EXPLICIT FORMULA FOR THE NATURAL AND PROJECTIVELY EQUIVARIANT QUANTIZATION

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Abstract. In [8], P. Lecomte conjectured the existence of a natural and projectively equivariant quantization. In [1], M. Bordemann proved this existence using the framework of Thomas-Whitehead connections. In [9], we gave a new proof of the same theorem thanks to the Cartan connections. After these works, there was no explicit formula for the quantization. In this paper, we give this formula using the formula in terms of Cartan connections given in [9]. This explicit formula constitutes the generalization to any order of the formulae at second and third orders soon published by Bouarroudj in [2] and [3].

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1. Introduction

A quantization can be defined as a linear bijection from the space $\mathcal{S}(M)$ of symmetric contravariant tensor fields on a manifold M (also called the space of *Symbols*) to the space $\mathcal{D}_{\frac{1}{2}}(M)$ of differential operators acting between half-densities.

It is known that there is no natural quantization procedure. In other words, the spaces of symbols and of differential operators are not isomorphic as representations of $\mathrm{Diff}(M)$.

The idea of equivariant quantization, introduced by P. Lecomte and V. Ovsienko in [7] is to reduce the group of local diffeomorphisms in the following way.

They considered the case of the projective group $PGL(m+1,\mathbb{R})$ acting locally on the manifold $M=\mathbb{R}^m$ by linear fractional transformations. They showed that the spaces of symbols and of differential operators are canonically isomorphic as representations of $PGL(m+1,\mathbb{R})$ (or its Lie algebra $sl(m+1,\mathbb{R})$). In other words, they showed that there exists a unique *projectively equivariant quantization.* In [5], the authors generalized this result to the spaces $\mathcal{D}_{\lambda\mu}(\mathbb{R}^m)$ of differential operators acting between λ- and

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μ-densities and to their associated graded spaces $\mathcal{S}_\delta$. They showed the existence and uniqueness of a projectively equivariant quantization, provided the shift value $\delta = \mu - \lambda$ does not belong to a set of critical values.

The problem of the $sl(m+1,\mathbb{R})$-equivariant quantization on $\mathbb{R}^m$ has a counterpart on an arbitrary manifold M. In [8], P. Lecomte conjectured the existence of a quantization procedure depending on a torsion-free connection, that would be natural (in all arguments) and that would be left invariant by a projective change of connection.

After the proof of the existence of such a *Natural and equivariant quantization* given by M. Bordemann in [1], we analysed in [9] the problem of this existence using Cartan connections. We obtained an explicit formula for the quantization map in terms of the normal Cartan connection associated to a projective equivalence class of torsion free-linear connections. This formula is nothing but the formula for the flat case given in [5] up to replacements of the partial derivatives by the invariant differentiation.

The goal of this paper is to obtain an explicit formula on M for the natural and projectively equivariant quantization. In order to do this, we develop the operators ∇^{ω^l} and Div^{ω^l} intervening in the formula given in [9] in terms of operators on M. This task can be realized using tools exposed in [4].

The paper is organized as follows. In the first section, we recall the fundamental notions necessary to understand the article. In the second part, we calculate the *deformation tensor,* the most important ingredient intervening in the developments of ∇^{ω^l} and Div^{ω^l}. In the third section, we give an algorithm that allows to compute these developments thanks to a general algorithm given in [4]. Finally, in the last part, we calculate the explicit developments of ∇^{ω^l} and Div^{ω^l} and we derive the explicit formula. We show that this formula generalizes the formulae at second and third orders soon published by Bouarroudj in [2] and [3].

2. Fundamental tools

For the sake of completeness, we briefly recall in this section the main notions and results of [9]. Throughout this note, we denote by M a smooth, Hausdorff and second countable manifold of dimension m.

2.1. **Tensor densities.**

The vector bundle of tensor densities $F_\lambda(M) \to M$ is a line bundle associated to the linear frame bundle :

$$F_\lambda(M) = P^1M \times_\rho \Delta^\lambda(\mathbb{R}^m),$$

where the representation ρ of the group $GL(m,\mathbb{R})$ on the one-dimensional vector space $\Delta^\lambda(\mathbb{R}^m)$ is given by

$$\rho(A)e = |detA|^{-\lambda}e, \quad \forall A \in GL(m,\mathbb{R}),\ \forall e \in \Delta^\lambda(\mathbb{R}^m).$$

As usual, we denote by $\mathcal{F}_\lambda(M)$ the space of smooth sections of this bundle. This is the space $C^\infty(P^1M, \Delta^\lambda(\mathbb{R}^m))_{GL(m,\mathbb{R})}$ of functions f such that

$$f(uA) = \rho(A^{-1})f(u) \quad \forall u \in P^1M,\ \forall A \in GL(m,\mathbb{R}).$$

2.2. **Differential operators and symbols.** We denote by $\mathcal{D}_{\lambda,\mu}(M)$ the space of differential operators from $\mathcal{F}_\lambda(M)$ to $\mathcal{F}_\mu(M)$. The space $\mathcal{D}_{\lambda,\mu}$ is filtered by the order of differential operators. We denote by $\mathcal{D}^k_{\lambda,\mu}$ the space of differential operators of order at most k. The space of *symbols* is then the associated graded space of $\mathcal{D}_{\lambda,\mu}$.

We denote by $S^l_\delta(\mathbb{R}^m)$ the vector space $S^l\mathbb{R}^m \otimes \Delta^\delta(\mathbb{R}^m)$. There is a natural representation ρ of $GL(m,\mathbb{R})$ on this space (the representation of $GL(m,\mathbb{R})$ on symmetric tensors is the natural one). We then denote by $S^l_\delta(M) \to M$ the vector bundle

$$P^1M \times_\rho S^l_\delta(\mathbb{R}^m) \to M,$$

and by $\mathcal{S}^l_\delta(M)$ the space of smooth sections of $S^l_\delta(M) \to M$, that is, the space $C^\infty(P^1M, S^l_\delta(\mathbb{R}^m))_{GL(m,\mathbb{R})}$.

Then if $\delta = \mu - \lambda$ the *principal symbol operator* $\sigma : \mathcal{D}^l_{\lambda,\mu}(M) \to \mathcal{S}^l_\delta(M)$ commutes with the action of diffeomorphisms and is a bijection from the quotient space $\mathcal{D}^l_{\lambda,\mu}(M)/\mathcal{D}^{l-1}_{\lambda,\mu}(M)$ to $\mathcal{S}^l_\delta(M)$.

2.3. **Projective equivalence of connections.** We denote by $\mathcal{C}_M$ the space of torsion-free linear connections on M. Two such connections are *Projectively equivalent* if there exists a one-form α on M such that their associated covariant derivatives ∇ and ∇' fulfill the relation

$$\nabla'_X Y = \nabla_X Y + \alpha(X)Y + \alpha(Y)X.$$

2.4. **Problem setting.** A *quantization* on M is a linear bijection Q_M from the space of symbols $\mathcal{S}_\delta(M)$ to the space of differential operators $\mathcal{D}_{\lambda,\mu}(M)$ such that

$$\sigma(Q_M(S)) = S, \quad \forall S \in \mathcal{S}^k_\delta(M),\ \forall k \in \mathbb{N}.$$

A *natural quantization* is a quantization which depends on a torsion-free connection and commutes with the action of diffeomorphisms (see [9] for a more precise definition).

A quantization Q_M is *projectively equivariant* if one has $Q_M(\nabla) = Q_M(\nabla')$ whenever ∇ and ∇' are projectively equivalent torsion-free linear connections on M.

2.5. **Projective structures and Cartan projective connections.** These tools were presented in detail in [9, Section 3]. We give here the most important ones for this paper to be self-contained.

We consider the group $G = PGL(m+1,\mathbb{R})$. We denote by H the subgroup

$$H = \{\begin{pmatrix} A & 0 \\ \xi & a \end{pmatrix} : A \in GL(m,\mathbb{R}), \xi \in \mathbb{R}^{m*}, a \neq 0\}/\mathbb{R}_0\mathrm{Id}. \tag{1}$$

The group H is the semi-direct product $G_0 \rtimes G_1$, where G_0 is isomorphic to $GL(m,\mathbb{R})$ and G_1 is isomorphic to $\mathbb{R}^{m*}$. The Lie algebra associated to H is $\mathfrak{g}_0 \oplus \mathfrak{g}_1$.

It is well-known that H can be seen as a subgroup of the group of 2-jets G^2_m.

A *Projective structure on* M is then a reduction of the second order frame bundle P^2M to the group H.
The following result ([6, p. 147]) is the starting point of our method :

Proposition 1 (Kobayashi-Nagano)**.** *There is a natural one to one correspondence between the projective equivalence classes of torsion-free linear connections on* M *and the projective structures on* M*.*

We now recall the definition of a projective Cartan connection :

Definition 1. Let $P \to M$ be a principal H-bundle. A projective Cartan connection on P is a $sl(m+1,\mathbb{R})$- valued 1-form ω such that

- There holds $R_a^*\omega = Ad(a^{-1})\omega, \quad \forall a \in H$,
- One has $\omega(k^*) = k \quad \forall k \in \mathfrak{h} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$,
- For all $u \in P$, $\omega_u : T_uP \to sl(m+1,\mathbb{R})$ is a linear bijection.

In general, if ω is a Cartan connection defined on a H-principal bundle P, then its curvature Ω is defined as usual by

$$\Omega = d\omega + \frac{1}{2}[\omega,\omega]. \tag{2}$$

We can define from Ω a function $\kappa \in C^\infty(P, \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}^* \otimes \mathfrak{g})$ by :

$$\kappa(u)(X,Y) := \Omega(u)(\omega^{-1}(X), \omega^{-1}(Y)).$$

The *Normal* Cartan connection has the following property (see [6, p. 136]):

$$\sum_i \kappa_{jil}^i = 0 \quad \forall j, \forall l.$$

Now, the following result ([6, p. 135]) gives the relationship between projective structures and Cartan connections :

Proposition 2. *A unique normal Cartan projective connection is associated to every projective structure* P*. This association is natural.*

The connection associated to a projective structure P is called the normal projective connection of the projective structure.

2.6. **Lift of equivariant functions.** If (V,ρ) is a representation of $GL(m,\mathbb{R})$, then we can define from it a representation (V,ρ') of H (see [9] section 3). If P is a projective structure on M, the natural projection $P^2M \to P^1M$ induces a projection $p : P \to P^1M$ and we have a well-known result:

Proposition 3. *If* (V,ρ) *is a representation of* $GL(m,\mathbb{R})$*, then the map*

$$p^* : C^\infty(P^1M, V) \to C^\infty(P,V) : f \mapsto f \circ p$$

defines a bijection from $C^\infty(P^1M,V)_{\mathrm{GL}(m,\mathbb{R})}$ *to* $C^\infty(P,V)_H$*.*

Subsequently, we will use the representation ρ'_* of the Lie algebra of H on V. If we recall that this algebra is isomorphic to $gl(m,\mathbb{R}) \oplus \mathbb{R}^{m*}$ then we have

$$\rho'_*(A,\xi) = \rho_*(A), \quad \forall A \in gl(m,\mathbb{R}), \xi \in \mathbb{R}^{m*}. \tag{3}$$

Recall that if $f \in C^\infty(P,V)_H$ then one has

$$L_{h^*}f(u)+\rho'_*(h)f(u)=0, \quad \forall h\in gl(m,\mathbb{R})\oplus\mathbb{R}^{m*}\subset sl(m+1,\mathbb{R}), \forall u\in P. \quad (4)$$

2.7. **The first explicit formula.** First, we give the definitions of operators used subsequently :

Definition 2. Let (V,ρ) be a representation of H. If $f\in C^\infty(P,V)$, then $\nabla^{\omega^k} f\in C^\infty(P,\otimes^k\mathbb{R}^{m*}\otimes V)$ is defined by

$$\nabla^{\omega^k} f(u)(X_1,\ldots,X_k)=L_{\omega^{-1}(X_k)}\circ\ldots\circ L_{\omega^{-1}(X_1)}f(u).$$

If we symmetrize this operation, we obtain the

Definition 3. If $f\in C^\infty(P,V)$, then $\nabla_s^{\omega^k} f\in C^\infty(P,S^k\mathbb{R}^{m*}\otimes V)$ is defined by :

$$\nabla_s^{\omega^k} f(u)(X_1,\ldots,X_k)=\frac{1}{k!}\sum_{\nu\in S_k}\nabla^{\omega^k} f(u)(X_{\nu_1},\ldots,X_{\nu_k}).$$

If $(e_1,\ldots,e_m)$ is the canonical basis of $\mathbb{R}^m$ and if $(\epsilon^1,\ldots,\epsilon^m)$ is the dual basis corresponding in $\mathbb{R}^{m*}$, the *divergence operator* is defined then by :

$$Div^\omega : C^\infty(P,S^k_\delta(\mathbb{R}^m))\to C^\infty(P,S^{k-1}_\delta(\mathbb{R}^m)) : S\mapsto\sum_{j=1}^m\nabla^\omega_{e_j}S(\epsilon^j).$$

If $\gamma\in C^\infty(P^1M,\Delta^\lambda(\mathbb{R}^m)\otimes S^l\mathbb{R}^{m*})$, one defines the symmetrized covariant derivative of γ, $\nabla_s\gamma\in C^\infty(P^1M,\Delta^\lambda(\mathbb{R}^m)\otimes S^{l+1}\mathbb{R}^{m*})$, by :

$$(\nabla_s\gamma)(X_1,\ldots,X_{l+1})=\frac{1}{(l+1)!}\sum_\nu(\nabla_{X_{\nu(1)}}\gamma)(X_{\nu(2)},\ldots,X_{\nu(l+1)}).$$

Recall now the definition of the numbers γ_{2k-l} :

$$\gamma_{2k-l}=\frac{m+2k-l-(m+1)\delta}{m+1}.$$

A value of δ is *critical* if there are $k,l\in\mathbb{N}$ such that $1\le l\le k$ and $\gamma_{2k-l}=0$.

Finally, we can recall the formula giving the natural and projectively equivariant quantization in terms of the normal Cartan connection (see [9], theorem 11) :

Theorem 4. *If δ is not critical, then the collection of maps $Q_M:\mathcal{C}_M\times\mathcal{S}_\delta(M)\to\mathcal{D}_{\lambda,\mu}(M)$ defined by*

$$Q_M(\nabla,S)(f)=p^{*^{-1}}(\sum_{l=0}^k C_{k,l}\langle Div^{\omega^l}p^*S,\nabla_s^{\omega^{k-l}}p^*f\rangle),\forall S\in\mathcal{S}^k_\delta(M) \quad (5)$$

defines a projectively invariant natural quantization if

$$C_{k,l}=\frac{(\lambda+\frac{k-1}{m+1})\cdots(\lambda+\frac{k-l}{m+1})}{\gamma_{2k-1}\cdots\gamma_{2k-l}}\begin{pmatrix}k\\ l\end{pmatrix},\forall l\ge 1,\quad C_{k,0}=1.$$

3. The deformation tensor

An Ehresmann connection γ on P^1M belonging to a projective structure P induces a $GL(m,\mathbb{R})$-equivariant section σ of $P \to P^1M$ (see [6] page 147). This correspondence establishes a bijection between the set of the connections belonging to the projective structure P and the set of the $GL(m,\mathbb{R})$-equivariant sections of $P \to P^1M$.

If σ is the section corresponding to a connection γ, one can define a map $\tau : P \to \mathfrak{g}_1$ in the following way :

$$u = \sigma(p(u)).\exp(\tau(u)).$$

If γ is a connection on P^1M corresponding to a section σ and if ω is the normal Cartan connection corresponding to the projective class of γ, one has the following result (see [4] page 43) :

Proposition 5. *There is a unique Cartan connection* $\tilde{\gamma} = \omega_{-1} \oplus \omega_0 \oplus \tilde{\gamma}_1$ *such as* $\tilde{\gamma}_1|(T\sigma(TP^1M)) = 0$.

This Cartan connection is called the *Cartan connection induced by* γ*.*

The normal Cartan connection ω and the Cartan connection $\tilde{\gamma}$ induced by γ differ only by their components in $\mathfrak{g}_1$. Moreover, as the difference $\omega - \tilde{\gamma}$ vanishes on vertical vector fields, there is a function $\Gamma \in C^\infty(P, \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_1)$ such as

$$\omega = \tilde{\gamma} - \Gamma \circ \omega_{-1}.$$

This function is H-equivariant and represents then a tensor of type $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ on M ; it is called the *deformation tensor* (see [4] page 45). This function has the following property (see [4] lemma 3.10) :

$$(\tilde{\kappa}_0 - \kappa_0)(u)(X,Y) = [X, \Gamma(u).Y] + [\Gamma(u).X, Y] \tag{6}$$

if $u \in P$, $X, Y \in \mathfrak{g}_{-1}$ and if $\tilde{\kappa}_0$ and κ_0 are the functions induced respectively by the curvatures of $\tilde{\gamma}$ and of ω.

One can compute the deformation tensor in the projective case exactly in the same way as it is calculated in the conformal case at page 63 of [4]. First we fix a basis e_i in $\mathfrak{g}_{-1}$, e^i_j in $\mathfrak{g}_0$, ϵ^i in $\mathfrak{g}_1$. We have then

$$\Gamma(u)(e_i) = \sum_j \Gamma(u)_{ji}\epsilon^j,$$

$$\kappa_0(u)(e_i, e_j) = \sum_{k,l} \kappa_0(u)^k_{lij} e^l_k$$

and

$$\tilde{\kappa}_0(u)(e_i, e_j) = \sum_{k,l} \tilde{\kappa}_0(u)^k_{lij} e^l_k.$$

One obtains then using the equality (6) the following relations :

$$(\kappa_0 - \tilde{\kappa_0})^l_{klj} = \Gamma_{jk} - m\Gamma_{kj}; \tag{7}$$

$$(\kappa_0 - \tilde{\kappa_0})^k_{kij} = (m+1)(\Gamma_{ji} - \Gamma_{ij}). \tag{8}$$

On one hand, the functions $(\kappa_0)^l_{klj}$ and $(\kappa_0)^k_{kij}$ vanish by normality of ω. On the other hand, the functions $(\tilde{\kappa_0})^k_{lij}$ are the components of the equivariant function on P that represents the curvature tensor corresponding to the connection γ. A straightforward computation allows then to obtain the expression of the deformation tensor from the relations (7) and (8) :

$$\Gamma_{jk} = \frac{\mathrm{Ric}_{kj}}{1-m} + \frac{m\ \mathrm{trR}_{jk}}{(m+1)(m-1)}, \tag{9}$$

where Ric and trR represent the equivariant functions on P corresponding respectively to the Ricci tensor and to the trace of the curvature.

4. Developments of ∇^{ω^l} and Div^{ω^l}

In order to obtain an explicit formula for the quantization, we need to know the developments of the operators ∇^{ω^l} and Div^{ω^l} in terms of operators on M. We first recall the developments of [4].

Let γ be an Ehresmann connection on P^1M corresponding to a covariant derivative ∇ and belonging to a projective structure P. We denote by ω the normal Cartan connection on P.

Let (V,ρ) be a representation of $GL(m,\mathbb{R})$ inducing a representation (V,ρ_*) of $gl(m,\mathbb{R})$. If we denote by $\rho_*^{(l)}$ the canonical representation on $\otimes^l\mathfrak{g}_{-1}^*\otimes V$ and if $s\in C^\infty(P^1M,V)_{GL(m,\mathbb{R})}$, then $F^ls := \nabla^{\omega^l}(p^*s) - p^*(\nabla^l s)$ is given by the following induction :

$$F^0s(u) = 0$$

$$\begin{aligned}
F^ls(u)(X_1,\ldots,X_l) = {} & \rho_*^{(l-1)}([X_l,\tau(u)])(F^{l-1}s(u))(X_1,\ldots,X_{l-1})\\
&+S_\tau(F^{l-1}s(u))(X_1,\ldots,X_{l-1})\\
&+S_\nabla(F^{l-1}s(u))(X_1,\ldots,X_{l-1})\\
&+S_\Gamma(F^{l-1}s(u))(X_1,\ldots,X_{l-1})\\
&+\rho_*^{(l-1)}([X_l,\tau(u)])(p^*(\nabla^{l-1}s)(u))(X_1,\ldots,X_{l-1}).
\end{aligned}$$

This expression expands into a sum of terms of the form

$$a\rho_*^{(t_1)}(\beta_1)\ldots\rho_*^{(t_i)}(\beta_i)p^*\nabla^js$$

where a is a scalar coefficient, the β_l are iterated brackets involving some arguments X_l, the iterated invariant differentials $\nabla^r\Gamma$ evaluated on some arguments X_l, and τ. Exactly the first t_j arguments $X_1,\ldots,X_{t_j}$ are evaluated after the action of $\rho_*^{(t_j)}(\beta_j)$, the other ones appearing on the right are evaluated before. The individual transformations in S_τ, S_∇ and S_Γ act as follows :

(1) The action of S_τ replaces each summand $a\rho_*^{(t_1)}(\beta_1)\dots\rho_*^{(t_i)}(\beta_i)p^*\nabla^j$ by a sum with just one term for each occurrence of τ where this τ is replaced by $[\tau,[\tau,X_l]]$ and the coefficient a is multiplied by $-\frac{1}{2}$.

(2) The transformation S_∇ replaces each summand in F^{l-1} by a sum with just one term for each occurrence of Γ and its differentials, where these arguments are replaced by their covariant derivatives ∇_{X_l}, and with one additional term where $\nabla^j s$ is replaced by $\nabla_{X_l}(\nabla^j s)$.

(3) The transformation S_Γ replaces each summand by a sum with just one term for each occurrence of τ where this τ is replaced by $\Gamma(u).X_l$.

In fact, this algorithm becomes easily linear in the following way :

Proposition 6. *The development of $\nabla^{\omega^l}(p^*s)(X_1,\dots,X_l)$ is obtained as follows :*

$$\begin{aligned}\nabla^{\omega^l}(p^*s)(X_1,\dots,X_l) &= \rho_*^{(l-1)}([X_l,\tau])(\nabla^{\omega^{l-1}}(p^*s))(X_1,\dots,X_{l-1})\\ &+S_\tau(\nabla^{\omega^{l-1}}(p^*s))(X_1,\dots,X_{l-1})\\ &+S_\nabla(\nabla^{\omega^{l-1}}(p^*s))(X_1,\dots,X_{l-1})\\ &+S_\Gamma(\nabla^{\omega^{l-1}}(p^*s))(X_1,\dots,X_{l-1}).\end{aligned}$$

Proposition 7. *If $f\in C^\infty(P^1M,\Delta^\lambda(\mathbb{R}^m))_{GL(m,\mathbb{R})}$, then $\nabla^{\omega^l}(p^*f)(X,\dots,X)$ is a linear combination of terms of the form*

$$(\otimes^{n_{-1}}\tau\otimes p^*(\otimes^{n_{l-2}}\nabla^{l-2}\Gamma\otimes\dots\otimes\otimes^{n_0}\Gamma\otimes\nabla^q f))(X,\dots,X).$$

If we denote by $T(n_{-1},\dots,n_{l-2},q)$ such a term, then $(\nabla^\omega T(n_{-1},\dots,n_{l-2},q))(X)$ is equal to

$$\begin{aligned}&(-\lambda(m+1)-2l+n_{-1})T(n_{-1}+1,\dots,n_{l-2},q)+T(n_{-1},\dots,n_{l-2},q+1)\\ &\qquad+\sum_{j=-1}^{l-2}n_jT(n_{-1},\dots,n_j-1,n_{j+1}+1,\dots,n_{l-2},q).\end{aligned}$$

Proof. One sees indeed easily that the application of the first part of the algorithm gives

$$(-\lambda(m+1)-2l)T(n_{-1}+1,\dots,n_{l-2},q).$$

The second part gives

$$n_{-1}T(n_{-1}+1,\dots,n_{l-2},q).$$

The third part contributes to

$$T(n_{-1},\dots,n_{l-2},q+1)+\sum_{j=0}^{l-2}n_jT(n_{-1},\dots,n_j-1,n_{j+1}+1,\dots,n_{l-2},q).$$

The fourth gives

$$n_{-1}T(n_{-1}-1, n_0+1, \dots, n_{l-2}, q).$$

□

From now, we will denote by r the following multiple of the tensor Ric :

$$r(X,Y) := \frac{1}{2(1-m)}(\mathrm{Ric}(X,Y) + \mathrm{Ric}(Y,X)).$$

One deduces easily from the proposition 7 the following corollary :

Proposition 8. *If $f \in C^\infty(P^1M, \Delta^\lambda(\mathbb{R}^m))_{GL(m,\mathbb{R})}$, $\nabla_s^{\omega^l}(p^*f)$ is a linear combination of terms of the form*

$$\tau^{n_{-1}} \vee p^*((\nabla_s^{l-2} r)^{n_{l-2}} \vee \dots \vee r^{n_0} \vee \nabla_s^q f).$$

If we denote by $T(n_{-1}, \dots, n_{l-2}, q)$ such a term, then $\nabla^\omega T(n_{-1}, \dots, n_{l-2}, q)$ is equal to

$$(-\lambda(m+1) - 2l + n_{-1})T(n_{-1}+1, \dots, n_{l-2}, q) + T(n_{-1}, \dots, n_{l-2}, q+1)$$

$$+ \sum_{j=-1}^{l-2} n_j T(n_{-1}, \dots, n_j - 1, n_{j+1}+1, \dots, n_{l-2}, q).$$

Proof. First we remark that the symmetric part of Γ is reduced to r by antisymmetry of the tensor trR. It suffices then to remark that if $\nabla^{\omega^l}(p^*f)(X, \dots, X)$ is equal to a linear combination of terms of the form

$$(\otimes^{n_{-1}}\tau \otimes p^*(\otimes^{n_{l-2}}\nabla^{l-2}\Gamma \otimes \dots \otimes \otimes^{n_0}\Gamma \otimes \nabla^q f))(X, \dots, X),$$

then $\nabla_s^{\omega^l}(p^*f)$ is equal to the corresponding linear combination of the terms of the form

$$\tau^{n_{-1}} \vee p^*((\nabla_s^{l-2} r)^{n_{l-2}} \vee \dots \vee r^{n_0} \vee \nabla_s^q f).$$

Indeed, the two last tensors are then equal because they are both symmetric and that they are equal when they are evaluated on X^l. □

Remark that the action of the algorithm on the generic term of the development of $\nabla_s^{\omega^l}(p^*f)$ can be summarized. Indeed, this action gives first

$$(-\lambda(m+1) - 2l + n_{-1})T(n_{-1}+1, \dots, n_{l-2}, q).$$

It gives next

$$n_{-1}T(n_{-1}-1, n_0+1, \dots, n_{l-2}, q).$$

Finally, it makes act the covariant derivative ∇_s on

$$(\nabla_s^{l-2} r)^{n_{l-2}} \vee \dots \vee r^{n_0} \vee \nabla_s^q f.$$

Proposition 9. *If* $S \in C^\infty(P^1M, \Delta^\delta\mathbb{R}^m \otimes S^k\mathbb{R}^m)_{GL(m,\mathbb{R})}$*, then* $Div^{\omega^l}(p^*S)$ *is a linear combination of terms of the form*

$$\langle \tau^{n_{-1}} \vee p^*((\nabla_s^{k-2} r)^{n_{k-2}} \vee \ldots \vee r^{n_0}), p^*(Div^q S)\rangle.$$

If we denote by $T(n_{-1}, \ldots, n_{l-2}, q)$ *such a term, then* $Div^\omega T(n_{-1}, \ldots, n_{l-2}, q)$ *is equal to*

$$(\gamma_{2(k-l)-1}(m+1) + n_{-1})T(n_{-1}+1, \ldots, n_{l-2}, q) + T(n_{-1}, \ldots, n_{l-2}, q+1)$$
$$+ \sum_{j=-1}^{l-2} n_j T(n_{-1}, \ldots, n_j - 1, n_{j+1}+1, \ldots, n_{l-2}, q).$$

Proof. We have to compute

$$(\nabla^{\omega^{l+1}}(p^*S)(e_{i_1}, \ldots, e_{i_{l+1}}))(\epsilon^{i_1}, \ldots, \epsilon^{i_{l+1}}).$$

As the first part of the development of

$$\nabla^{\omega^{l+1}}(p^*S)(e_{i_1}, \ldots, e_{i_{l+1}})$$

according to the algorithm is

$$(\rho_*^{(l)}([e_{i_{l+1}}, \tau(u)])\nabla^{\omega^l}(p^*S)(u))(e_{i_1}, \ldots, e_{i_l}),$$

we must first calculate

$$[(\rho_*^{(l)}([e_{i_{l+1}}, \tau(u)])\nabla^{\omega^l}(p^*S)(u))(e_{i_1}, \ldots, e_{i_l})](\epsilon^{i_1}, \ldots, \epsilon^{i_{l+1}}).$$

This latter expression is equal to

$$[\rho_*([e_{i_{l+1}}, \tau(u)])(\nabla^{\omega^l}(p^*S)(u)(e_{i_1}, \ldots, e_{i_l}))](\epsilon^{i_1}, \ldots, \epsilon^{i_{l+1}})$$
$$- \sum_{j=1}^{l}(\nabla^{\omega^l}(p^*S)(u)(e_{i_1}, \ldots, [e_{i_{l+1}}, \tau(u)]e_{i_j}, \ldots, e_{i_l}))(\epsilon^{i_1}, \ldots, \epsilon^{i_{l+1}}),$$

i.e. to

$$[\rho'_*([e_{i_{l+1}}, \tau(u)])(\nabla^{\omega^l}(p^*S)(u)(e_{i_1}, \ldots, e_{i_l})(\epsilon^{i_1}, \ldots, \epsilon^{i_l}))](\epsilon^{i_{l+1}})$$
$$+ \sum_{j=1}^{l}(\nabla^{\omega^l}(p^*S)(u)(e_{i_1}, \ldots, e_{i_l}))(\epsilon^{i_1}, \ldots, \epsilon^{i_j}[e_{i_{l+1}}, \tau(u)], \ldots, \epsilon^{i_{l+1}})$$
$$- \sum_{j=1}^{l}(\nabla^{\omega^l}(p^*S)(u)(e_{i_1}, \ldots, [e_{i_{l+1}}, \tau(u)]e_{i_j}, \ldots, e_{i_l}))(\epsilon^{i_1}, \ldots, \epsilon^{i_{l+1}}),$$

if ρ' denotes the action of $GL(m,\mathbb{R})$ on symbols of degree $k-l$. The second and third lines of the previous expression give respectively $2l$ and $-2l$ terms in which n_{-1} is replaced by $n_{-1}+1$. Their contributions vanish. One sees easily that the first line gives

$$\gamma_{2(k-l)-1}(m+1)T(n_{-1}+1, \ldots, n_{l-2}, q).$$

One can see that the substitutions intervening in the third last parts of the algorithm "commute" with the valuations in $\epsilon^{i_1}, \ldots, \epsilon^{i_l}$ thanks to the general form of $\nabla^{\omega^l}(p^*S)(X_1,\dots,X_l)$. Indeed, one can show easily that $\nabla^{\omega^l}(p^*S)(X_1,\dots,X_l)$ is a linear combination of terms constructed in the following way. One applies first $p^*(\nabla^q S)$ on some X_i and one contracts the result several times with τ. One contracts then the obtained symbol with tensors of degree 1 obtained contracting some $p^*(\nabla^t\Gamma)$ with $t+1$ arguments X_i. One multiplies symmetrically the result by others X_i. Finally, one multiplies the result by numbers obtained applying τ on some X_i and some $p^*(\nabla^t\Gamma)$ on $t+2$ arguments X_i.

One sees then that the second part of the algorithm gives n_{-1} terms where n_{-1} becomes $n_{-1}+1$. One sees too that the third part contributes to

$$T(n_{-1},\dots,n_{l-2},q+1)+\sum_{j=0}^{l-2} n_j T(n_{-1},\dots,n_j-1,n_{j+1}+1,\dots,n_{l-2},q).$$

The fourth gives

$$n_{-1}T(n_{-1}-1,n_0+1,\dots,n_{l-2},q).$$

□

Remark that the action of the algorithm on the generic term of the development of $Div^{\omega^l}(p^*S)$ can be summarized. Indeed, this action gives first

$$(\gamma_{2(k-l)-1}(m+1)+n_{-1})T(n_{-1}+1,\dots,n_{l-2},q).$$

It gives next

$$n_{-1}T(n_{-1}-1,n_0+1,\dots,n_{l-2},q).$$

Finally, it makes act the divergence Div on

$$\langle(\nabla_s^{k-2}r)^{n_{k-2}}\vee\dots\vee r^{n_0}, Div^q S\rangle.$$

5. The main result

Because of the previous propositions, the quantization can be written as a linear combination of terms of the form

$$\langle\langle\tau^{n_{-1}}\vee p^*((\nabla_s^{k-2}r)^{n_{k-2}}\vee\dots\vee r^{n_0}), p^*(Div^q S)\rangle, p^*(\nabla_s^l f)\rangle.$$

In this expression, it suffices to consider the terms for which $n_{-1}=0$. Indeed, suppose that the expression

$$\sum_{j=0}^{k}\langle a_j,\tau^j\rangle \tag{10}$$

in which the functions a_j are H-equivariant is H-equivariant. First note that $L_{h^*}\tau = h$ for all $h\in\mathfrak{g}_1$ (see [4], page 48). The fact that the application of L_{h^*} to (10) gives 0 for all $h\in\mathfrak{g}_1$ implies that

$$\sum_{j=1}^{k}\langle j a_j,\tau^{j-1}\rangle$$

is equal to zero, hence H-equivariant. Repeating the process, one finds finally that $a_k = 0$. One deduces then progressively that the functions a_j are equal to zero for j equal to $1, \dots, k$.

In the sequel, we will need two operators that we will call T_1 and T_2.

If T is a tensor of type $\begin{pmatrix} 0 \\ j \end{pmatrix}$ with values in the λ-densities, then

$$T_1 T = (-\lambda(m+1) - j)(j+1) r \vee T.$$

If S is a symbol of degree j, then

$$T_2 S = ((m+1)\gamma_{2k-1} - k + j)(k - j + 1) i(r) S.$$

The following results give the explicit developments of $\nabla_s^{\omega^l}(p^* f)$ and of $Div^{\omega^l}(p^* S)$:

Proposition 10. *The term of degree t in τ in the development of $\nabla_s^{\omega^l}(p^* f)$ is equal to*

$$\begin{pmatrix} l \\ t \end{pmatrix} \prod_{j=1}^{t} (-\lambda(m+1) - l + j) p^* (\pi_{l-t}(\sum_{j=0}^{l-t} (\nabla_s + T_1)^j) f),$$

where π_{l-t} denotes the projection on the operators of degree $l-t$ (the degree of ∇_s is 1 whereas the degree of T_1 is 2). We set $\prod_{j=1}^{t}(-\lambda(m+1) - l + j)$ equal to 1 if $t = 0$.

Proof. In order to simplify the notations, denote by β the number $-\lambda(m+1)$. The formula is true if l and t are equal to 0. Suppose that the formula is satisfied for all t until the order $l-1$. If $l - t \geq 2$ and if $t \geq 2$, then the term of degree t in τ at the order l is equal using the induction procedure to :

$$(t+1) \begin{pmatrix} l-1 \\ t+1 \end{pmatrix} \prod_{j=1}^{t+1} (\beta - l + 1 + j) p^* (r \vee \pi_{l-t-2}(\sum_{j=0}^{l-t-2} (\nabla_s + T_1)^j) f)$$

$$+ \begin{pmatrix} l-1 \\ t \end{pmatrix} \prod_{j=1}^{t} (\beta - l + 1 + j) p^* (\nabla_s (\pi_{l-t-1}(\sum_{j=0}^{l-t-1} (\nabla_s + T_1)^j)) f)$$

$$+ \begin{pmatrix} l-1 \\ t-1 \end{pmatrix} (\prod_{j=1}^{t-1} (\beta - l + 1 + j))(\beta - 2l + t + 1)$$

$$p^* (\pi_{l-t}(\sum_{j=0}^{l-t} (\nabla_s + T_1)^j) f).$$

Note that

$$(\beta - l + t + 2)(l - t - 1) r \vee \pi_{l-t-2}(\sum_{j=0}^{l-t-2} (\nabla_s + T_1)^j)$$

is equal to

$$\pi_{l-t}(T_1(\sum_{j=0}^{l-t-2}(\nabla_s+T_1)^j)).$$

The sum of the three terms above is then equal to a multiple of

$$p^*(\pi_{l-t}(\sum_{j=0}^{l-t}(\nabla_s+T_1)^j)f),$$

this multiple being equal to

$$\prod_{j=2}^{t}(\beta-l+j)(\binom{l-1}{t}(\beta-l+t+1)+\binom{l-1}{t-1}(\beta-2l+t+1)),$$

i.e. to

$$\prod_{j=2}^{t}(\beta-l+j)$$

$$((\beta-l+1)(\binom{l-1}{t}+\binom{l-1}{t-1})+t\binom{l-1}{t}+(t-l)\binom{l-1}{t-1}).$$

We conclude using the formula of the Pascal's triangle.

We deal with the cases $l-t\geq 2$ & $t<2$, $l-t<2$ & $t\geq 2$ and $l-t<2$ & $t<2$ in a same way. □

Proposition 11. *The term of degree t in τ in the development of $Div^{\omega^l}(p^*S)$ is equal to*

$$\binom{l}{t}\prod_{j=1}^{t}(\gamma_{2k-1}(m+1)-l+j)p^*(\pi_{t-l}(\sum_{j=0}^{l-t}(Div+T_2)^j)S),$$

where π_{t-l} denotes the projection on the operators of degree $t-l$ (the degree of Div is -1 whereas the degree of T_2 is -2). We set the product $\prod_{j=1}^{t}(\gamma_{2k-1}(m+1)-l+j)$ equal to 1 if $t=0$.

Proof. The proof is completely similar to the one of the previous proposition. □

We can remark that the formula of the previous proposition is very similar to the equation (6.15) of [1], p.28, this equation giving the tensors intervening in the lift of the symbol S.

We can now write the explicit formula giving the natural projectively equivariant quantization of [9] :

Theorem 12. *The quantization Q_M of* [9] *is given by the following formula :*

$$Q_M(\nabla,S)(f)=\sum_{l=0}^{k}C_{k,l}\langle\pi_l(\sum_{j=0}^{l}(Div+T_2)^j)S,\pi_{k-l}(\sum_{j=0}^{k-l}(\nabla_s+T_1)^j)f\rangle.$$

One can easily derive from this formula the formula at the third order given by Bouarroudj in [3]. Indeed, if we denote by D, T, ∂T the operators ∇_s, $r\vee$ and $(\nabla_s r)\vee$ (resp. Div, $i(r)$ and $i(\nabla_s r)$) and if we denote by β the number $-\lambda(m+1)$ (resp. $\gamma_5(m+1)$), one obtains :

$$\pi_1(\sum_{j=0}^{1}(D+T)^j) = D, \quad \pi_2(\sum_{j=0}^{2}(D+T)^j) = D^2 + \beta T,$$

$$\pi_3(\sum_{j=0}^{3}(D+T)^j) = D^3 + \beta DT + 2(\beta-1)TD = D^3 + (3\beta-2)TD + \beta(\partial T).$$

We can then write the formula at the third order :

$$\begin{aligned}&\langle S, (\nabla_s^3 - (3(m+1)\lambda+2)r\vee\nabla_s - \lambda(m+1)(\nabla_s r))f\rangle\\ &+C_{3,1}\langle DivS, (\nabla_s^2 - \lambda(m+1)r)f\rangle + C_{3,2}\langle (Div^2+\gamma_5(m+1)i(r))S, \nabla_s f\rangle\\ &+C_{3,3}\langle (Div^3 + (3\gamma_5(m+1)-2)i(r)Div + \gamma_5(m+1)i(\nabla_s r))S, f\rangle.\end{aligned}$$

At the second order, the formula is simply :

$$\langle S, (\nabla_s^2 - \lambda(m+1)r)f\rangle + C_{2,1}\langle DivS, \nabla_s f\rangle + C_{2,2}\langle (Div^2+\gamma_3(m+1)i(r))S, f\rangle.$$

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