A new look at the $[\mathbf{70},1^-]$ baryon multiplet in the $1/N_c$ expansion

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So far, the masses of excited states of mixed orbital symmetry and in particular those of nonstrange $[\mathbf{70},1^-]$ baryons derived in the $1/N_c$ expansion were based on the separation of a system of N_c quarks into a symmetric core and an excited quark. Here we avoid this separation and show that an advantage of this new approach is to substantially reduce the number of linearly independent operators entering the mass formula. A novelty is that the isospin-isospin term becomes as dominant in Δ as the spin-spin term in N resonances.

I. INTRODUCTION

In 1974 't Hooft [1] suggested a perturbative expansion of QCD in terms of the parameter $1/N_c$ where N_c is the number of colors. This suggestion, together with the power counting rules of Witten [2] has lead to the $1/N_c$ expansion method which allows to systematically analyze baryon properties. The current research status is described, for example, in Ref. [3]. The success of the method stems from the discovery that the ground state baryons have an exact contracted SU($2N_f$) symmetry when $N_c \to \infty$ [4, 5], N_f being the number of flavors. A considerable amount of work has been devoted to the ground state baryons [5, 6, 7, 8, 9, 10, 11]. For $N_c \to \infty$ the baryon masses are degenerate. For finite N_c the mass splitting starts at order $1/N_c$. Operator reduction rules simplify the $1/N_c$ expansion [6, 7]. It is customary to drop higher order corrections of order $1/N_c^2$.

It is thought that 't Hooft's suggestion [1] would lead to an $1/N_c$ expansion to hold in all QCD regimes. Accordingly, the applicability of the approach to excited states is a subject of current investigation.

In the language of the constituent quark model the excited states can be grouped into excitation bands with N = 1, 2, 3, etc. units of excitation energy. Among them, the N = 1 band, or equivalently the $[\mathbf{70},1^-]$ multiplet, has been most extensively studied, either for $N_f = 2$ [12, 13, 14, 15, 16, 17, 18, 19] or for $N_f = 3$ [20]. In the latter case, first order corrections in SU(3) symmetry breaking were also included. In either case, the conclusion was that the splitting starts at order N_c^0.

The N = 2 band contains the $[\mathbf{56'},0^+]$, $[\mathbf{56},2^+]$, $[\mathbf{70},\ell^+]$ (ℓ = 0, 2) and $[\mathbf{20},1^+]$ multiplets. There are no physical resonances associated to $[\mathbf{20},1^+]$. The few studies related to the N = 2 band concern the $[\mathbf{56'},0^+]$ for $N_f = 2$ [21], $[\mathbf{56},2^+]$ for $N_f = 3$ [22] and $[\mathbf{70},\ell^+]$ for $N_f = 2$ [23], later extended to $N_f = 3$ [24]. The method has also been applied to highly excited nonstrange and strange baryons belonging to $[\mathbf{56},4^+]$ [25] which is the lowest of the 17 multiplets of the N = 4 band [26].

The mass operator M is defined as a linear combination of independent operators O_i

$$M = \sum_i c_i O_i, \tag{1}$$

where the coefficients c_i are reduced matrix elements that encode the QCD dynamics and are determined from a fit to the existing data. Here we are concerned with nonstrange baryons only. The building blocks of the operators O_i are the SU($2N_f$) generators S_i, T_a and G_{ia} and the SO(3) generators ℓ_i. Their general form is

$$O_i = \frac{1}{N_c^{n-1}} O_\ell^{(k)} \cdot O_{SF}^{(k)}, \tag{2}$$

where $O_\ell^{(k)}$ is a k-rank tensor in SO(3) and $O_{SF}^{(k)}$ a k-rank tensor in SU(2)-spin, but invariant in SU(N_f). Thus O_i are rotational invariant. For the ground state one has $k = 0$. The excited states also require $k = 1$ and $k = 2$ terms.

The spin-flavor (SF) operators $O_{SF}^{(k)}$ are combinations of SU($2N_f$) generators, the lower index i in the left hand side of (2) representing a specific combination. Each n-body operator is multiplied by an explicit factor of $1/N_c^{n-1}$ resulting from the power counting rules. Some compensating N_c factors may arise in the matrix elements when O_i contains a coherent operator such as G^{ia} or T^a.

The excited states belonging to $[\mathbf{56},\ell]$ multiplets are rather simple and can be studied by analogy with the ground state. In this case both the orbital and the spin-flavor parts of the wave function are symmetric. Naturally, it turned out that the splitting starts at order $1/N_c$ [22, 25], as for the ground state.

The states belonging to $[\mathbf{70},\ell]$ multiplets are apparently more difficult. So far, the general practice was to decouple the baryon into an excited quark and a symmetric core. This means that each generator of SU($2N_f$) must be written as a sum of two terms, one acting on the excited quark and the other on the core. As a consequence, the number of linearly independent operators O_i increases tremendously and the number of coefficients c_i, to be determined, becomes much larger than the experimental data available. For example, for the $[\mathbf{70},1^-]$

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multiplet with $N_f = 2$ one has 13 linearly independent operators up to order $1/N_c$ included [16], instead of 7 (see below). We recall that there are only 7 nonstrange resonances belonging to this band. Consequently, selecting the most dominant operators is very difficult so that one risks to make an arbitrary choice [16].

In this practice the matrix elements of the excited quark are straightforward, as being described by single-particle operators. The matrix elements of the core operators S^i_c, T^a_c are also simple to calculate, while those of G^{ia}_c are more involved. Analytic formulas for the matrix elements of all SU(4) generators have been derived in Ref. [27]. Every matrix element is factorized according to a generalized Wigner-Eckart theorem into a reduced matrix element and an SU(4) Clebsch-Gordan coefficient. These matrix elements have been used in nuclear physics, which is governed by the SU(4) symmetry. Recently we have extended the approach of Ref. [27] to SU(6) [28] and obtained matrix elements of all SU(6) generators between symmetric $[N_c]$ states.

Here we propose a method where no decoupling is necessary. All one needs to know are the matrix elements of the SU($2N_f$) generators between mixed symmetric states $[N_c - 1, 1]$. For SU(4) they were obtained by Hecht and Pang [27]. They can be easily applied to a system of N_c nonstrange quarks. To our knowledge such matrix elements are yet unknown for $N_f = 3$.

II. THE WAVE FUNCTION

We deal with a system of N_c quarks having one unit of orbital excitation. Then the orbital wave function must have a mixed symmetry $[N_c - 1, 1]$. Its spin-flavor part must have the same symmetry in order to obtain a totally symmetric state in the orbital-spin-flavor space. The general form of such a wave function is [29]

$$|[N_c]\rangle = \frac{1}{\sqrt{d_{[N_c-1,1]}}} \sum_Y |[N_c-1,1]Y\rangle_O |[N_c-1,1]Y\rangle_{FS} \tag{3}$$

where $d_{[N_c-1,1]} = N_c - 1$ is the dimension of the representation $[N_c - 1, 1]$ of the permutation group S_{N_c} and Y is a symbol for a Young tableau (Yamanouchi symbol). The sum is performed over all possible standard Young tableaux. In each term the first basis vector represents the orbital space (O) and the second the spin-flavor space (FS). In this sum there is only one Y (the normal Young tableau) where the last particle is in the second row and $N_c - 2$ terms where the last particle is in the first row. All these terms were neglected in the procedure of decoupling the excited quark, which implies that the permutation symmetry S_{N_c} was broken, *i.e.* the orbital-spin-flavor wave function was no more symmetric, as it should be. One can easily prove the above assertion by looking at the expression of the wave function, Eqs. (3.4)-(3.5) in the second paper of Ref. [16], for example. This definition contains the coefficients $c_{\rho\eta}$ which are defined as coefficients of an "orthogonal rotation". In Ref. [23] we have shown that $c_{\rho\eta}$ are some specific isoscalar factors of the permutation group S_{N_c} [29]. These are factors of the Clebsch-Gordan coefficients, factorized as isoscalar factors times Clebsch-Gordan coefficients of the group S_{N_c-1}. In the case under concern the isoscalar factors incorporate the position of the N_c-th particle in a Young tableau. By identifying our expressions with those of Ref. [16] we found that they correspond to the term where the last particle is located in the second row of the Young tableau of the representation $[N_c-1, 1]$. Thus the other $N_c - 2$ terms of the wave function, with the N_c-th particle in the first row, are missing. In Appendix A we show explicitly which are the missing terms for $N_c = 3$ in the sectors ^{28}, ^{48} and ^{210}. In addition, as an example, the orbital basis vectors of configuration s^4p, containing one unit of orbital excitation, which span the invariant subspace of the irreducible representation [41] of S_5 are given in Appendix B. The definition and the orthogonality properties together with examples of isoscalar factors can be found in Ref. [30]. In Sec. VI we discuss the validity of the approximate (asymmetric) wave function of Ref. [16].

If there is no decoupling, there is no need to specify Y, the matrix elements being identical for all Y's, due to Weyl's duality between a linear group and a symmetric group in a given tensor space [34]. Then the explicit form of a wave function of total angular momentum $\vec{J} = \vec{\ell} + \vec{S}$ and isospin I is

$$|\ell S I I_3; J J_3\rangle = \sum_{m_\ell, S_3} \left(\begin{array}{cc|c} \ell & S & J \\ m_\ell & S_3 & J_3 \end{array} \right) \times |[N_c-1,1]\ell m_\ell\rangle \; |[N_c-1,1] S S_3 I I_3\rangle, \tag{4}$$

each term containing an SU(2) Clebsch-Gordan (CG) coefficient, an orbital part $|[N_c - 1, 1]\ell m_\ell\rangle$ an a spin-flavor part $|[N_c - 1, 1]SS_3II_3\rangle$.

III. SU(4) GENERATORS AS TENSOR OPERATORS

The SU(4) generators S_i, T_a and G_{ia}, globally denoted by E_{ia} [27], are components of an irreducible tensor operator which transform according to the adjoint representation [211] of dimension **15** of SU(4). We recall that the SU(4) algebra is

$$\begin{gathered} [S_i, T_a] = 0, \qquad [S_i, G_{ja}] = i\varepsilon_{ijk} G_{ka}, \\ [T_a, G_{ib}] = i\varepsilon_{abc} G_{ic}, \\ [S_i, S_j] = i\varepsilon_{ijk} S_k, \qquad [T_a, T_b] = i\varepsilon_{abc} T_c, \\ [G_{ia}, G_{jb}] = \frac{i}{4}\delta_{ij}\varepsilon_{abc} T_c + \frac{i}{4}\delta_{ab}\varepsilon_{ijk} S_k. \end{gathered} \tag{5}$$

As one can see, the tensor operators E_{ia} are of three types: E_i (i = 1,2,3) which form the subalgebra of SU(2)-spin, E_a (a = 1,2,3) which form the subalgebra of SU(2)-isospin and E_{ia} which act both in the spin and the isospin spaces. They are related to S_i, T_a and G_{ia} ($i = 1,2,3;\ a = 1,2,3$) by

$$E_i = \frac{S_i}{\sqrt{2}}; \quad E_a = \frac{T_a}{\sqrt{2}}; \quad E_{ia} = \sqrt{2}G_{ia}. \tag{6}$$

The matrix elements of every E_{ia} between states belonging to the representation $[N_c-1,1]$ can be expressed as a generalized Wigner-Eckart theorem which reads [27]

$$\begin{aligned}&\langle [N_c-1,1]I'I'_3S'S'_3|E_{ia}|[N_c-1,1]II_3SS_3\rangle = \\ &\sqrt{C^{[N_c-1,1]}(\mathrm{SU}(4))}\left(\begin{array}{cc|c} S & S^i & S' \\ S_3 & S^i_3 & S'_3\end{array}\right)\left(\begin{array}{cc|c} I & I^a & I' \\ I_3 & I^a_3 & I'_3\end{array}\right) \\ &\times \left(\begin{array}{cc||c} [N_c-1,1] & [211] & [N_c-1,1] \\ SI & S^iI^a & S'I'\end{array}\right)_{\rho=1},\end{aligned} \tag{7}$$

where $C^{[N_c-1,1]}(\mathrm{SU}(4)) = N_c(3N_c+4)/8$ is the eigenvalue of the SU(4) Casimir operator for the representation $[N_c-1,1]$. The other three factors are: an SU(2)-spin CG coefficient, an SU(2)-isospin CG coefficient and an isoscalar factor of SU(4). Note that the isoscalar factor carries a lower index $\rho = 1$. In general, this index is necessary to distinguish between irreducible representations, whenever the multiplicity in the inner product $[N_c-1,1] \times [211] \to [N_c-1,1]$ is larger than one. In that case, the matrix elements of the SU(4) generators in a fixed irreducible representation $[f]$ are defined such as the reduced matrix elements take the following values [27]

$$\langle [f]||\mathrm{E}||[f]\rangle = \begin{cases} \sqrt{C^{[N_c-1,1]}(\mathrm{SU}(4))} & \text{for} \quad \rho = 1 \\ 0 & \text{for} \quad \rho \neq 1\end{cases}. \tag{8}$$

Thus the knowledge of the matrix elements of SU(4) generators amounts to the knowledge of the corresponding SU(4) isoscalar factors. In Ref. [27] a variety of isoscalar factors were obtained. We need those for $[f] = [N_c-1,1]$. They are reproduced in Table I in terms of our notation and typographical errors corrected. They contain the phase factor introduced in Eq. (35) of Ref. [27]. As compared to the symmetric $[N_c]$ representation, where $I = S$ always, here one has $I = S$ (13 cases) but also $I \neq S$ (10 cases). Some of the properties of these isoscalar factors are given in Appendix C.

One can easily identify the matrix elements associated to the generators of SU(4). One has $S_2I_2 = 10$ for S_i, $S_2I_2 = 01$ for T_a and $S_2I_2 = 11$ for G_{ia}, where 1 or 0 is the rank of the SU(2)-spin or SU(2)-isospin tensor contained in the generator. The generalized Wigner-Eckart theorem (7) is used to calculate the matrix elements of O_i needed for the mass operator, as described below.

IV. THE MASS OPERATOR

As specified in the introduction, here we are concerned with nonstrange baryons only. Table II contains the seven independent operators up to order $1/N_c$ appearing in the mass operator Eq. (1). As already mentioned, the building blocks of O_i are S^i, T^a, G^{ia} and ℓ^i. We also need the rank $k = 2$ tensor operator

$$\ell^{(2)ij} = \frac{1}{2}\left\{\ell^i, \ell^j\right\} - \frac{1}{3}\delta_{i,-j}\vec{\ell}\cdot\vec{\ell}, \tag{9}$$

which, like ℓ^i, acts on the orbital wave function $|\ell m_\ell\rangle$ of the whole system of N_c quarks (see Ref. [23] for the normalization of $\ell^{(2)ij}$).

In Table II the first nontrivial operator is the spin-orbit operator O_2. In the spirit of the Hartree picture [2], generally adopted for the description of baryons, we identify the spin-orbit operator with the single-particle operator

$$\ell \cdot s = \sum_{i=1}^{N_c} \ell(i)\cdot s(i). \tag{10}$$

Accordingly, its matrix elements are of order N_c^0. For simplicity we ignore the two-body part of the spin-orbit operator, denoted by $1/N_c\,(\ell\cdot S_c)$ in Ref. [16], as being of a lower order (the lower case indicates operators acting on the excited quark and the subscript c is attached to those acting on the core).

The operators O_3 and O_4 are two-body and linearly independent. However, in the decoupling procedure the corresponding isospin-isospin operator $t^aT^a_c/N_c$ has always been avoided in the numerical analysis [16, 20].

To be consistent with Ref. [16] we assume that the operators O_5 and O_6 are dominantly two-body, which means that they carry a factor $1/N_c$. Moreover, as G^{ia} sums coherently, it introduces an extra factor N_c and makes the matrix elements of O_5 and O_6 of order N_c^0 as well (what it matters in the mass operator are the products c_5O_5 and c_6O_6 and it will turn out that their contribution is small in any case).

We have also included in the fit the following operator

$$O_7 = \frac{3}{N_c^2}S^iT^aG^{ia}, \tag{11}$$

an SU(4) invariant built from products of all generators of SU(4), S_i, T_a and G_{ia}. In the core plus excited quark procedure its counterpart was listed in Table I of the second paper of Ref. [16] as $O_{16} = gS_cT_c/N_c^2$ but completely ignored in the numerical fit, one reason being that the number of operators in the mass formula was much too large as compared to the data. The operator O_{16} is only a part of O_7, as it can be easily seen. As shown below, its matrix elements are of order $1/N_c$, like those of the pure spin O_3 or pure isospin O_4 operators. Therefore there is no a priori reason to ignore it.

Naturally, one should also include the operator

$$O_8 = \frac{1}{N_c}\ell^{(2)}S\cdot S, \tag{12}$$

also of order $1/N_c$. However, in our basis we found a proportionality relation between expectation values of two different operators

$$\langle \ell^{(2)ij}S^iS^j\rangle = 12\langle \ell^{(2)ij}G^{ia}G^{ia}\rangle, \tag{13}$$

TABLE I: Isoscalar factors of SU(4) for $[N_c-1,1]\times[211]\to[N_c-1,1]$ defined by Eq. (7).

S_1	I_1	$S_2 I_2$	SI	$\left(\begin{array}{cc\|c} [N_c-1,1] & [211] & [N_c-1,1] \\ S_1 I_1 & S_2 I_2 & SI \end{array}\right)_{\rho=1}$
$S+1$	$S+1$	11	SS	$\sqrt{\frac{S(S+2)(2S+3)(N_c-2-2S)(N_c+2+2S)}{(2S+1)(S+1)^2N_c(3N_c+4)}}$
$S+1$	S	11	SS	$-\frac{1}{S+1}\sqrt{\frac{(2S+3)(N_c+2+2S)}{(2S+1)(3N_c+4)}}$
S	$S+1$	11	SS	
S	S	11	SS	$-\frac{N_c-(N_c+2)S(S+1)}{S(S+1)\sqrt{N_c(3N_c+4)}}$
S	$S-1$	11	SS	$-\frac{1}{S}\sqrt{\frac{(2S-1)(N_c-2S)}{(2S+1)(3N_c+4)}}$
$S-1$	S	11	SS	
$S-1$	$S-1$	11	SS	$\frac{1}{S}\sqrt{\frac{(S-1)(S+1)(2S-1)(N_c+2S)(N_c-2S)}{(2S+1)N_c(3N_c+4)}}$
$S+1$	S	10	SS	0
S	$S+1$	01	SS	
$S-1$	S	10	SS	0
S	$S-1$	01	SS	
S	S	10	SS	$\sqrt{\frac{4S(S+1)}{N_c(3N_c+4)}}$
S	S	01	SS	
$S+1$	S	11	$SS-1$	$\sqrt{\frac{(2S+3)(N_c+2+2S)(N_c-2S)}{(2S+1)N_c(3N_c+4)}}$
S	S	11	$SS-1$	$\frac{1}{S}\sqrt{\frac{N_c-2S}{3N_c+4}}$
S	$S-1$	11	$SS-1$	$\frac{1}{S}\sqrt{\frac{(S-1)(S+1)N_c}{3N_c+4}}$
$S-1$	S	11	$SS-1$	$-\frac{N_c+4S^2}{S\sqrt{(2S-1)(2S+1)N_c(3N_c+4)}}$
$S-1$	$S-1$	11	$SS-1$	$\frac{1}{S}\sqrt{\frac{N_c+2S}{3N_c+4}}$
$S-1$	$S-2$	11	$SS-1$	$\sqrt{\frac{(2S-3)(N_c+2-2S)(N_c+2S)}{(2S-1)N_c(3N_c+4)}}$
S	$S-1$	10	$SS-1$	$\sqrt{\frac{4S(S+1)}{N_c(3N_c+4)}}$
$S-1$	$S-1$	10	$SS-1$	0
S	S	10	$SS-1$	0
S	$S-1$	01	$SS-1$	$\sqrt{\frac{4(S-1)S}{N_c(3N_c+4)}}$

TABLE II: List of operators and the coefficients resulting from numerical fits. The values of c_i are indicated under the headings Fit n, in each case.

Operator	Fit 1 (MeV)	Fit 2 (MeV)	Fit 3 (Mev)	Fit 4 (MeV)	Fit 5 (MeV)	Fit 6 (MeV)
$O_1 = N_c \mathbb{1}$	481 ± 5	482 ± 5	484 ± 4	484 ± 4	498 ± 3	495 ± 3
$O_2 = \ell^i s^i$	-31 ± 26	-20 ± 23	-12 ± 20	3 ± 15	38 ± 34	-30 ± 25
$O_3 = \frac{1}{N_c} S^i S^i$	161 ± 16	149 ± 11	163 ± 16	150 ± 11	156 ± 16	
$O_4 = \frac{1}{N_c} T^a T^a$	169 ± 36	170 ± 36	141 ± 27	139 ± 27		
$O_5 = \frac{15}{N_c} \ell^{(2)ij} G^{ia} G^{ja}$	-29 ± 31		-34 ± 30		-34 ± 31	-32 ± 29
$O_6 = \frac{3}{N_c} \ell^i T^a G^{ia}$	32 ± 26	35 ± 26			-67 ± 30	28 ± 20
$O_7 = \frac{3}{N_c^2} S^i T^a G^{ia}$						649 ± 61
$\chi^2_{\rm dof}$	0.43	0.68	0.94	1.04	11.5	0.24

TABLE III: Matrix elements of O_i for all states belonging to the $[\mathbf{70}, 1^-]$ multiplet.

	O_1	O_2	O_3	O_4	O_5	O_6	O_7
$^2N_{\frac{1}{2}}$	N_c	$-\frac{2N_c-3}{3N_c}$	$\frac{3}{4N_c}$	$\frac{3}{4N_c}$	0	$\frac{N_c-6}{4N_c}$	$-\frac{3(N_c-6)}{16N_c^2}$
$^4N_{\frac{1}{2}}$	N_c	$-\frac{5}{6}$	$\frac{15}{4N_c}$	$\frac{3}{4Nc}$	$\frac{25}{24}$	$-\frac{5}{8}$	$\frac{15}{16N_c}$
$^2N_{\frac{3}{2}}$	N_c	$\frac{2N_c-3}{6N_c}$	$\frac{3}{4N_c}$	$\frac{3}{4N_c}$	0	$-\frac{N_c-6}{8N_c}$	$-\frac{3(N_c-6)}{16N_c^2}$
$^4N_{\frac{3}{2}}$	N_c	$-\frac{1}{3}$	$\frac{15}{4N_c}$	$\frac{3}{4N_c}$	$-\frac{5}{6}$	$-\frac{1}{4}$	$\frac{15}{16N_c}$
$^4N_{\frac{5}{2}}$	N_c	$\frac{1}{2}$	$\frac{15}{4N_c}$	$\frac{3}{4N_c}$	$\frac{5}{24}$	$\frac{3}{8}$	$\frac{15}{16N_c}$
$^2\Delta_{\frac{1}{2}}$	N_c	$\frac{1}{3}$	$\frac{3}{4N_c}$	$\frac{15}{4N_c}$	0	$-\frac{5}{4}$	$\frac{15}{16N_c}$
$^2\Delta_{\frac{3}{2}}$	N_c	$-\frac{1}{6}$	$\frac{3}{4N_c}$	$\frac{15}{4N_c}$	0	$\frac{5}{8}$	$\frac{15}{16N_c}$
$^4N_{\frac{1}{2}} - {}^2N_{\frac{1}{2}}$	0	0	0	0	$-\frac{25}{12N_c}\sqrt{\frac{N_c(N_c+3)}{2}}$	$-\frac{1}{2N_c}\sqrt{\frac{N_c(N_c+3)}{2}}$	0
$^4N_{\frac{3}{2}} - {}^2N_{\frac{3}{2}}$	0	0	0	0	$\frac{5}{24N_c}\sqrt{5N_c(N_c+3)}$	$-\frac{1}{4N_c}\sqrt{5N_c(N_c+3)}$	0

TABLE IV: The partial contribution and the total mass (MeV) predicted by the $1/N_c$ expansion using Fit 1. The last two columns give the empirically known masses, name and status.

	Part. contrib. (MeV)						Total (MeV)	Exp. (MeV)	Name, status
	c_1O_1	c_2O_2	c_3O_3	c_4O_4	c_5O_5	c_6O_6			
$^2N_{\frac{1}{2}}$	1444	10	40	42	0	-8	1529 ± 11	1538 ± 18	$S_{11}(1535)$****
$^4N_{\frac{1}{2}}$	1444	26	201	42	-31	-20	1663 ± 20	1660 ± 20	$S_{11}(1650)$****
$^2N_{\frac{3}{2}}$	1444	-5	40	42	0	4	1525 ± 8	1523 ± 8	$D_{13}(1520)$****
$^4N_{\frac{3}{2}}$	1444	10	201	42	25	-8	1714 ± 45	1700 ± 50	$D_{13}(1700)$***
$^4N_{\frac{5}{2}}$	1444	-16	201	42	-6	12	1677 ± 8	1678 ± 8	$D_{15}(1675)$****
$^2\Delta_{\frac{1}{2}}$	1444	-10	40	211	0	-40	1645 ± 30	1645 ± 30	$S_{31}(1620)$****
$^2\Delta_{\frac{3}{2}}$	1444	5	40	211	0	20	1720 ± 50	1720 ± 50	$D_{33}(1700)$****

for all states belonging to the $[\mathbf{70}, 1^-]$ multiplet. This implies that we cannot include O_8 independently in the fit to the experimental spectrum, because its expectation values are proportional to those of O_5.

The operators O_5, O_6 and O_7 are normalized to allow their coefficients c_i to have a natural size [20, 31]. The normalization factors follow from the matrix elements of O_i presented in Table III. These matrix elements have been calculated for all available states of the multiplet $[\mathbf{70}, 1^-]$ starting from the wave function (4) and using the isoscalar factors of Table I. The general analytic expressions of O_5, O_6 and O_7, up to an obvious factor, are given in Appendix D. For completeness, in Table III, we also indicate the off-diagonal matrix elements of O_5 and O_6.

V. RESULTS

We have implemented the matrix elements of Table III into the mass formula (1) and have performed several distinct fits of the theoretical masses to the experiment [32]. Each of the six fits corresponds to a selection of operators O_i used in Eq. (1), such as to cover the most relevant possibilities, in our view.

In this way we have obtained sets of values for the dynamical coefficients c_i presented in Table II. In Tables IV and V we present the masses of the nonstrange resonances belonging to the $[\mathbf{70}, 1^-]$ multiplet obtained from the coefficients resulting from the Fit 1 and the Fit 6, which correspond to the lowest values of $\chi^2_{\rm dof}$.

In Tables IV and V we have also indicated the partial contribution (without error bars) of each term present in the total mass. These are obtained from the values of c_i of Table II and the values of $\langle O_i \rangle$ of Table III. The Fit 1, containing all operators but O_7, is indeed excellent, giving $\chi^2_{\rm dof} \simeq 0.43$. From Table II one can see that the values of the coefficients c_3 and c_4 are closed to each other, which shows the importance of including O_4, besides the usual O_3. In addition, one can see that O_3 is dominant for the ^4N_J resonances while O_4 is dominant for the $^2\Delta_J$ resonances, the contribution being of about 200 MeV in both cases. This brings a new aspect into the description of excited states studied so far, where the dominant term was always the spin-spin term [23], the isospin term being absent in the numerical analysis. To get a better idea about the role of the operator O_4 we have also made a fit by removing it from the definition of the mass operator (1). The result is shown in Table II column Fit 5. The $\chi^2_{\rm dof}$ deteriorates considerably, becoming 11.5 instead of 0.43. This clearly shows that O_4 is crucial in the fit.

The coefficient c_2 of the spin-orbit term is small and its magnitude and sign remains comparable to that of Ref. [24] obtained in the analysis of the $[\mathbf{70}, \ell^+]$ multiplet. The value of c_2 implies a small spin-orbit contribution to the total mass, in agreement with the general pattern observed for the excited states [23] and in agreement with constituent quark models.

The error bars of c_5 and c_6 are comparable to their central values. However, the removal of O_5 and/or O_6 from the mass operator does not deteriorate the fit too badly, as shown in Table II, Fits 2–4, the $\chi^2_{\rm dof}$ becoming at most 1.04. The contribution of O_5 or of O_6 is comparable to that of the spin-orbit operator. Note that the structure of O_6 is related to that of the spin-orbit term, which makes its small contribution entirely plausible. Thus the contribution of all operators containing angular momentum is small, which may be a dynamic effect.

Table V shows explicitly the role of the operator O_7, never included so far in numerical fits. One can see that this operator plays a dominant role in ^4N_J and $^2\Delta_J$, where it contributes with about 200 MeV to the mass, value comparable to that of O_3 or O_4 in the Fit 1. Including O_3, O_4 and O_7 together, their contributions remains equally large but c_7 changes sign and $\chi^2_{\rm dof}$ increases to from 0.24 to about 2. This suggests that O_7 somehow compensates for the pure spin $S \cdot S$ or pure isospin $T \cdot T$ operators, or in other words, plays a kind of common role with O_3 and O_4. We consider that more theoretical work is needed to better understand the algebraic relations between various O_i operators, in particular to find new operator identities for mixed symmetric states.

VI. VALIDITY OF THE APPROXIMATE WAVE FUNCTION

In Sec. II it was mentioned that all previous studies of the $[\mathbf{70}, 1^-]$ multiplet were performed with the asymmetric wave function (3.4) of the second paper of Ref. [16]. Here we discuss the validity of this approximation by comparing matrix elements of the same operators calculated both with the exact (symmetric) and the approximate (asymmetric) wave function.

First we consider the operator O_2, common to previous and present calculations. It is a one-body operator, defined by Eq. (10). Its matrix elements can be written as

$$\langle \ell \cdot s \rangle = N_c \langle \ell(N_c) \cdot s(N_c) \rangle, \tag{14}$$

because the orbital-spin-flavor wave function is symmetric. Thus it is enough to know the matrix element of a single quark operator, say N_c. Let us illustrate the case $N_c = 5$, for which the components of the orbital wave function are given in Table VII. One can see that only the first basis vector, associated to the normal Young tableau, gives a nonvanishing contribution to $\langle \ell(N_c) \cdot s(N_c) \rangle$ and this comes only from the term $ssssp$, because it is the only one where the particle 5 is in a p state. The generalization of this argument to an arbitrary N_c is obvious and equally good. Thus in the case of a single excited quark it is equally well to calculate $\langle \ell \cdot s \rangle$ with the exact or with the approximate wave function of

TABLE V: The partial contribution and the total mass (MeV) predicted by the $1/N_c$ expansion using Fit 6. The last two columns give the empirically known masses, name and status.

	Part. contrib. (MeV)					Total (MeV)	Exp. (MeV)	Name, status
	c_1O_1	c_2O_2	c_5O_5	c_6O_6	c_7O_7			
$^2N_{\frac{1}{2}}$	1486	10	0	-7	41	1529 ± 11	1538 ± 18	$S_{11}(1535)$****
$^4N_{\frac{1}{2}}$	1486	25	-33	-18	203	1663 ± 20	1660 ± 20	$S_{11}(1650)$****
$^2N_{\frac{3}{2}}$	1486	-5	0	4	41	1525 ± 7	1523 ± 8	$D_{13}(1520)$****
$^4N_{\frac{3}{2}}$	1486	10	26	-7	203	1718 ± 41	1700 ± 50	$D_{13}(1700)$***
$^4N_{\frac{5}{2}}$	1486	-15	7	11	203	1677 ± 8	1678 ± 8	$D_{15}(1675)$****
$^2\Delta_{\frac{1}{2}}$	1486	-10	0	-35	203	1643 ± 29	1645 ± 30	$S_{31}(1620)$****
$^2\Delta_{\frac{3}{2}}$	1486	5	0	18	203	1711 ± 24	1720 ± 50	$D_{33}(1700)$****

TABLE VI: Matrix elements of operators from the decoupling scheme [16] corresponding to the $[\mathbf{70}, 1^-]$ multiplet. The columns *asym* reproduce results obtained with the asymmetric wave function of Ref. [16] and the columns *sym* show results obtained with the exact wave function (3), detailed in Appendix A.

	$\langle s \cdot S_c \rangle$		$\langle S_c^2 \rangle$		$\langle t \cdot T_c \rangle$		$\langle T_c^2 \rangle$	
	asym	sym	asym	sym	asym	sym	asym	sym
^{28}	$-\frac{1}{2}$	$-\frac{1}{2}$	1	1	$-\frac{1}{2}$	$-\frac{1}{2}$	1	1
^{48}	$\frac{1}{2}$	$\frac{1}{2}$	2	2	-1	$-\frac{1}{2}$	2	1
^{210}	-1	$-\frac{1}{2}$	2	1	$\frac{1}{2}$	$\frac{1}{2}$	2	2

Ref. [16], because the missing terms in the latter function do not contribute. Note however that the spin-orbit operator has a negligible contribution to the mass in all previous and present calculations and in practice it can be neglected.

Next we consider a few of the two-body operators of Ref. [16] $s \cdot S_c$, S_c^2, $t \cdot T_c$ and T_c^2 and restrict the discussion to the case of physical interest, $N_c = 3$, which is enough for our purpose. Appendix A gives the approximate wave functions [16] and the exact wave functions for the submultiplets ^{28}, ^{48} and ^{210}. The calculated matrix elements are shown in Table VI. One can see that for every operator there is a case where the approximation fails. This failure is related to those missing parts of the wave function, where the core has $I_c \neq S_c$. Moreover, the approximate matrix elements of the operators $s \cdot S_c$ and S_c^2 turn out to be isospin dependent and the approximate matrix elements of the operators $t \cdot T_c$ and T_c^2 are spin dependent. Using the exact wave function, this anomaly disappears.

For an arbitrary N_c we expect that the exact wave function would generally give a dependence on N_c for the matrix elements of operators from previous works, entirely different from that of Table II and III of Ref. [16]. As a by-product, one can also see that the operator T_c^2, always ignored previously, has matrix elements comparable to those of S_c^2. This is consistent with our result that the isospin-isospin becomes as dominant in Δ as the spin-spin in N resonances.

VII. CONCLUSIONS

In this work we have studied the $[\mathbf{70}, 1^-]$ multiplet in the $1/N_c$ expansion by using a simple approach which avoids the separation of the system into a core and an excited quark. This allows us to use an exact wave function of a system of N_c quarks where both the orbital and spin-flavor parts are in the mixed representation $[N_c - 1, 1]$. Previously the wave function was truncated to a single term which implied that the symmetry S_{N_c} was broken. That asymmetric wave function was associated with describing the system as being decoupled into an excited quark and a core with $N_c - 1$ quarks in the ground state. Such a description involves an excessively large number of linearly independent operators in the mass formula and the only possible way to make a fit was to arbitrarily select some of them. Not surprisingly, the asymmetric wave function fails to reproduce the exact values of the matrix elements of some dominant operators in the decoupling scheme itself.

The present approach, based on a wave function with the correct permutation symmetry, sheds an entirely new light on the description of the baryon multiplet $[\mathbf{70}, 1^-]$ in the $1/N_c$ expansion. We have shown that the isospin operator O_4 is crucial in the fit to the existing data and its contribution to the mass of the Δ resonances is as important as that of the spin operator O_3 for N resonances. Also we found that the operator O_7, never included previous fits and containing products of all generators of SU(4), Eqs. (5), plays by itself a dominant role in ^4N and $^2\Delta$, states where the spin and isospin are different. By contrast, all operators containing the O(3) generators ℓ_i bring negligible contributions to the mass.

A comment is in order regarding Refs. [18, 19] where a submultiplet structure (distinct towers of states) has been found, in the procedure of decoupling the system into a core plus an excited quark. The present analysis would give similar results. The reason is that the existence of three towers of states in the $[\mathbf{70}, 1^-]$ multiplet is due to the the presence of three operators when working up to order N_c^0: $\mathbb{1}$ (of order N_c) and $\ell \cdot s$ and $\ell^{(2)}GG/N_c$ (of order N_c^0). The meson-baryon scattering analysis of Ref. [19] proves the compatibility between the three towers and three resonance poles in the scattering amplitude with quantum numbers corresponding to the states in the $[\mathbf{70}, 1^-]$ multiplet.

It would be interesting to reconsider the study of higher excited baryons, for example those belonging to $[\mathbf{70}, \ell^+]$ multiplets, in the spirit of the present approach.

In practical terms, the extension to three flavors would involve a considerable amount of work on isoscalar factors of SU(6) generators for mixed symmetric representations.

APPENDIX A

We consider the particular case of $N_c = 3$ to prove that the wave function given by Eq. (3.4) of the first paper of Ref. [16] breaks S_3 symmetry.

The basis vectors which span the invariant subspace of the mixed symmetric representation correspond to the following Young tableaux

$$X^\lambda \rightarrow \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array}\,, X^\rho \rightarrow \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}\,, \qquad \text{(A1)}$$

where $X = R, S, F$ and FS are the orbital, spin, flavor and flavor-spin wave functions respectively.

In the spin space one can construct $|S^\lambda\rangle$ and $|S^\rho\rangle$ by first coupling the spin of quarks 1 and 2 to S_c followed by the coupling of S_c to the spin of the third quark. We explicitly have

$$|S^\lambda\rangle = \sum_{m_1,m_2} \left(\begin{array}{cc|c} 1 & 1/2 & 1/2 \\ m_1 & m_2 & S_3 \end{array} \right) |S^c = 1; m_1\rangle \left| \frac{1}{2}; m_2 \right\rangle, \qquad \text{(A2)}$$

and

$$|S^\rho\rangle = |S^c = 0; m_1 = 0\rangle \left| \frac{1}{2}; m_2 = S_3 \right\rangle, \qquad \text{(A3)}$$

and equivalently in the isospin space

$$|F^\lambda\rangle = \sum_{\alpha_1,\alpha_2} \left(\begin{array}{cc|c} 1 & 1/2 & 1/2 \\ \alpha_1 & \alpha_2 & I_3 \end{array} \right) |I^c = 1; \alpha_1\rangle \left| \frac{1}{2}; \alpha_2 \right\rangle, \qquad \text{(A4)}$$

and

$$|F^\rho\rangle = |I^c = 0; \alpha_1 = 0\rangle \left|\frac{1}{2}; \alpha_2 = I_3\right\rangle . \qquad \text{(A5)}$$

In the standard notation the states $(FS)^\lambda$ and $(FS)^\rho$ can be written as (see *e.g.* [29])

$$(FS)^\lambda_{I=3/2;S=1/2} = F^{\rm S}S^\lambda, \qquad \text{(A6)}$$
$$(FS)^\rho_{I=3/2;S=1/2} = F^{\rm S}S^\rho, \qquad \text{(A7)}$$
$$(FS)^\lambda_{I=1/2;S=3/2} = F^\lambda S^{\rm S}, \qquad \text{(A8)}$$
$$(FS)^\rho_{I=1/2;S=3/2} = F^\rho S^{\rm S}, \qquad \text{(A9)}$$
$$(FS)^\lambda_{I=1/2;S=1/2} = \sqrt{\frac{1}{2}}\left(F^\lambda S^\lambda - F^\rho S^\rho\right), \quad \text{(A10)}$$
$$(FS)^\rho_{I=1/2;S=1/2} = -\sqrt{\frac{1}{2}}\left(F^\lambda S^\rho + F^\rho S^\lambda\right). \text{(A11)}$$

where F^S and S^S denote symmetric states in isospin and spin respectively. In this notation the orbital-flavor-spin wave function of a baryon, which must be symmetric under S_3, is a particular case of Eq. (3) and can be written as

$$|[3]\rangle = \frac{1}{\sqrt{2}}[R^\lambda(FS)^\lambda + R^\rho(FS)^\rho]. \qquad \text{(A12)}$$

We wish to rewrite the flavor-spin part of the wave function (3.4) of the first paper of Ref. [16], denoted by $|II_3; SS_3\rangle$ in the above notation.

Let us first consider the case $I = 3/2, S = 1/2$. One has

$$|3/2\ I_3; 1/2\ S_3\rangle = \sum_{m_1,m_2,\alpha_1,\alpha_2} \left(\begin{array}{cc|c} S_c & 1/2 & 1/2 \\ m_1 & m_2 & S_3 \end{array}\right) \left(\begin{array}{cc|c} I_c & 1/2 & 3/2 \\ \alpha_1 & \alpha_2 & I_3 \end{array}\right) c^{\rm MS}_{--}\,|S^c = I^c = 1; m_1\alpha_1\rangle\,|1/2, m_2; 1/2, \alpha_2\rangle, \quad \text{(A13)}$$

where $c^{\rm MS}_{--} = 1$. The spin-flavor states are factorisable into spin and isospin, so that due to (A2) this state is identical to (A6). For the case $I = 1/2, S = 3/2$, one has

$$|1/2\ I_3, 3/2\ S_3\rangle = \sum_{m_1,m_2,\alpha_1,\alpha_2} \left(\begin{array}{cc|c} S_c & 1/2 & 3/2 \\ m_1 & m_2 & S_3 \end{array}\right) \left(\begin{array}{cc|c} I_c & 1/2 & 1/2 \\ \alpha_1 & \alpha_2 & I_3 \end{array}\right) c^{\rm MS}_{++}\,|S^c = I^c = 1; m_1\alpha_1\rangle\,|1/2, m_2; 1/2; \alpha_2\rangle, \quad \text{(A14)}$$

where $c^{\rm MS}_{++} = 1$. Due to (A4) this state is identical to (A8).

Next we consider the case $I = 1/2, S = 1/2$,

$$\begin{aligned} |1/2\ I_3, 1/2\ S_3\rangle &= \sum_{m_1,\alpha_1,\eta} \left(\begin{array}{cc|c} S_c & 1/2 & 1/2 \\ m_1 & m_2 & S_3 \end{array}\right) \left(\begin{array}{cc|c} I_c & 1/2 & 1/2 \\ \alpha_1 & \alpha_2 & I_3 \end{array}\right) c^{\rm MS}_{0\eta} \left|S^c = I^c = \frac{1}{2} + \frac{\eta}{2}; m_1\alpha_1\right\rangle |1/2, m_2; 1/2, \alpha_2\rangle \\ &= \sqrt{\frac{1}{2}}\Bigg\{\sum_{m_1,\alpha_1} \left(\begin{array}{cc|c} 1 & 1/2 & 1/2 \\ m_1 & m_2 & S_3 \end{array}\right) \left(\begin{array}{cc|c} 1 & 1/2 & 1/2 \\ \alpha_1 & \alpha_2 & I_3 \end{array}\right) |S^c = I^c = 1; m_1\alpha_1\rangle\,|1/2, m_2; 1/2, \alpha_2\rangle \\ &\quad - |S^c = I^c = 0; m_1 = \alpha_1 = 0\rangle\,|1/2; m_2 = \alpha_2 = 1/2\rangle\Bigg\}, \end{aligned} \qquad \text{(A15)}$$

where we have introduced $c^{\rm MS}_{0+} = \sqrt{\frac{1}{2}}$ and $c^{\rm MS}_{0-} = -\sqrt{\frac{1}{2}}$ after the second equality sign. Due to (A2)-(A5) this state is identical to (A10). This proves that in (A13), (A14) and (A15) the second term of Eq. (A12) is missing. Thus the wave function of Ref. [16]. is truncated. It contains only one term instead of two as required by the S_3 symmetry. In Sec. VI we show that the missing terms (A7), (A9) and (A11) have a considerable contribution to the matrix elements of some operators used in the $1/N_c$ expansion mass formula.

APPENDIX B

As an example, in this Appendix we present the orbital basis vectors which span the invariant subspace of the representation [41] of S_5.

An exact orbital-spin-flavor wave function of five fermions (for which the color part is totally antisymmetric) having the configuration s^4p, *i.e.* a single quark excited to the p shell, has to be built from 4 independent basis vectors, each having a distinct Young tableau, both in the orbital and spin-flavor spaces. The basis vectors in the orbital space are shown in Table VII [33]. Note that every term in each state implies the normal order of particles: $1, 2, 3, 4, 5$. One can see that the first basis vector, with the 5-th particle in the second row contains the configuration *ssssp*, *i.e.* it is the only part of all these basis vectors which has the first four quarks in the ground state and the 5-th in a p state. One can see that in fact any quark can be excited to the p shell in a properly symmetrized state. Thus the wave function used in previous literature should contain only this *ssssp* term [13, 14, 15, 16, 18, 20] if the core was unexcited. The truncation of the spin-flavor part was discussed in Section II.

APPENDIX C

The grouping in Table I is justified by the observation that the isoscalar factors obey the following orthogonality relation

$$\sum_{S_1I_1S_2I_2} \left(\begin{array}{cc|c} [N_c-1,1] & [211] & [N_c-1,1] \\ S_1I_1 & S_2I_2 & SI \end{array} \right)_\rho \left(\begin{array}{cc|c} [N_c-1,1] & [211] & [N_c-1,1] \\ S_1I_1 & S_2I_2 & S'I' \end{array} \right)_\rho = \delta_{SS'}\delta_{II'}, \tag{C1}$$

which can be easily checked. For example, by taking $S = S'$ and $I = I'$ one can find that the squares of the first 13 coefficients sum up to one.

For completeness also note that the isoscalar factors obey the following symmetry property

$$\left(\begin{array}{cc|c} [N_c-1,1] & [211] & [N_c-1,1] \\ I_1S_1 & I_2S_2 & (S-1)S \end{array} \right)_\rho = \left(\begin{array}{cc|c} [N_c-1,1] & [211] & [N_c-1,1] \\ S_1I_1 & S_2I_2 & S(S-1) \end{array} \right)_\rho . \tag{C2}$$

APPENDIX D

Here we present the analytic form of the matrix elements of operators proportional to O_5 and O_6. They have been obtained following the approach described in Sec. III. In that notation we have

$$\begin{aligned} &\langle \ell' S' J' J_3'; I' I_3' | \ell^{(2)ij} G^{ia} G^{ja} | \ell S J J_3; I I_3 \rangle = \delta_{J'J}\delta_{J_3J_3'}\delta_{\ell'\ell}\delta_{I'I}\delta_{I_3'I_3} \\ &\times (-1)^{J+\ell-S} \frac{N_c(3N_c+4)}{16} \sqrt{2S'+1} \sqrt{\frac{5\ell(\ell+1)(2\ell-1)(2\ell+1)(2\ell+3)}{6}} \left\{ \begin{array}{ccc} \ell & \ell & 2 \\ S & S' & J \end{array} \right\} \sum_{S''I''} \left\{ \begin{array}{ccc} 1 & 1 & 2 \\ S & S' & S'' \end{array} \right\} \\ &\sqrt{\frac{(2S''+1)(2I''+1)}{2I+1}} \times \left(\begin{array}{cc|c} [N_c-1,1] & [21^2] & [N_c-1,1] \\ SI & 11 & S''I'' \end{array} \right)_1 \left(\begin{array}{cc|c} [N_c-1,1] & [21^2] & [N_c-1,1] \\ S''I'' & 11 & S'I \end{array} \right)_1 , \end{aligned} \tag{D1}$$

TABLE VII: Young tableaux and the corresponding basis vectors of the irrep [41] of S_5 for the configuration s^4p [29].

Young tableau	Young-Yamanouchi basis vectors of [41]
1 2 3 4 5	$\frac{1}{\sqrt{20}}(4ssssp - sssps - sspss - spsss - pssss)$
1 2 3 5 4	$\frac{1}{\sqrt{12}}(3sssps - sspss - spsss - pssss)$
1 2 4 5 3	$\frac{1}{\sqrt{6}}(2sspss - spsss - pssss)$
1 3 4 5 2	$\frac{1}{\sqrt{2}}(spsss - pssss)$

$$\langle \ell' S' J' J_3'; I' I_3' | \ell^i T^a G^{ia} | \ell S J J_3; I I_3 \rangle = \delta_{J'J}\delta_{J_3 J_3'}\delta_{\ell'\ell}\delta_{I'I}\delta_{I_3' I_3}(-1)^{J+\ell+S'}\frac{N_c(3N_c+4)}{8}\sqrt{2S'+1}$$
$$\times\sqrt{\ell(\ell+1)(2\ell+1)}\begin{Bmatrix} \ell & \ell & 1 \\ S' & S & J \end{Bmatrix}\left(\begin{array}{cc|c} [N_c-1,1] & [21^2] & [N_c-1,1] \\ SI & 11 & S'I \end{array}\right)_1 \left(\begin{array}{cc|c} [N_c-1,1] & [21^2] & [N_c-1,1] \\ S'I & 01 & S'I \end{array}\right)_1, \quad \text{(D2)}$$

and

$$\langle \ell' S' J' J_3'; I' I_3' | S^i T^a G^{ia} | \ell S J J_3; I I_3 \rangle = \delta_{J'J}\delta_{J_3' J_3}\delta_{\ell'\ell}\delta_{S'S}\delta_{S_3' S_3}\delta_{I'I}\delta_{I_3' I_3}$$
$$\times\frac{1}{4}\sqrt{N_c(3N_c+4)}\sqrt{I(I+1)}\sqrt{S(S+1)}\left(\begin{array}{cc|c} [N_c-1,1] & [21^2] & [N_c-1,1] \\ SI & 11 & S'I' \end{array}\right)_1. \quad \text{(D3)}$$

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[1] G. 't Hooft, Nucl. Phys. **72** (1974) 461.
[2] E. Witten, Nucl. Phys. **B160** (1979) 57.
[3] Proceedings of the Workshop *Large N_c QCD 2004*, Trento, Italy, July 5-11 2004, eds. J. L. Goity, R. F. Lebed, A. Pich, C. L. Schat and N. N. Scoccola (World Scientific, Singapore, 2005).
[4] J. L. Gervais and B. Sakita, Phys. Rev. Lett. **52** (1984) 87; Phys. Rev. D **30** (1984) 1795.
[5] R. Dashen and A. V. Manohar, Phys. Lett. **B315** (1993) 425; ibid **B315** (1993) 438.
[6] R. Dashen, E. Jenkins, and A. V. Manohar, Phys. Rev. **D49** (1994) 4713.
[7] R. Dashen, E. Jenkins, and A. V. Manohar, Phys. Rev. **D51** (1995) 3697.
[8] C. D. Carone, H. Georgi and S. Osofsky, Phys. Lett. **B322** (1994) 227.
M. A. Luty and J. March-Russell, Nucl. Phys. **B426** (1994) 71.
M. A. Luty, J. March-Russell and M. White, Phys. Rev. **D51** (1995) 2332.
[9] E. Jenkins, Phys. Lett. **B315** (1993) 441.
[10] E. Jenkins and R. F. Lebed, Phys. Rev. **D52** (1995) 282.
[11] J. Dai, R. Dashen, E. Jenkins, and A. V. Manohar, Phys. Rev. **D53** (1996) 273.
[12] C. D. Carone, H. Georgi, L. Kaplan and D. Morin, Phys. Rev. **D50** (1994) 5793.

[13] J. L. Goity, Phys. Lett. **B414** (1997) 140.
[14] D. Pirjol and T. M. Yan, Phys. Rev. **D57** (1998) 1449.
[15] D. Pirjol and T. M. Yan, Phys. Rev. **D57** (1998) 5434.
[16] C. E. Carlson, C. D. Carone, J. L. Goity and R. F. Lebed, Phys. Lett. **B438** (1998) 327; Phys. Rev. **D59** (1999) 114008.
[17] C. E. Carlson and C. D. Carone, Phys. Lett. **B441** (1998) 363; Phys. Rev. **D58** (1998) 053005.
[18] D. Pirjol and C. Schat, Phys. Rev. **D67** (2003) 096009.
[19] T. D. Cohen and R. F. Lebed, Phys. Rev. Lett. **91**, 012001 (2003); Phys. Rev. **D67**, 096008 (2003);
[20] C. L. Schat, J. L. Goity and N. N. Scoccola, Phys. Rev. Lett. **88** (2002) 102002; J. L. Goity, C. L. Schat and N. N. Scoccola, Phys. Rev. **D66** (2002) 114014.
[21] C. E. Carlson and C. D. Carone, Phys. Lett. **B484** (2000) 260.
[22] J. L. Goity, C. L. Schat and N. N. Scoccola, Phys. Lett. **B564** (2003) 83.
[23] N. Matagne and Fl. Stancu, Phys. Lett **B631** (2005) 7.
[24] N. Matagne and F. Stancu, Phys. Rev. **D74** (2006) 034014.
[25] N. Matagne and Fl. Stancu, Phys. Rev. **D71** (2005) 014010.
[26] P. Stassart and Fl. Stancu, Z. Phys. **A359** (1997) 321.
[27] K. T. Hecht and S. C. Pang, J. Math. Phys. **10** (1969) 1571.
[28] N. Matagne and F. Stancu, Phys. Rev. **D73** (2006) 114025.
[29] Fl. Stancu, *Group Theory in Subnuclear Physics*, Clarendon Press, Oxford, (1996) Ch. 4.
[30] Fl. Stancu and S. Pepin, Few-Body Systems **26** (2004) 113.
[31] N. N. Scoccola, J. L. Goity and N. Matagne, Phys. Lett. **B663** (2008) 222.
[32] W.-M. Yao *et al.* [Particle Data Group], J. Phys. **G33** (2006) 1.
[33] N. Matagne, *Baryon resonances in large N_c QCD*, PhD Thesis, University of Liège (2006) [arXiv:hep-ph/0701061].
[34] see Ref. [29], Sec 4.5.