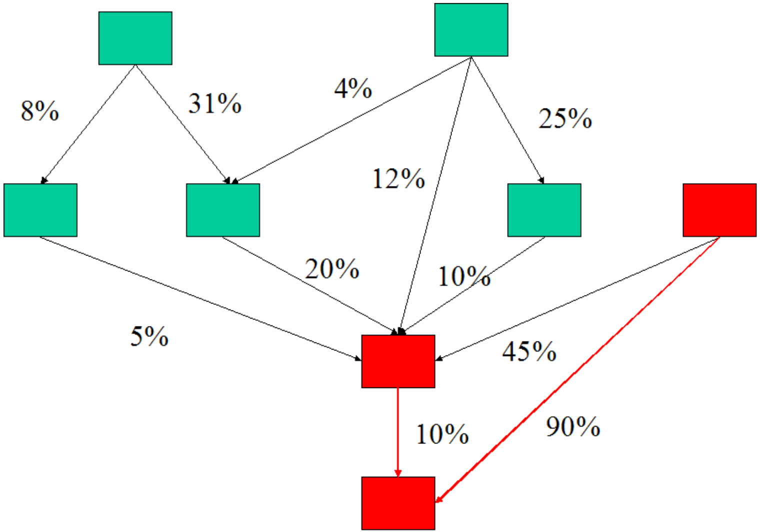
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Preprocessing

- preprocess large networks by identifying all firms which can influence a given target (for all targets, in succession), determining the tree structure and the strongly connected components of the network rooted at the target, and eliminating firms which are fully controlled.

Monte Carlo Simulation

- Note that

$$Z_k = \frac{1}{2^{n-1}} \cdot \sum_{k \in T \subseteq N} \left(v(T) - v(T \setminus \{k\})\right)$$

is the expected value of $\left(v(T) - v(T \setminus \{k\})\right)$ when T is drawn uniformly at random.

- Handle large networks by simulating votes to estimate this expected value.

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- Question: can we account for the float in a more efficient way?

Cycles

- Handle cycles by generating iterated sequences of votes (looking for "fixed point" patterns, or sampling from the resulting distribution)
- We return later to this issue.

Computational experiments

Integrated computer code:

- takes as input a database of shareholdings
- returns the Banzhaf indices of ultimate shareholders for every firm
- first approach allowing to handle large corporate networks in a systematic fashion

Applications

Stock exchanges: Paris, London, Seoul, Kuala Lumpur

- automatic identification of corporate groups (groups of firms controlled by a same firm)
- use of control indices in econometric models of financial performance
- computation of market liquidity indices

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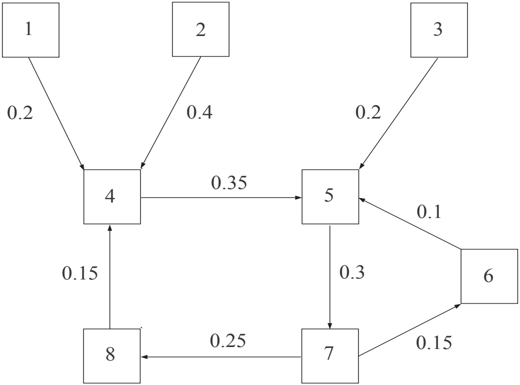

The problem with the float. . .

- The shareholders' game is not deterministic. Some of the players are not modeled like Bernoulli players.
- Can we extend the definition of the Banzhaf index in this case?

Earlier attempts

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  (1) when its value depends on the (random) vote of the float,
  (2) or when the players in T are drawn randomly.

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Consider a shareholding network $G = (N, A)$.

- for a player $j \in N$ and a subset T of its (identified) shareholders, let $v_j(T)$ denote the outcome of the simple game played by the shareholders.
- $v_j(T)$ may be a (Bernoulli) random variable, e.g.,
  (1) when its value depends on the (random) vote of the float,
  (2) or when the players in T are drawn randomly.

Note: under assumption (2), the Banzhaf index of shareholder k in the game v_j is

$$Z_v(k) = \mathrm{Prob}\left(v_j(T \cup k) - v_j(T) = 1\right)$$

where T is a random uniform subset of the shareholders of j.

New attempt: float

The same definition

$$Z_v(k) = \mathrm{Prob}\left(v_j(T \cup k) - v_j(T) = 1\right)$$

can be used more generally when we view $v_j(X)$ as a random game; e.g., when the float votes randomly.

Note that equivalently, the Banzhaf index of shareholder k in the game v_j is

$$Z_v(k) = \mathrm{Prob}\left(v_j(X \vee e_k) - v_j(X) = 1\right),$$

or

$$\mathrm{E}(v_j(X \vee e_k)) - \mathrm{E}(v_j(X)),$$

where X is a random uniform vector on $\{0, 1\}^N$.

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New attempt: cycles

We would like to interpret again the definition

$$Z_v(k) = \mathrm{Prob}\left(v_j(X \vee e_k) - v_j(X) = 1\right),$$

in the presence of cycles.

New attempt: cycles

Behavior of firms under the influence of their shareholders: Crama and Leruth (2007) propose the following model.

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Updating rule

Given a vector of votes $X \in \{0,1\}^N$, each firm j updates its vote according to the rule $X \to Y$ with $y_j = v_j(X)$ (simultaneously).

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Updating rule

Given a vector of votes $X \in \{0,1\}^N$, each firm j updates its vote according to the rule $X \to Y$ with $y_j = v_j(X)$ (simultaneously).

When v_j is random (e.g., because the float votes randomly), this model defines a Markov chain with state space $\{0,1\}^N$ and with transition matrix $P(X, Y)$, where

$$P(X,Y) = \prod_{j=1}^{N} \mathrm{Prob}(v_j(X) = y_j).$$

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If the network is acyclic and there is no float, then:

- the Markov chain converges to a unique distribution of votes $u(X)$ for each initial distribution of votes X (as in the straightforward simple game model).

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- the Markov chain converges to a unique distribution of votes $u(X)$ for each initial distribution of votes X (as in the straightforward simple game model).
- For each firm j, $u_j(X)$ can be viewed as a simple game.
- As before, the Banzhaf index of player k in u_j is

$$\mathrm{Prob}\left(u_j(X \vee e_k) - u_j(X) = 1\right),$$

or

$$\mathrm{E}(u_j(X \vee e_k)) - \mathrm{E}(u_j(X)),$$

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These define a probabilistic simple game u_j for each player j: for a vector $X \in \{0,1\}^N$ (initial votes of all firms),

$$\mathrm{Prob}(u_j(X) = 1) = \sum \{q(Y, X) \mid y_j = 1\}$$

(long-term probability that node j take value 1).

New attempt: cycles

Define again the Banzhaf index of player k in the game u_j as

$$\text{Prob}\left(u_j(X \vee e_k) - u_j(X) = 1\right),$$

where X is a random uniform vector on $\{0,1\}^N$.

Main result

The above definition extends the definition of the Banzhaf index in the case of cyclic networks with float.

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In an acyclic shareholding network G (with or without float), the long-run probabilities exist and the Banzhaf index is well defined.

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The above definition extends the definition of the Banzhaf index in the case of cyclic networks with float.

Proposition

In an acyclic shareholding network G (with or without float), the long-run probabilities exist and the Banzhaf index is well defined.

Note:

Acyclicity of G is a sufficient condition, but it is not necessary for the index to be well defined.

More details and sufficient conditions

- $\{0,1\}^N$ can be partitioned into $T \cup C_1 \cup \ldots \cup C_s$, where $C_1, \ldots, C_s$ are the irreducible classes of the Markov chain.

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Proposition

$\lim_{m\to\infty} p(C_i, X, m)$ exists for each irreducible class C_i and for each initial state X.

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Proposition

If the shareholders' network G is acyclic, then the irreducible classes C_i are aperiodic.

Acyclicity of G is a sufficient condition, but it is not necessary.

Summary

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- Such indices can be efficiently computed, even for real-world large-size networks.
- Interesting theoretical questions emerge in connection with cyclic and stochastic voting networks.
- Interesting links with Boolean functions.

Outlook

This work is in progress.
Open questions:

- Links among various models: this one, Hu-Shapley, Gambarelli-Owen, recent work by Grabisch, Rusinowska, De Swart et al., Van den Brink and Steffen, etc.

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- Which shareholding networks give rise to aperiodic Markov chains?
- Extensions to other power indices.

For Further Reading I

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