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Hotelling's T^2 control chart with two adaptive sample sizes

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Abstract Some quality control schemes have been developed when several related quality characteristics are to be monitored. The familiar multivariate process monitoring and control procedure is the Hotelling's T^2 control chart for monitoring the mean vector of the process. It is a direct analog of the univariate shewhart $\bar{x}$ chart. As in the case of univariate, the ARL improvements are very important particularly for small process shifts. In this paper, we study the T^2 control chart with two-state adaptive sample size, when the shift in the process mean does not occur at the beginning but at some random time in the future. Further, the occurrence time of the shift is assumed to be exponentially distributed random variable.

Keywords Hotelling's T^2 control chart · Adjusted average time to signal (AATS) · Adaptive sample size

1 Introduction

Nowadays there is growing interest in multivariate statistical quality control, i.e. the simultaneous control of several related quality characteristics of a process, because quality control problems in industry may involve more than a single quality (a vector of) characteristics. This has formed the basis of extensive work performed in the field of multivariate quality control. Shewhart, who is famous for the development of the statistical control chart (Shewhart charts) first recognized the need to consider quality control problems as multivariate in character. A great deal of work on multivariate statistical control procedures was performed in the 1930s and in the 1940s by Hotelling (1947). Jackson (1985) mentioned in his paper that

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the multivariate techniques should posse three important properties: (1) they produce a single answer to the question that whether the process is in-control, and (2) whether the specified type I error has been mentioned, and finally (3) the technique must take into account the relationship between the variables. The Hotelling's T^2 chart, satisfies the above three properties and also has the advantage of its simplicity. Consider p correlated characteristics are being measured simultaneously and are being controlled jointly. It is assumed that the joint probability distribution of the p quality characteristics is the p-variate normal distribution. The procedure requires computing the samples mean for each of the p quality characteristics from a sample of size n_0. This set of quality characteristic means is represented by the vector $\bar{\mathbf{x}}' = (\bar{x}_1, \ldots, \bar{x}_p)$.

The subgroup statistics $T^2 = \mathbf{n_0}(\bar{\mathbf{x}} - \boldsymbol{\mu_0})' \sum^{\mathbf{-1}} (\bar{\mathbf{x}} - \boldsymbol{\mu_0})$ are plotted on a control chart in sequential order. Here, $\boldsymbol{\mu}'_{\mathbf{0}} = (\mu_{01}, \ldots, \mu_{0p})$ is the vector of in-control means for the p quality characteristics. For the sake of simplicity, we assume here that $\boldsymbol{\mu_0}$ and $\sum$ are known. The statistic T^2 is distributed as a χ^2 variable with p degrees of freedom. The upper limit on the control chart is $UCL = x^2_\alpha(p)$. When the process is in control, with a probability α, the statistic T^2 exceeds the UCL, so that the overall error rate (type I) can be maintained exactly at the level α. If the process is out of control, the chart statistic is distributed as a non-central χ^2 distribution with p degrees of freedom with non-centrality parameter $\lambda = n_0 d^2$, where d is the Mahalanobis distance that is used as a measure of process shift in multivariate statistical quality control. It is usually assumed that the variance-covariance structure of the quality characteristics being charted does not change and that the assignable cause is manifested by a shift or drift in at least one component of the mean vector of the process. The magnitude of this shift is often expressed by $d^2 = (\boldsymbol{\mu} - \boldsymbol{\mu_0})' \sum^{\mathbf{-1}} (\boldsymbol{\mu} - \boldsymbol{\mu_0})$, and $\boldsymbol{\mu}$ is the p-characteristics mean vector. We denote in-control state with $d = 0$, hence we have $\boldsymbol{\alpha} = \mathbf{P}(\mathbf{T^2} > \mathbf{UCL}|\mathbf{d} = \mathbf{0})$ and so the Average Run Length (ARL) is given by ARL $= \frac{1}{\alpha}$. When the process is out of control with shift $d \neq 0$ the ARL is $\frac{1}{1-\beta}$, where β is the probability of the type II errors, i.e. $\beta = P(T^2 < UCL|d \neq 0)$.

The traditional practice in applying a control chart to monitor a process is to obtain samples of fixed size, n_0. Therefore, traditional control charts use a Fixed Ratio Sampling (FRS) scheme. Variable Sample Size procedure (VSS) is a scheme that varies the sampling size as a function of prior sample results.

The design of univariate Shewhart charts with adaptive sample sizes studied by Burr (1969), Daudin (1992), Prabhu et al. (1993), and Costa (1994). In these studies, two sample sizes are used and this procedure shows better power in detecting shifts in the mean. Zimmer et al. (1998) present a three-state adaptive sample size control chart and compared it with both the standard Shewhart control chart and the developed two-state adaptive sample size control chart. They conclude that three-state procedure is only slightly better than the two-state scheme, and so the two-state scheme is likely adequate in most applications.

In the Multivariate SPC, The T^2-*FRS* control chart shows a good performance to detect large shifts in the process mean. However, in many practical situations it is necessary to be able to detect even moderate shifts in the process mean. In such cases, the statistical efficiency of the T^2-*FRS* chart (in terms of the speed with which process mean shifts are detected) is poor.

Aparisi (1996) studied the T^2-*VSS* control chart. He considered an adaptive strategy for the subgroup size based on the data trends. He divided the area between the UCL and the origin, into two zones for the use of two different sample sizes. If the current sample value falls in a particular zone, then the corresponding sample size is to be applied for the successive sampling. He showed that the Hotelling's T^2 control chart with VSS scheme, significantly improves the efficiency of the standard Hotelling's T^2 control chart in detecting small changes in the process mean. For moderate shifts the obtained ARL values are about 1/3 of FRS scheme. However, Aparisi (1996) found that, when the shifts are large, ARL for the FRS chart is smaller or almost equal for the VSS chart.

In the present paper, we study the T^2 control chart with two-state adaptive sample size, when the shift in the process mean does not occur at the beginning but at some random time in the future. Further, the occurrence time of the shift is assumed to be exponentially distributed random variable. Considering the adjusted average time to signal (AATS) as a measure of performance, we show that T^2 control chart with two-state adaptive sample size control chart is always quicker than T^2 control chart in detecting all but very large shifts in the process mean. In Sect. 2, we take a brief look into T^2-*VSS* control chart introduced by Aparisi (1996). In Sect. 3, we introduce the T^2 control chart with two-state adaptive sample size. In Sect. 4, we discuss the measure of performance. In Sect. 5, we discuss the T^2-*VSS* chart with fixed n_2, and finally in Sect. 6, we make concluding remarks.

2 Reviewing the Hotelling's T^2 control chart with variable sample sizes (T^2-*VSS*)

As is defined in Aparisi (1996), the T^2 control chart with variable sample sizes contains a warning line which is denoted by W besides the UCL. If $0 < T_{i-1}^2 < W$ then the ith sample size will be n_1. If $W < T_{i-1}^2 < UCL$ then ith sample size is n_2 and if $T_{i-1}^2 > UCL$ then we say that the process is out of control. The values of n_1 and n_2 are selected in conjunction with W to obtain an average sample size n_0 which is the sample size for standard T^2 control chart ($n_1 < n_0 < n_2$). Therefore the T^2-VSS is defined as follows:

Upper control limit: $UCL = x_\alpha^2(p)$

$$\text{Sample size: } n_i = \begin{cases} n_1 & 0 < T_{i-1}^2 < W \\ n_2 & W < T_{i-1}^2 < UCL. \end{cases} \tag{1}$$

Aparisi (1996) used Markov chain to calculate the ARL values of T^2-VSS depends on the d parameter. To explain this, let

State 1 represent the state where $0 < T_{i-1}^2 < W$.
State 2 represent the state where W $< T_{i-1}^2 < UCL$.
State 3 represent the state where $T_{i-1}^2 > UCL$.

This state is an absorbing state. Consider the following transition probability matrix when a shift d is occurred:

$$P = \begin{bmatrix} p_{11}^d & p_{12}^d & p_{13}^d \\ p_{21}^d & p_{22}^d & p_{23}^d \\ 0 & 0 & 1 \end{bmatrix}$$

where

$$p_{j1}^d = \Pr(T_i^2 < W|n_j, d)$$
$$p_{j2}^d = \Pr(W < T_i^2 < UCL|n_j, d), \quad j = 1, 2$$

for example, The notation $\Pr(T_i^2 < W|n_j, d)$ is the probability that a subgroup statistic is under the warning line with sample size n_j when a shift d occurred.

Since state 3 is the out-of-control state, it is the absorbing state and, with a Markov process, the expected number of trials until the absorbing state is reached can be obtained from $B(I - Q_d)^{-1}$ where Q_d is the transition probability matrix with the rows and columns of the absorbing state deleted and B is the vector of initial probabilities. Brook and Evans (1972) applied this property to calculate the ARL of a control chart as follows:

$$ARL_d = B(I - Q_d)^{-1}1 \tag{2}$$

where $1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. Hence $b_1 + b_2 = 1$. Reasonable choice for b_1, b_2 are the proportions of time spent in state 1 and 2 before the control chart signals, and therefore $b_1 = \Pr(x_p^2 < W)$. With these choices for b_1, b_2, the restriction that the on target average sample size for the adaptive-sample-size chart is n_0 implies that:

$$n_1 b_1 + n_2 b_2 = n_0 \tag{3}$$

So we can write Eq. 2 as follows:

$$ARL(p, d, W, n_1, n_2) = (F(W, p, 0) \quad 1 - F(W, p, 0)) \\ \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} F(W, p, \lambda_1) & F(UCL, p, \lambda_1) - F(W, p, \lambda_1) \\ F(W, p, \lambda_2) & F(UCL, p, \lambda_2) - F(W, p, \lambda_2) \end{bmatrix}\right)^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{4}$$

where $\lambda_1 = n_1 d^2$, $\lambda_2 = n_2 d^2$ and $F(x, p, \lambda)$ is the cumulative probability function of a χ^2 distribution with non-centrality parameter λ. The direct minimization of Eq. 4 with the considered restrictions seems to be impossible. Aparisi (1996) has minimized Eq. 4 with using the following b_1 values: 0.95, 0.9, 0.85, 0.80, 0.75, 0.70, 0.65, 0.60, 0.55, 0.50, 0.45 and 0.40. He summarized results in tables. The tables show the ARL values using a variable sample size versus the ARL using a fixed sample size. In these tables when the process shift is important (i.e. $d = 2$), the ARL values are slightly larger for the double sample size scheme. However, he used $(b_1 = F(W, p, 0), b_2 = 1 - F(W, p, 0))$ as the proportions of time spent in states 1 and 2 before the control chart signals. Since the state 3 represents the state where $T_{i-1}^2 > UCL$ and This state is an absorbing state a reasonable choice for b_1, b_2 is $\left(b_1 = \frac{F(W,p,0)}{F(UCl,p,0)},\ b_2 = 1 - \frac{F(W,p,0)}{F(UCL,p,0)}\right)$, the proportion of time that process remains in states 1 and 2 respectively when the process is in control.

3 Description of the Hotelling's T^2 control chart with two adaptive sample sizes

Throughout this article, it is assumed that the process starts in a state of statistical control with mean vector $\boldsymbol{\mu} = \boldsymbol{\mu_0}$ and known covariance matrix $\sum$. The occurrence of assignable cause results in a shift in the process mean, which is measured by d. The time before the assignable cause occurs has an exponential distribution with parameter λ. Thus, the mean time that the process remains in state of statistical control is λ^{-1}.

We use two sample sizes $n_1 < n_0 < n_2$. The position of each sample point on the chart establishes the size of the next sample. The Procedure is as follows:

- If $0 \leq T_{i-1}^2 < W$ then the next sample is taken with size n_1.
- If $W \leq T_{i-1}^2 < UCL$ then the next sample is taken a with size n_2.

When the process is just starting, or after a false alarm, the first sample size and sampling interval is chosen at random.

Control Limit: $UCL = \chi_p^2(\alpha = 0.005)$

$$(n(T_i^2)) = \begin{cases} (n_2) & \text{if} \quad W \le T_{i-1}^2 < UCL \\ (n_1) & \text{if} \quad 0 \le T_{i-1}^2 < W \end{cases}$$

If $T_{i-1}^2 \ge UCL$, then the chart will show that the process is out of control. However, if $\boldsymbol{\mu} = \boldsymbol{\mu_0}$ then the signal is a false alarm.

4 Performance measure

Following Costa (1997), the speed with which a control chart detects process mean shifts measures its statistical efficiency. When the intervals between samples are fixed, the speed can be measured by the average run length (ARL). However, when the process starts in a state of statistical control and the time before the assignable cause occurs has an exponential distribution with parameter λ, it must be measured by modification of ARL, namely the adjusted average time to signal (AATS), which is some times called the steady state ATS. The ARL is the expected number of samples before the chart produces a signal, and the AATS is the average time from the process means shift until the chart produces a signal.

The average time of the cycle (ATC) is the average time from the start of the production until the first signal after the process shift. If the assignable cause occurs according to an exponential distribution with parameter λ then

$$AATS = ATC - \frac{1}{\lambda} \tag{5}$$

The memoryless property of the exponential distribution allows the computation of the ATC using the Markov chain approach. At each sampling stage, one of the following six transient states is reached according to the status of the process (in or out of control) and size of the sample (small or large):

State 1: $0 \le T^2 < W$ and the process is in control;
State 2: $\mathrm{W} \le T^2 < UCL$ and the process is in control;
State 3: $0 \le T^2 < W$ and the process is out of control;
State 4: $\mathrm{W} \le T^2 < UCL$ and the process is out of control.

The control chart produces a signal when $T^2 \ge UCL$. If the current state is 1 or 2, the signal is a false alarm; if the current state is 3 or 4, the signal is a true alarm. The absorbing state, state 5, is reached when the true alarm occurs. The transition probability matrix is given by

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} & p_{15} \\ p_{21} & p_{22} & p_{23} & p_{24} & p_{25} \\ 0 & 0 & p_{33} & p_{34} & p_{35} \\ 0 & 0 & p_{43} & p_{44} & p_{45} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where p_{ij} denotes the transition probability that i is the prior state and j is the current state. In what follows, $\eta_1 = n_1 d^2$, $\eta_2 = n_2 d^2$ and $F(x, p, \eta)$ will denote cumulative probability distribution function of a non-central χ^2 distribution with p-degrees of freedom and non-centrality parameter η. Then, p_{ij}'s which are probabilities conditional on the prior states are

$$
\begin{aligned}
p_1 = p_{11} = p_{21} &= \Pr(0 \le T^2 < W|T^2 < UCL) \times e^{-\lambda t_0} \\
&= \frac{F(W, p, \eta = 0)}{F(UCL, p, \eta = 0)} \times e^{-\lambda t_0} \\
p_2 = p_{12} = p_{22} &= \Pr(W \le T^2 < UCL|T^2 < UCL) \times e^{-\lambda t_0} \\
&= \frac{F(UCL, p, 0) - F(W, p, 0)}{F(UCL, p, \eta = 0)} \times e^{-\lambda t_0}
\end{aligned}
$$

$$
\begin{aligned}
p_{13} &= \Pr(0 \le T^2 < W) \times (1 - e^{\lambda t_0}) = F(W, p, \eta_1) \times (1 - e^{-\lambda t_0}) \\
p_{23} &= \Pr(0 \le T^2 < W) \times (1 - e^{\lambda t_0}) = F(W, p, \eta_2) \times (1 - e^{-\lambda t_0}) \\
p_{14} &= \Pr(W \le T^2 < UCL) \times (1 - e^{-\lambda t_0}) = [F(UCL, p, \eta_1) - F(W, p, \eta_1)] \times (1 - e^{-\lambda t_0}) \\
p_{24} &= \Pr(W \le T^2 < UCL) \times (1 - e^{-\lambda t_0}) = [F(UCL, p, \eta_2) - F(W, p, \eta_2)] \times (1 - e^{-\lambda t_0}) \\
p_{33} &= \Pr(0 \le T^2 < W|T^2 \sim \chi^2(p, \eta_1)) = F(W, p, \eta_1) \\
p_{43} &= \Pr(0 \le T^2 < W|T^2 \sim \chi^2(p, \eta_2)) = F(W, p, \eta_2) \\
p_{34} &= \Pr(W \le T^2 < UCL|T^2 \sim \chi^2(p, \eta_1)) = F(UCL, p, \eta_1) - F(W, p, \eta_1) \\
p_{44} &= \Pr(W \le T^2 < UCL|T^2 \sim \chi^2(p, \eta_2)) = F(UCL, p, \eta_2) - F(W, p, \eta_2)
\end{aligned}
$$

It is well known(see, e.g., Cinlar 1975), that the expected number of trials needed to reach the absorbing state can be obtained from $\mathbf{B}'(I - \mathbf{Q})^{-\mathbf{1}}$ where $\mathbf{Q}$ is the 4×4 matrix obtained from $\mathbf{P}$ on deleting the elements corresponding to the absorbing state, $\mathbf{I}$ is the identity matrix of order 4 and $\mathbf{B}' = (p_1, p_2, p_3, p_4)$ is a vector of initial probabilities, with $\sum_{i=1}^{4} p_i = 1$. Hence,

$$
ATC = \mathbf{B}'(\mathbf{I} - \mathbf{Q})^{-\mathbf{1}}\mathbf{t} \tag{6}
$$

where $\mathbf{t}' = (t_0, t_0, t_0, t_0)$ is the vector of sampling time intervals. In this paper the vector $\mathbf{B}'$ is set to (0,1,0,0), for providing an extra protection and preventing problems that are encountered during start-up.

The values of n_1 and n_2 are chosen so as to obtain an average sample size n_0, thus guaranteeing a meaningful comparison of T^2-FRS and T^2-VSS charts. The two charts should require the same average number of items (ANI) to be inspected during the in-control period.

By the elementary properties of Markov chains

$$
ANI = \mathbf{B}'(\mathbf{I} - \mathbf{Q})^{-\mathbf{1}}\mathbf{N} \tag{7}
$$

where $\mathbf{N}' = (n_1, n_2, n_1, n_2)$. The ANI and ATC for the T^2-FRS chart are easily determined because in this case, if m_1 and m_2 represent the number of samples before and after the process shift, then $m_1 + 1$ and m_2 are geometrically distributed with parameters q and $\bar{p}$, respectively. Hence for the T^2-FRS chart

$$
ANI = \left(\frac{q}{1-q}\right) n_0 \tag{8}
$$

$$
ATC = \left(\frac{q}{1-q} + \frac{1}{\bar{p}}\right) t_0 \tag{9}
$$

where, $q = e^{-\lambda t_0}$ and $\bar{p} = \Pr(T^2 > UCL|T^2 \sim \chi^2(p, \eta = n_0 d^2))$.

4.1 Optimal process

We illustrate the minimization of *AATS* by considering the following case: $p = 2,\ n_0 = 2$, $d = 0.5(0.5)2,\ \lambda = 0.0001$. Following Minimization of Eq. 6 analytically with the assigned constraints seems to be impossible. So, a SAS/IML computer program (available upon request) is needed to minimize the $AATS$ values for different shift sizes.

Considering the set of parameters(W, n_1, n_2), the practitioner can fix only two of them. The remaining parameter is a function of the others. We have fixed W, n_1. Then n_2 is determined by equating Eqs. 7 and 8, i.e.,

$$n_2 = \frac{F(UCL, p, 0)n_0 - F(W, p, 0)n_1}{F(UCL, p, 0) - F(W, p, 0)} \tag{10}$$

Aparisi (1996) has used some specific b_1 values. There is no mathematical support that why he used these values so we have used all possible values of b_1 in the interval $[0, 1-\alpha)$. The possible values of n_1 are $n_1 = 1, 2, \ldots, n_0 - 1$ and n_2 takes a value from Eq. 10 for a given n_1 and W.

Table 1 provides the adjusted average time to signal (AATS) for the T^2-VSS chart. We have found the values of W, n_1, n_2 that minimize the Eq. 6 and hence the Eq. 5.

5 Comparing approaches

Aparisi (1996) studied the T^2-*VSS* control chart, but the assumptions of their model are different from our assumptions. They do not consider the process as starting with the mean adjusted on target (i.e., $d = 0$) and then the shift occurring at some random time in the future.

Table 1 Comparison of T^2-VSS and T^2-FRS using AATS performance for $p = 2, \alpha = 0.005$ and $\lambda = 0.0001$

n_0	d	n_1	n_2	W_N	AATS_{VSS}	AATS_{FRS}
2	0.5	1	38	6.87	30.28	76.36
	1.0	1	10	4.21	5.39	17.98
	1.5	1	6	2.98	2.24	5.27
	2.0	1	4	1.82	1.32	2.01
3	0.5	1	36	5.54	16.89	54.82
	1.0	1	11	3.08	3.42	10.01
	1.5	2	7	2.98	1.57	2.68
	2.0	1	4	0.45	0.90	1.06
4	0.5	1	34	4.67	11.72	41.42
	1.0	2	13	3.28	2.57	6.38
	1.5	3	8	2.98	1.22	1.66
	2.0	3	5	0.81	0.70	0.73
5	0.5	1	35	4.18	8.99	32.44
	1.0	3	14	3.28	2.09	4.43
	1.5	4	8	2.49	1.00	1.17
	2.0	4	6	0.81	0.59	0.60

Table 2 Comparison of T^2-VSS and T^2-FRS using AATS performance for $p = 4$, $\alpha = 0.005$ and $\lambda = 0.0001$

n_0	d	n_1	n_2	W_N	$AATS_{VSS}$	$AATS_{FRS}$
2	0.5	1	53	11.21	40.51	100.77
	1.0	1	12	7.79	7.44	28.20
	1.5	1	6	5.66	2.84	8.32
	2.0	1	4	4.03	1.60	3.00
3	0.5	1	47	9.55	22.67	76.87
	1.0	1	13	6.3	4.41	15.96
	1.5	1	7	4.31	1.93	4.09
	2.0	1	4	1.65	1.11	1.46
4	0.5	1	44	8.49	15.60	60.46
	1.0	1	13	5.23	3.28	10.13
	1.5	2	8	4.31	1.49	2.43
	2.0	3	5	2.37	0.83	0.92
5	0.5	1	45	7.87	11.84	48.69
	1.0	2	15	5.46	2.61	6.95
	1.5	4	10	6.19	1.21	1.63
	2.0	4	6	2.37	0.68	0.70

They adopted the simplifying assumption that the process starts with $d > 0$. So, they did not develop a model involving a prior distribution for the amount of time the process remains in control. They expressed the statistical efficiency of control charts in terms of ARL. The ARl and AATS measure the chart's sensitivity in detecting shifts in the process mean, but they cannot be compared because they were obtained under different assumptions. So, we generated numerical results for the above-mentioned schemes using the AATS as the measure of the time required to detect a shift.

Tables 1, 2 provide optimal control scheme for T^2-*VSS* along with a comparison with the T^2-*FRS* control chart in term of AATS performance. Full Tables of numerical solutions for $p = 2, 3, 4, 5, 10, 20$, $n_0 = 2, 3, 4, 5, 7$, $d = 0.5(0.5)2$, and $\lambda = 0.0001, 0.001, 0.01, 0.1$ are available upon request.

According to Tables 1, 2, the T^2-*VSS* chart has lower AATS-values for detecting shifts in comparison to the other procedure and hence, one can expect great savings due to the reduction in the production of non-conforming parts in an out-of-control condition. For example, consider the case where $p = 2, n_0 = 2,\ t_0 = 20'$ and $d = 1$. In this case, two-state adaptive sample size procedure reduces the time interval from 359.60 to 107.80 min i.e., our proposed chart alarms 251.80 min sooner than the FRS procedure for a shift of 1. In this case, a reduction of 70% is obtained.

We end this section by addressing two common questions. One may mention that the Tables in VSS procedure have imposed different structures (plans) for detecting different shift sizes. So in order to select a plan to be applied from the Tables, the magnitude of the process shift has been decisive. As Aparisi (1997) mentioned, we should not forget that the importance of a process shift depends on the process capability. If a process is very capable, small process shifts hardly influence the amount of nonconforming items. If we have to choose a plan for these very capable processes we should select the one able to quickly detect large process shifts. On the other hand, for a process of small capability even a small shift can produce a large amount of nonconforming items. Therefore, we should be quick to detect these small shifts. Thus, we need to clarify which shift sizes are "important" for control purposes. A good reference on multivariate capability index is Taam et al. (1993).

6 The T^2-VSS chart with fixed n_2

According to the Tables 1, 2, the magnitude of the process shift is decisive. But, we should not ignore the importance of process ability in producing the required sample size n_2 in t_0 units of time. Thus a table is not useful, if a process cannot provide the required sample size n_2 in t_0 units of time. As an example, consider the case $d = 0.5$, $n_0 = 2$ in the Table 1. To be able to use this table, it is necessary to produce a sample of size 38 units in t_0 units of time, when the sample value falls in the corresponding zone. What if a process cannot produce such a sample size in the requested time interval? Shouldn't there be a cost factor attached to using sample size 38 or larger? Many processes do not allow for such large sample sizes. Hence, we have developed a procedure to take the sample size limitation into account. Based on the maximum allowable sample size n_2 (to be considered as a fixed value), $n_2 = 5(5)20$, we minimized the Eq. 5 under the proposed restrictions. As an example, Table 3 presents the VSS scheme with fixed sample size n_2 and AATS-values as functions of the magnitude of the shift $d$in the process mean.

With fixed n_1, n_2, an expression for the calculation of W is obtained as follows:

$$W = F^{-1}\left[\frac{(n_0 - n_2)\,F(UCL, p, 0)}{(n_1 - n_2)}, p\right]$$

Table 3 The AATS values for T^2-VSS chart with fixed n_2 and T^2-FRS for $p = 2$, $\alpha = 0.005$ and $\lambda = 0.0001$

n_0	d	n_1	n_2	W_N	AATS$_{VSS}$	AATS$_{FRS}$
2	0.5	1	5	2.74	62.22	76.36
		1	10	4.32	49.21	
		1	15	5.15	41.22	
		1	20	5.72	36.24	
3	0.5	1	5	1.38	44.89	54.82
		1	10	2.97	31.45	
		1	15	3.83	24.53	
		1	20	4.42	20.69	
4	0.5	1	5	0.57	36.83	41.42
		1	10	2.18	23.74	
		1	15	3.04	17.67	
		1	20	3.64	14.52	
5	0.5	1	5	0.00	32.44	32.44
		1	10	1.61	19.62	
		1	15	2.48	14.08	
		1	20	3.08	11.33	

We have developed similar tables for the other cases. These tables are available upon request. Improvements in AATS-values with respect to FRS scheme are substantial, even if n_2 is predetermined.

7 Concluding remarks

In this paper, a T^2 control chart with two-adaptive sample sizes has been developed. It is assumed that the amount of time the process remains in control has exponential distribution. The user has been provided with tables of these schemes that easily allow an optimal plan. The proposed schemes result in more rapid detection of lack of control and hence, reduce the costs associated with nonconforming products.

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