

The wave function of $[70, 1^-]$ baryons in the $1/N_c$ expansion

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Abstract

Much work has been devoted to the study of nonstrange baryons belonging to the $[70, 1^-]$ multiplet in the framework of the $1/N_c$ expansion. Using group theoretical arguments here we examine the relation between the exact wave function and the approximate one, customarily used in applications where the system is separated into a ground state core and an excited quark. We show that the exact and approximate wave functions globally give similar results for all of mass operators presented in this work. However we find that the inclusion of operators acting separately on the core and on the excited quark deteriorates the fit and leads to unsatisfactory values for the coefficients which encode the quark dynamics. Much better results are obtained when we include operators acting on the whole system, both for the exact and the approximate wave function.

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I. INTRODUCTION

The $1/N_c$ expansion of QCD [1, 2, 3] is an interesting and systematic approach to study baryon spectroscopy. It has been applied to the ground state baryons [4, 5, 6, 7, 8, 9, 10] as well as to excited states, in particular to the negative parity $[70, 1^-]$ multiplet ($N = 1$ band) [11, 12, 13, 14, 15, 16], to the positive parity Roper resonance $[56', 0^+]$ ($N = 2$ band) [17], to the $[56, 2^+]$ [18] and the $[70, \ell^+]$ multiplets ($\ell = 0$ and 2) [19, 20], both belonging to the $N = 2$ band and to the $[56, 4^+]$ multiplet ($N = 4$ band) [21]. Estimates for the lowest multiplet $[70, 3^-]$ of the $N = 3$ band have also been made [22]. In this approach the main features of the constituent quark model emerge naturally [23, 24] as for example the dominant role of the spin-spin term and the smallness of the spin-orbit term.

The study of excited states belonging to the symmetric representation $[56]$ is similar to that of the ground state. The introduction of an orbital part in the wave function does not affect the procedure, inasmuch as the flavor-spin part remains symmetric. The study of excited states which are mixed symmetric, both in the orbital and flavor-spin space is more complicated and it became controversial. Presently we discuss the $[70, 1^-]$ multiplet, which is the simplest one, having 7 experimentally known resonances [25] in the nonstrange sector. The standard procedure [13] is to separate the system into of a ground state core of $N_c - 1$ quarks and an orbitally excited ($\ell = 1$) quark. Then, in the spirit of a Hartree picture the system is described by an approximate wave function, where the orbital part has a configuration of type $s^{N_c-1}p$ (no antisymmetrization) which is combined with an approximate spin-flavor part. There is also a more straightforward procedure where the system is treated as a whole [26]. This procedure is in the spirit of the *spectrum generating algebra* method, introduced by Gell-Mann and Ne'eman. It has been applied to the ground state, where use of the generators describing the entire system is made. In this procedure the exact wave function is needed.

Here we reanalyze the $[70, 1^-]$ multiplet ($N = 1$ band) without any prejudice. We start with the standard procedure based on the core+excited quark separation. We show the relation between the approximate wave function and the exact one, constructed explicitly in the next section. To test the validity of the approximate wave function [13] we first compare the analytical expressions of the matrix elements of various operators entering the mass formula. Next we perform a numerical fit to the data. For practical reasons we choose a

simple mass operator containing the most dominant terms in order to fit the 7 experimentally known masses and ultimately the two mixing angles also.

The next section is devoted to the analysis of the wave function and its construction by using isoscalar factors. In Sec. III we briefly introduce the mass operator. In Sec. IV we perform fits to the data, by including various sets of invariant operators constructed from the generators of SU(4) and O(3). Sec. V is devoted to mixing angles. The last section contains our conclusions. Appendix A describes the procedure used to construct the needed isoscalar factors of the permutation group. Appendix B introduces the fractional parentage coefficients needed to separate the orbital wave function into a part describing the core formed of $N_c - 1$ quarks and a single quark. Appendix C provides analytic expressions of the matrix elements necessary to calculate the isoscalar factors.

II. THE EXACT WAVE FUNCTION

The system under concern has four degrees of freedom: orbital (O), flavor (F), spin (S) and color (C). It is then useful to construct its wave function with the help of inner products of S_{N_c} in order to fulfill the Fermi statistics. The color part being antisymmetric, the orbital-spin-flavor part must be symmetric. As in the mass formula there are no color operators, the color part being integrated out, we are concerned with the orbital-spin-flavor part only.

A convenient way to construct a wave function of given symmetry $[f]$ is to use the inner product $[f'] \times [f'']$ of two irreducible representations (irreps) of S_{N_c} , generally reducible into a Clebsch-Gordan series with a given number of irreps $[f]$. A basis vector of $[f]$ is denoted by $|[f]Y\rangle$, where Y is the corresponding Young tableau (or Yamanouchi symbol). Its general form is [27]

$$|[f]Y\rangle = \sum_{Y', Y''} S([f']Y'[f'']Y''|[f]Y)|[f']Y'\rangle|[f'']Y''\rangle, \quad (1)$$

where $S([f']Y'[f'']Y''|[f]Y)$ are Clebsch-Gordan (CG) coefficients of S_{N_c} .

Here we deal with a system of N_c quarks having one unit of orbital excitation. Hence the orbital part must have a mixed symmetry $[N_c - 1, 1]$. To get a totally symmetric state $[[N_c]1]$, where for brevity the Yamanouchi symbol 1...1 (N_c times) is denoted by 1, the FS part must belong to the irrep $[N_c - 1, 1]$ as well. Then the wave function takes the form

$$|[N_c]1\rangle = \left(\frac{1}{N_c - 1}\right)^{1/2} \sum_Y |[N_c - 1, 1]Y\rangle_O |[N_c - 1, 1]Y\rangle_{FS}, \quad (2)$$

where the coefficient in front is the CG coefficient needed to construct a symmetric state of N_c quarks. The sum is performed over all possible $N_c - 1$ standard Young tableaux. In this sum there is only one Y (the normal Young tableau) where the last particle is in the second row and $N_c - 2$ terms with the N_c -th particle in the first row. If we specify the row p of the N_c -th quark and the row q of the $(N_c - 1)$ -th quark, and denote by y the distribution of the $N_c - 2$ remaining quarks, we can write Y more explicitly as

$$Y = (pqy). \quad (3)$$

For illustration let us consider the case $N_c = 5$. Expressed in terms of Young tableaux, the wave function (2) reads

$$\begin{aligned} \boxed{1\ 2\ 3\ 4\ 5} = & \\ \frac{1}{\sqrt{4}} \left[& \left(\begin{array}{c} \boxed{1\ 2\ 3\ 4} \\ \boxed{5} \end{array} \right)_O \left(\begin{array}{c} \boxed{1\ 2\ 3\ 4} \\ \boxed{5} \end{array} \right)_{FS} + \left(\begin{array}{c} \boxed{1\ 2\ 3\ 5} \\ \boxed{4} \end{array} \right)_O \left(\begin{array}{c} \boxed{1\ 2\ 3\ 5} \\ \boxed{4} \end{array} \right)_{FS} \\ & + \left(\begin{array}{c} \boxed{1\ 2\ 4\ 5} \\ \boxed{3} \end{array} \right)_O \left(\begin{array}{c} \boxed{1\ 2\ 4\ 5} \\ \boxed{3} \end{array} \right)_{FS} + \left(\begin{array}{c} \boxed{1\ 3\ 4\ 5} \\ \boxed{2} \end{array} \right)_O \left(\begin{array}{c} \boxed{1\ 3\ 4\ 5} \\ \boxed{2} \end{array} \right)_{FS} \right]. \quad (4) \end{aligned}$$

In this notation, in the left hand side one has $p = q = 1$ because the N_c -th particle, *i.e.* particle 5, and the $(N_c - 1)$ -th, *i.e.* particle 4, are both in the first row. The first term in the right hand side of (4) has $p = 2$ and $q = 1$ both for (O) and (FS) parts. This is the only term considered in Ref. [13]. The second term has $p = 1$ and $q = 2$ and the last two terms have $p = q = 1$. In the following we shall mostly concentrate on the $p = 1$ terms in the right hand side, neglected in Ref. [13] and compare results derived from the exact wave function (2) with those obtained in Ref. [13]. To include the neglected terms we have to explicitly consider the (FS) part of the wave function, as follows.

At its turn, the (FS) part can be decomposed into a sum of products of spin (S) and flavor (F) parts using Eq. (1) where in the left hand side we now have $|[f]Y\rangle = |[N_c - 1, 1]Y\rangle_{FS}$ and in the right hand side the appropriate CG coefficients. For these coefficients it is useful to use a factorization property (Racah's *factorization lemma*) which helps to decouple the N_c -th particle from the rest. Accordingly, every CG coefficient of S_{N_c} can be factorized into an isoscalar factor times a CG coefficient of $S_{N_c - 1}$, and so on. We apply this property to $S_{N_c} \supset S_{N_c - 1}$ only. To do so, it is necessary to specify the row p of the N_c -th and the row q of the $(N_c - 1)$ -th quarks, as above. If the isoscalar factor is denoted by $K([f']p'[f'']p''|[f]p)$ the factorization property reads

$$S([f']p'q'y'[f'']p''q''y''|[f]pqy) = K([f']p'[f'']p''|[f]p)S([f'_p]q'_y'[f''_p]q''y''|[f_p]qy), \quad (5)$$

where the second factor in the right hand side is a CG coefficient of S_{N_c-1} containing the partitions $[f'_{p'}]$, $[f''_{p''}]$ and $[f_p]$ obtained after the removal of the N_c -th quark.

In Ref. [19] we have derived the isoscalar factors associated to the normal Young tableau of $[[N_c - 1, 1]Y]_{FS}$, *i.e.* for $p = 2$, $q = 1$. We have also shown that the coefficients $c_{p\eta}$ appearing in the wave function of Ref. [13] correspond precisely to these isoscalar factors. In the present work we derive the isoscalar factors of the other $N_c - 2$ terms, having $p = 1$, for the first time. Details are given in Appendices A and C. The results are summarized in Tables I, II and III. In each table the column corresponding to $p = 1$ is new. For completeness, we also reproduce the results of Ref. [19] for $p = 2$. They are both used in the calculation of matrix elements of various operators entering the mass formula. As shown below, the total wave function needed to calculate these matrix elements contains a spin-flavor part, Eq. (8), constructed with the help of the isoscalar factors of Tables I, II and III.

The isoscalar factors derived here can be applied to the study of other $[70, \ell^P]$ multiplets with $\ell \neq 1$ and parity $P = +$ or $P = -$, or to other physical systems having N fermions. They satisfy a recurrence relation described in Refs. [27, 28] which has been used to check the presently derived analytic results in the case of $N_c = 5, 7$ and 9 quarks. The results from Tables I, II and III can also be interpreted as isoscalar factors of $SU(4)$, related to CG coefficients appearing in the chain $SU(4) \supset SU_S(2) \times SU_I(2)$ with the specific values of S and I presented here.

For the orbital part of the wave function (2) one can use one-body fractional parentage coefficients (cfp) to decouple the N_c -th quark from the rest. Together with the factorization property (5) it leads to a form of the wave function which is a sum of products of a wave function describing a system of $N_c - 1$ quarks, called ‘‘core’’ times the wave function of the N_c -th quark. If couplings are introduced this gives

$$|lSJJ_3; II_3\rangle = \sum_{p, \ell_c, \ell_q} a(p, \ell_c, \ell_q) |\ell_c \ell_q l S J J_3; II_3\rangle_p, \quad (6)$$

where $a(p, \ell_c, \ell_q)$ are the one-body cfp given in Appendix B, with ℓ_c and ℓ_q representing the angular momentum of the core and of the decoupled quark respectively. One can write

$$|\ell_c \ell_q l S J J_3; II_3\rangle_p = \sum_{m_c, m_q, m_\ell, S_3} \begin{pmatrix} \ell_c & \ell_q & l \\ m_c & m_q & m_\ell \end{pmatrix} \begin{pmatrix} \ell & S & J \\ m_\ell & S_3 & J_3 \end{pmatrix} |\ell_c m_c\rangle |\ell_q m_q\rangle |[N_c - 1, 1]p; SS_3; II_3\rangle, \quad (7)$$

containing the spin-flavor part

$$|[N_c - 1, 1]p; SS_3; II_3\rangle = \sum_{p'p''} K([f']p'[f'']p''|[N_c - 1, 1]p)|SS_3; p'\rangle|II_3; p''\rangle \quad (8)$$

where

$$|SS_3; p'\rangle = \sum_{m_1, m_2} \begin{pmatrix} S_c & \frac{1}{2} \\ m_1 & m_2 \end{pmatrix} \begin{matrix} S \\ S_3 \end{matrix} |S_c m_1\rangle |1/2 m_2\rangle, \quad (9)$$

with $S_c = S - 1/2$ if $p' = 1$ and $S_c = S + 1/2$ if $p' = 2$ and

$$|II_3; p''\rangle = \sum_{i_1, i_2} \begin{pmatrix} I_c & \frac{1}{2} \\ i_1 & i_2 \end{pmatrix} \begin{matrix} I \\ I_3 \end{matrix} |I_c i_1\rangle |1/2 i_2\rangle, \quad (10)$$

with $I_c = I - 1/2$ if $p'' = 1$ and $I_c = I + 1/2$ if $p'' = 2$. Thus p' and p'' represent here the position of the N_c -th quark in the spin and isospin parts of the wave function $|SS_3; p'\rangle|II_3; p''\rangle$ respectively. Note that the wave function (8) should also contain the second factor $S([f'_p]q'y'[f''_p]q''y''|[f_p]qy)$ of Eq. (5) together with sums over $q'y'$ and $q''y''$ which ensures the proper permutation symmetry of the spin-flavor part of the core wave function after the removal of the N_c -th quark. However, these CG coefficients of S_{N_c-1} do not need to be known and for simplicity we ignore them because their squares add up together to 1, by the standard normalization of CG coefficients. Equations (6)-(10) are needed in the calculation of operators as $s \cdot S_c$, S_c^2 , $t \cdot T_c$, T_c^2 , etc. (see next section) which involve the core explicitly.

Let us illustrate, for example, the use of Table I, when $p = 1$. Then Eq. (8) becomes

$$\begin{aligned} & |[N_c - 1, 1]p = 1; \frac{1}{2}S_3; \frac{1}{2}I_3\rangle = \\ & \sqrt{\frac{N_c - 3}{2(N_c - 2)}} \left| \frac{1}{2}S_3; p' = 2 \right\rangle \left| \frac{1}{2}I_3; p'' = 2 \right\rangle \\ & - \sqrt{\frac{N_c - 1}{4(N_c - 2)}} \left| \frac{1}{2}S_3; p' = 2 \right\rangle \left| \frac{1}{2}I_3; p'' = 1 \right\rangle - \sqrt{\frac{N_c - 1}{4(N_c - 2)}} \left| \frac{1}{2}S_3; p' = 1 \right\rangle \left| \frac{1}{2}I_3; p'' = 2 \right\rangle, \end{aligned} \quad (11)$$

where S_c and I_c are determined as explained below Eqs. (9) and (10).

In the following we shall compare results obtained from the exact wave function (6) with those obtained from a truncated wave function, representing only the term with $p = 2$ in the sum (6). In addition, by taking $a(2, \ell_c = 0, \ell_q = 1) = 1$, instead of the value given in Appendix B, and renormalizing the resulting wave function, we recover the wave function of

$[f']p'[f'']p''$	$[N_c - 1, 1]1$	$[N_c - 1, 1]2$
$\left[\frac{N_c+1}{2}, \frac{N_c-1}{2}\right]_1 \left[\frac{N_c+1}{2}, \frac{N_c-1}{2}\right]_1$	0	$-\sqrt{\frac{3(N_c-1)}{4N_c}}$
$\left[\frac{N_c+1}{2}, \frac{N_c-1}{2}\right]_2 \left[\frac{N_c+1}{2}, \frac{N_c-1}{2}\right]_2$	$\sqrt{\frac{N_c-3}{2(N_c-2)}}$	$\sqrt{\frac{N_c+3}{4N_c}}$
$\left[\frac{N_c+1}{2}, \frac{N_c-1}{2}\right]_2 \left[\frac{N_c+1}{2}, \frac{N_c-1}{2}\right]_1$	$-\frac{1}{2}\sqrt{\frac{N_c-1}{N_c-2}}$	0
$\left[\frac{N_c+1}{2}, \frac{N_c-1}{2}\right]_1 \left[\frac{N_c+1}{2}, \frac{N_c-1}{2}\right]_2$	$-\frac{1}{2}\sqrt{\frac{N_c-1}{N_c-2}}$	0

TABLE I: Isoscalar factors $K([f']p'[f'']p''|[f]p)$ for $S = I = 1/2$, corresponding to 2_8 when $N_c = 3$. The second column gives results for $p = 1$ and the third for $p = 2$.

$[f']p'[f'']p''$	$[N_c - 1, 1]1$	$[N_c - 1, 1]2$
$\left[\frac{N_c+3}{2}, \frac{N_c-3}{2}\right]_1 \left[\frac{N_c+1}{2}, \frac{N_c-1}{2}\right]_1$	$\frac{1}{2}\sqrt{\frac{(N_c-1)(N_c+3)}{N_c(N_c-2)}}$	0
$\left[\frac{N_c+3}{2}, \frac{N_c-3}{2}\right]_2 \left[\frac{N_c+1}{2}, \frac{N_c-1}{2}\right]_2$	$\frac{1}{2}\sqrt{\frac{5(N_c-1)(N_c-3)}{2N_c(N_c-2)}}$	0
$\left[\frac{N_c+3}{2}, \frac{N_c-3}{2}\right]_1 \left[\frac{N_c+1}{2}, \frac{N_c-1}{2}\right]_2$	$\frac{1}{2}\sqrt{\frac{(N_c-3)(N_c+3)}{2N_c(N_c-2)}}$	1
$\left[\frac{N_c+3}{2}, \frac{N_c-3}{2}\right]_2 \left[\frac{N_c+1}{2}, \frac{N_c-1}{2}\right]_1$	0	0

TABLE II: Isoscalar factors $K([f']p'[f'']p''|[f]p)$ for $S = 3/2$, $I = 1/2$, corresponding to 4_8 when $N_c = 3$. The second column gives results for $p = 1$ and the third for $p = 2$.

Ref. [13]. As seen from Appendix B, it is entirely reasonable to take $a(2, \ell_c = 0, \ell_q = 1) = 1$ when N_c is large. Then the core is in its ground state. It is in fact the validity of this wave function, representing the Hartree approximation, we wish to analyze as compared to the exact wave function in the description of the $[70, 1^-]$ multiplet.

$[f']p'[f'']p''$	$[N_c - 1, 1]1$	$[N_c - 1, 1]2$
$\left[\frac{N_c + 1}{2}, \frac{N_c - 1}{2}\right]_1 \left[\frac{N_c + 3}{2}, \frac{N_c - 3}{2}\right]_1$	$\frac{1}{2} \sqrt{\frac{(N_c - 1)(N_c + 3)}{N_c(N_c - 2)}}$	0
$\left[\frac{N_c + 1}{2}, \frac{N_c - 1}{2}\right]_2 \left[\frac{N_c + 3}{2}, \frac{N_c - 3}{2}\right]_2$	$\frac{1}{2} \sqrt{\frac{5(N_c - 1)(N_c - 3)}{2N_c(N_c - 2)}}$	0
$\left[\frac{N_c + 1}{2}, \frac{N_c - 1}{2}\right]_2 \left[\frac{N_c + 3}{2}, \frac{N_c - 3}{2}\right]_1$	$\frac{1}{2} \sqrt{\frac{(N_c - 3)(N_c + 3)}{2N_c(N_c - 2)}}$	1
$\left[\frac{N_c + 1}{2}, \frac{N_c - 1}{2}\right]_1 \left[\frac{N_c + 3}{2}, \frac{N_c - 3}{2}\right]_2$	0	0

TABLE III: Isoscalar factors $K([f']p'[f'']p''|[f]p)$ for $S = 1/2$, $I = 3/2$, corresponding to ²10 when $N_c = 3$. The second column gives results for $p = 1$ and the third for $p = 2$.

III. THE MASS OPERATOR

The mass operator M is a linear combination of independent operators O_i

$$M = \sum_i c_i O_i, \quad (12)$$

where the coefficients c_i are reduced matrix elements that encode the QCD dynamics and are determined from a fit to the existing data. The operators O_i are constructed from the SU(4) generators S_i , T_a and G_{ia} and the SO(3) generators ℓ_i . For the purpose of the present analysis it is enough to restrict the choice of the operators O_i to a few selected dominant operators. Our previous analysis [26] suggests that the dominant operators up to order $\mathcal{O}(1/N_c)$ included are those constructed from the SU(4) generators exclusively, the operators containing angular momentum having a minor role. Among them, the spin-orbit, although weak, is however considered here. Samples are given in Tables VI-X. (For convenience and clarity we denote some operators in the sum (12) by O'_i . The meaning will be obvious later on.)

In Tables IV and V we present the analytic expressions of the matrix elements of the operators containing spin and isospin, used in this study. They have been obtained either with the approximate or with the exact wave function. One can see that the analytic forms are different in the two cases. This implies that at $N_c = 3$ the values obtained for c_i from the fit to data are expected to be also different, as illustrated in the next section. The

	$\langle s \cdot S_c \rangle$		$\langle S_c^2 \rangle$	
	approx. w.f.	exact w.f.	approx. w.f.	exact w.f.
2_8	$-\frac{N_c + 3}{4N_c}$	$-\frac{3(N_c - 1)}{4N_c}$	$\frac{N_c + 3}{2N_c}$	$\frac{3(N_c - 1)}{2N_c}$
4_8	$\frac{1}{2}$	$-\frac{3(N_c - 5)}{4N_c}$	2	$\frac{3(3N_c - 5)}{2N_c}$
$^2_{10}$	-1	$-\frac{3(N_c - 1)}{4N_c}$	2	$\frac{3(N_c - 1)}{2N_c}$

TABLE IV: Matrix elements of the spin operators calculated with the approximate and the exact wave functions.

	$\langle t \cdot T_c \rangle$		$\langle T_c^2 \rangle$	
	approx. w.f.	exact w.f.	approx. w.f.	exact w.f.
2_8	$-\frac{N_c + 3}{4N_c}$	$-\frac{3(N_c - 1)}{4N_c}$	$\frac{N_c + 3}{2N_c}$	$\frac{3(N_c - 1)}{2N_c}$
4_8	-1	$-\frac{3(N_c - 1)}{4N_c}$	2	$\frac{3(N_c - 1)}{2N_c}$
$^2_{10}$	$\frac{1}{2}$	$-\frac{3(N_c - 5)}{4N_c}$	2	$\frac{3(3N_c - 5)}{2N_c}$

TABLE V: Matrix elements of the isospin operators calculated with the approximate and the exact wave functions.

matrix elements of the spin-orbit operator are identical for the exact and an approximate wave function and can be found in Table II of Ref. [13].

IV. FIT AND DISCUSSION

In the numerical fit we have considered the masses of the seven experimentally known nonstrange resonances belonging to the $[70, 1^-]$ multiplet. Neglecting the mixing $^4N - ^2N$ we identify them as: $^2N_{1/2}(1538 \pm 18)$, $^4N_{1/2}(1660 \pm 20)$, $^2N_{3/2}(1523 \pm 8)$, $^4N_{3/2}(1700 \pm 50)$, $^4N_{5/2}(1678 \pm 8)$, $^2\Delta_{1/2}(1645 \pm 30)$ and $^2\Delta_{3/2}(1720 \pm 50)$.

The results from various fits are presented in Tables VI-IX. In Table VI the first four operators are among those considered in Ref. [13], where the system is decoupled into a ground state core and an excited quark. To them we have added the isospin operator which we expect to play an important role. The value of χ_{dof}^2 is satisfactory but the value of the

coefficient c_1 is much smaller than in previous studies where it generally appears to be of the order of 500 MeV. The spin-orbit coefficient is small, as expected, but the coefficients c_3 and c_4 are exceedingly large in absolute values and have opposite signs, which suggest some compensation. The coefficient c_5 is also very large and negative for the approximate wave function but it has a reasonable value for the exact wave function. A solution is either to eliminate the isospin operator, but this cannot be theoretically justified or to make some combinations of the above operators. In Table VII we consider the linear combination $2s^i S_c^i + S_c^i S_c^i + \frac{3}{4} = S^2$, *i.e.* we use the total spin operator of the system, entirely legitimate in the $1/N_c$ expansion approach. The coefficient c_1 acquires the expected value and the spin coefficient c'_3 has a reasonable value, being about 70 MeV smaller for the exact wave function than for the approximate one. The isospin coefficient is equal in both cases and also has a reasonable value. By analogy, in Table VIII we include the linear combination $2t^a T_c^a + T_c^a T_c^a + \frac{3}{4} = I^2$ *i.e.* we use the total isospin and restrict the spin contribution to $\frac{1}{N_c} s \cdot S_c$. Then the isospin coefficient c'_5 acquires values comparable to the spin contribution in Table VII. The spin coefficient c_3 has now values comparable to the isospin coefficient c_5 in Table VII. One can infer that the spin and isospin play a similar role in the mass formula. This statement is clearly proved in Table IX where we include the spin and isospin contributions on an equal footing, *i.e.* we include operators proportional to S^2 and I^2 . The situation is entirely identical for the exact and approximate wave function, with reasonable values for all coefficients and a $\chi^2_{\text{dof}} = 1.04$. The identity of the results is natural because both the approximate and the exact wave function describe a system of a given spin S and isospin I .

V. MIXING ANGLES

So far we have discussed the mass spectrum. Additional empirical information come from the mixing angles extracted from the electromagnetic and strong decays, in particular from the dominance of $N\eta$ decay of $N_{1/2}(1538 \pm 18)$. These angles are defined as

$$\begin{aligned}
 |N_J(\text{upper})\rangle &= \cos \theta_J |^4 N_J\rangle + \sin \theta_J |^2 N_J\rangle, \\
 |N_J(\text{lower})\rangle &= \cos \theta_J |^2 N_J\rangle - \sin \theta_J |^4 N_J\rangle.
 \end{aligned}
 \tag{13}$$

Operator		Approx. w.f. (MeV)	Exact w.f. (MeV)
$O_1 = N_c \mathbb{1}$	$c_1 =$	211 ± 23	299 ± 20
$O_2 = \ell^i s^i$	$c_2 =$	3 ± 15	3 ± 15
$O_3 = \frac{1}{N_c} s^i S_c^i$	$c_3 =$	-1486 ± 141	-1096 ± 125
$O_4 = \frac{1}{N_c} S_c^i S_c^i$	$c_4 =$	1182 ± 74	1545 ± 122
$O_5 = \frac{1}{N_c} t^a T_c^a$	$c_5 =$	-1508 ± 149	417 ± 79
χ_{dof}^2		1.56	1.56

TABLE VI: List of operators and coefficients obtained in the numerical fit to the 7 known experimental masses of the lowest negative parity resonances (see text). For the operators O_3 , O_4 and O_5 we use the matrix elements from Tables IV and V.

Operator		Approx. w.f. (MeV)	Exact w.f. (MeV)
$O_1 = N_c \mathbb{1}$	$c_1 =$	513 ± 4	519 ± 5
$O_2 = \ell^i s^i$	$c_2 =$	3 ± 15	3 ± 15
$O'_3 = \frac{1}{N_c} \left(2s^i S_c^i + S_c^i S_c^i + \frac{3}{4} \right)$	$c'_3 =$	219 ± 19	150 ± 11
$O_5 = \frac{1}{N_c} t^a T_c^a$	$c_5 =$	417 ± 80	417 ± 80
χ_{dof}^2		1.04	1.04

TABLE VII: Same as Table VI but for O'_3 , which combines O_3 and O_4 instead of using them separately.

Operator		Approx. w.f. (MeV)	Exact w.f. (MeV)
$O_1 = N_c \mathbb{1}$	$c_1 =$	516 ± 3	522 ± 3
$O_2 = \ell^i s^i$	$c_2 =$	3 ± 15	3 ± 15
$O_3 = \frac{1}{N_c} s^i S_c^i$	$c_3 =$	450 ± 33	450 ± 33
$O'_5 = \frac{1}{N_c} \left(2t^a T_c^a + T_c^a T_c^a + \frac{3}{4} \right)$	$c'_5 =$	214 ± 28	139 ± 27
χ_{dof}^2		1.04	1.04

TABLE VIII: Same as Table VII but combining isospin operators instead of spin operators.

Operator	Approx. w.f. (MeV)	Exact w.f. (MeV)
$O_1 = N_c \mathbb{1}$	$c_1 = 484 \pm 4$	484 ± 4
$O_2 = \ell^i s^i$	$c_2 = 3 \pm 15$	3 ± 15
$O'_3 = \frac{1}{N_c} \left(2s^i S_c^i + S_c^i S_c^i + \frac{3}{4} \right)$	$c'_3 = 150 \pm 11$	150 ± 11
$O'_5 = \frac{1}{N_c} \left(2t^a T_c^a + T_c^a T_c^a + \frac{3}{4} \right)$	$c'_5 = 139 \pm 27$	139 ± 27
χ_{dof}^2	1.04	1.04

TABLE IX: Same as Table VI but with operators proportional to the SU(2)-spin and SU(2)-isospin Casimir operators (see text).

Experimentally one finds $\theta_{1/2}^{\text{exp}} \approx -0.56$ rad and $\theta_{3/2}^{\text{exp}} \approx 0.10$ rad [29]. In our case, the only operator with non-vanishing matrix elements ${}^4N - {}^2N$, thus contributing to the mixing, is the spin-orbit operator. Its off-diagonal matrix elements are $\langle {}^4N_{1/2} | \ell \cdot s | {}^2N_{1/2} \rangle = -1/3$ and $\langle {}^4N_{3/2} | \ell \cdot s | {}^2N_{3/2} \rangle = -\sqrt{5/18}$, compatible with Ref. [13] when $N_c = 3$. In this notation the physical masses become

$$\begin{aligned}
M_J(\text{upper}) &= M({}^4N_J) + c_2 \langle {}^4N_J | \ell \cdot s | {}^2N_J \rangle \tan \theta_J, \\
M_J(\text{lower}) &= M({}^2N_J) - c_2 \langle {}^4N_J | \ell \cdot s | {}^2N_J \rangle \tan \theta_J.
\end{aligned}
\tag{14}$$

Due to the fact that the diagonal contribution of $O_2 = \ell \cdot s$ is very small, we expect its contribution to the mixing to be small as well. By including the experimental values of θ_J in Eqs. (14) and applying again the minimization procedure, we found indeed that the effect of mixing is negligible. The only coefficient being slightly affected is c_2 and the χ_{dof}^2 remains practically the same. Next we varied θ_J to see if the value of χ_{dof}^2 remains stable. Indeed it does, for variations of θ_J of ± 0.05 rad. In Table X we show the optimal set of coefficients c_i associated to the experimental values of θ_J . In a more extended analysis as that of Ref. [26] the operators $O_5 = \frac{15}{N_c} \ell^{(2)ij} G^{ia} G^{ja}$ and $O_6 = \frac{3}{N_c} \ell^i T^a G^{ia}$ would also contribute. However all these operators contain angular momentum while in constituent quark models where the predictions are generally good, the source of mixing are either the spin-spin or the tensor interactions (no angular momentum), contrary to the present case. We therefore believe that this important issue deserves further work in the future, combined with a strong decay analysis, like in Ref. [30].

Operator	Approx. w.f. (MeV)	Exact w.f. (MeV)
$O_1 = N_c \mathbb{1}$	$c_1 = 484 \pm 4$	484 ± 4
$O_2 = \ell^i s^i$	$c_2 = -9 \pm 15$	-9 ± 15
$O'_3 = \frac{1}{N_c} \left(2s^i S_c^i + S_c^i S_c^i + \frac{3}{4} \right)$	$c'_3 = 150 \pm 11$	150 ± 11
$O'_5 = \frac{1}{N_c} \left(2t^a T_c^a + T_c^a T_c^a + \frac{3}{4} \right)$	$c'_5 = 139 \pm 27$	139 ± 27
χ_{dof}^2	1.05	1.05

TABLE X: Fit with state mixing for $N_{1/2}$ and $N_{3/2}$ due to the spin-orbit coupling $\ell \cdot s$.

VI. CONCLUSIONS

This study shows that the exact and the approximate wave function give identical χ_{dof}^2 in this simplified fit and rather similar results for the dynamical coefficients c_i entering in the mass formula. It also shows that the contribution of the isospin operator $O'_5 = \frac{1}{N_c} I^2$ is as important as that of the spin operator $O'_3 = \frac{1}{N_c} S^2$. In addition, it turned out that the separation of S^2 and of I^2 into independent parts containing core and excited quark operators is undesirable because it seriously deteriorates the fit. The only satisfactory way to describe the spectrum is to include in Eq. (12) the Casimir operators of $SU_S(2)$ and $SU_I(2)$ acting on the entire system.

The difficulty with the approximate wave function is that it cannot distinguish between S_c^2 and I_c^2 , which act on the core only. In practice it means that the contribution of $\frac{1}{N_c} I_c^2$ to the mass is hidden in the coefficient c_i associated to $\frac{1}{N_c} S_c^2$. This also happens for $\frac{1}{N_c} I^2$ for states described by a symmetric flavor-spin wave function where its contribution is absorbed by the coefficient of $\frac{1}{N_c} S^2$. In other words S^2 and I^2 should share their contribution to the mass by about the same amount.

In the decoupling scheme the isospin can be introduced only through the operator $\frac{1}{N_c} t \cdot T_c$ which manifestly deteriorates the fit, as seen in Table VI. This may explain why this operator has been avoided in numerical fits of previous studies based on core + quark separation [13].

While preserving the S_{N_c} symmetry in these calculations, our conclusion is at variance with that of Pirjol and Schat [31] who formally claim that the inclusion of core and excited quark operators is necessary, as a consequence of constraints on the mass operator, resulting

from the same S_{N_c} symmetry.

The dynamical coefficients c_i encode the quark dynamics in hadrons. Therefore, to find their correct values and their evolution with the excitation energy is a very important task. They can serve to test the validity of quark models and quantitatively determine the contribution of gluon and Goldstone boson exchange interactions, both justified by the two-scale picture of Manohar and Georgi [32].

As a byproduct we have obtained isoscalar factors of the permutation group which may be useful in further studies of mixed symmetric multiplets within the $1/N_c$ expansion or in any other approach where one fermion is separated from the rest.

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APPENDIX A: ISOSCALAR FACTORS

Here we shortly describe the derivation of the isoscalar factors of Tables I,II and III. We first recall that the expressions of $K([f']p'[f'']p''|[N_c - 1, 1]p)$ with $p = 2$ were obtained in Ref. [19]. We are presently concerned with the calculation of $K([f']p'[f'']p''|[N_c - 1, 1]p)$ for $p = 1$.

In the derivation below we use the symmetry property

$$K([f']p'[f'']p''|[f]p) = K([f'']p''[f']p'|[f]p), \quad (\text{A1})$$

and the orthogonality relation

$$\sum_{p'p''} K([f']p'[f'']p''|[f]p) K([f']p'[f'']p''|[f_1]p_1) = \delta_{ff_1} \delta_{pp_1}. \quad (\text{A2})$$

1. The case $S = I = 1/2$

In this case we have

$$[f'] = [f''] = \left[\frac{N_c + 1}{2}, \frac{N_c - 1}{2} \right], \quad (\text{A3})$$

and there are three nonzero coefficients necessary to express the FS state $|[N_c - 1, 1]1\rangle$, as linear combinations of S and F states. These are $K([f']2[f'']2|[N_c - 1, 1]1)$, $K([f']1[f'']2|[N_c - 1, 1]1)$ and $K([f']2[f'']1|[N_c - 1, 1]1)$. But due to (A1) the latter two are equal. Then we are

left with two unknown isoscalar factors which following (A2) are normalized as

$$(K([f']2[f'']2|[N_c - 1, 1]1))^2 + 2(K([f']1[f'']2|[N_c - 1, 1]1))^2 = 1. \quad (\text{A4})$$

To find the solution we need one more equation. This is provided by the matrix elements of the operator G^{ia} written as

$$\langle G^{ia} \rangle = \langle G_c^{ia} \rangle + \langle g^{ia} \rangle \quad (\text{A5})$$

with G_c^{ia} and g^{ia} acting on the core and on the excited quark respectively. According to Eqs. (C1) and (C3) the matrix elements of the SU(4) irrep $[N_c - 1, 1]$ with $S = I = 1/2$ are

$$\begin{aligned} \left\langle \frac{1}{2}S'_3; \frac{1}{2}I'_3 \left| G^{ia} \right| \frac{1}{2}S_3; \frac{1}{2}I_3 \right\rangle = \\ -\frac{N_c - 6}{12} \begin{pmatrix} 1/2 & 1 & | & 1/2 \\ S_3 & i & | & S'_3 \end{pmatrix} \begin{pmatrix} 1/2 & 1 & | & 1/2 \\ I_3 & a & | & I'_3 \end{pmatrix}. \end{aligned} \quad (\text{A6})$$

Taking $p = 1$ in Eqs. (C4) and (C5) one obtains for $\langle G_c^{ia} \rangle$ and $\langle g^{ia} \rangle$ the following expressions

$$\begin{aligned} \left\langle [N_c - 1, 1]1; \frac{1}{2}S'_3; \frac{1}{2}I'_3 \left| G_c^{ia} \right| [N_c - 1, 1]1; \frac{1}{2}S_3; \frac{1}{2}I_3 \right\rangle = \\ \frac{1}{6} \left[\frac{N_c + 3}{2} (K([f']2[f'']2|[N_c - 1, 1]1))^2 + (N_c + 3) (K([f']1[f'']2|[N_c - 1, 1]1))^2 \right. \\ \left. + 4\sqrt{\frac{(N_c - 3)(N_c - 1)}{2}} K([f']2[f'']2|[N_c - 1, 1]1)K([f']1[f'']2|[N_c - 1, 1]1) \right] \\ \times \begin{pmatrix} 1/2 & 1 & | & 1/2 \\ S_3 & i & | & S'_3 \end{pmatrix} \begin{pmatrix} 1/2 & 1 & | & 1/2 \\ I_3 & a & | & I'_3 \end{pmatrix}, \end{aligned} \quad (\text{A7})$$

and

$$\begin{aligned} \left\langle [N_c - 1, 1]1; \frac{1}{2}S'_3; \frac{1}{2}I'_3 \left| g^{ia} \right| [N_c - 1, 1]1; \frac{1}{2}S_3; \frac{1}{2}I_3 \right\rangle = \\ \left[\frac{1}{12} (K([f']2[f'']2|[N_c - 1, 1]1))^2 - \frac{1}{2} (K([f']1[f'']2|[N_c - 1, 1]1))^2 \right] \\ \times \begin{pmatrix} 1/2 & 1 & | & 1/2 \\ S_3 & i & | & S'_3 \end{pmatrix} \begin{pmatrix} 1/2 & 1 & | & 1/2 \\ I_3 & a & | & I'_3 \end{pmatrix} \end{aligned} \quad (\text{A8})$$

respectively. Then Eq. (A5) leads to the following relation

$$\begin{aligned} \frac{N_c + 4}{12} (K([f']2[f'']2|[N_c - 1, 1]1))^2 + \frac{N_c}{6} (K([f']1[f'']2|[N_c - 1, 1]1))^2 \\ + \frac{2}{3} \sqrt{\frac{(N_c - 3)(N_c - 1)}{2}} K([f']2[f'']2|[N_c - 1, 1]1)K([f']1[f'']2|[N_c - 1, 1]1) \\ + \frac{N_c - 6}{12} = 0, \end{aligned} \quad (\text{A9})$$

which together with the normalization relation (A4) forms a system of two nonlinear equations for the unknown isoscalar factors. The solution is exhibited in Table I column 2. The phase convention for the isoscalar factors is the same as in Refs. [27, 28].

2. The case $S = 3/2, I = 1/2$

These values of S and I imply

$$[f'] = \left[\frac{N_c + 3}{2}, \frac{N_c - 3}{2} \right], \quad [f''] = \left[\frac{N_c + 1}{2}, \frac{N_c - 1}{2} \right], \quad (\text{A10})$$

and, as above, there are three nonzero coefficients necessary to express the FS state $|[N_c - 1, 1]1\rangle$, as linear a combination of S and F states. In this case, these are $K([f']2[f'']2|[N_c - 1, 1]1)$, $K([f']1[f'']2|[N_c - 1, 1]1)$ and $K([f']1[f'']1|[N_c - 1, 1]1)$. One needs three equations to find the solution.

First, Eq. (A2) gives the normalization relation

$$(K([f']2[f'']2|[N_c - 1, 1]1))^2 + (K([f']1[f'']2|[N_c - 1, 1]1))^2 + (K([f']1[f'']1|[N_c - 1, 1]1))^2 = 1. \quad (\text{A11})$$

Second, Eq. (A5) gives

$$\begin{aligned} & \frac{3(3N_c - 1)}{10N_c} (K([f']2[f'']2|[N_c - 1, 1]1))^2 + \frac{N_c + 1}{2N_c} (K([f']1[f'']2|[N_c - 1, 1]1))^2 \\ & + \frac{3}{N_c} (K([f']1[f'']1|[N_c - 1, 1]1))^2 \\ & - \sqrt{\frac{(N_c - 1)(N_c + 3)}{5N_c^2}} K([f']2[f'']2|[N_c - 1, 1]1) K([f']1[f'']2|[N_c - 1, 1]1) \\ & + \sqrt{\frac{2(N_c - 3)(N_c - 1)}{N_c}} K([f']1[f'']2|[N_c - 1, 1]1) K([f']1[f'']1|[N_c - 1, 1]1) \\ & + \sqrt{\frac{2(N_c - 3)(N_c + 3)}{5N_c^2}} K([f']2[f'']2|[N_c - 1, 1]1) K([f']1[f'']1|[N_c - 1, 1]1) = 1. \quad (\text{A12}) \end{aligned}$$

Third, for the irrep $[f] = [N_c - 1, 1]$ the Casimir operator identity becomes (see Eq. (C3))

$$\frac{\langle S^2 \rangle}{2} + \frac{\langle T^2 \rangle}{2} + 2 \langle G^2 \rangle = \frac{N_c(3N_c + 4)}{8}, \quad (\text{A13})$$

from which one can derive the following equation

$$\langle s \cdot S_c \rangle + \langle t \cdot T_c \rangle + 4 \langle g \cdot G_c \rangle = \frac{3N_c - 7}{4}. \quad (\text{A14})$$

Using Eqs (C6)–(C8) we obtain

$$\begin{aligned}
& \frac{3N_c - 23}{8} (K([f']2[f'']2|[N_c - 1, 1]1))^2 - \frac{N_c + 7}{8} (K([f']1[f'']2|[N_c - 1, 1]1))^2 \\
& + \frac{\sqrt{5(N_c - 1)(N_c + 3)}}{4} K([f']2[f'']2|[N_c - 1, 1]1)K([f']1[f'']2|[N_c - 1, 1]1) \\
& + \frac{1}{2} \sqrt{\frac{5(N_c - 3)(N_c + 3)}{2}} K([f']2[f'']2|[N_c - 1, 1]1)K([f']1[f'']1|[N_c - 1, 1]1) \\
& + \frac{1}{2} \sqrt{\frac{(N_c - 1)(N_c - 3)}{2}} K([f']1[f'']2|[N_c - 1, 1]1)K([f']1[f'']1|[N_c - 1, 1]1) \\
& + \frac{1}{2} (K([f']1[f'']1|[N_c - 1, 1]1))^2 = \frac{3N_c - 7}{4}, \tag{A15}
\end{aligned}$$

which is the third equation needed to derive the isoscalar factors presented in the second column of Table II.

3. The case $\mathbf{S} = 1/2, \mathbf{I} = 3/2$

This case is similar to the previous one. The results can be obtained by interchanging S with I .

APPENDIX B: FRACTIONAL PARENTAGE COEFFICIENTS

We are in the case where the orbital wave function $|[N_c - 1, 1]pqy\rangle_O$ contains only one excited quark with the structure $s^{N_c-1}p$. The one-body fractional parentage coefficients (cfp) help to decouple the system into a core of $N_c - 1$ quarks and a single quark.

For $p = 2$ (obviously $q = 1$ and y is fixed) the decoupling into a core and a quark gives

$$|[N_c - 1, 1]2\rangle_O = \sqrt{\frac{N_c - 1}{N_c}} R_{[N_c-1]}^c(s^{N_c-1}) R_{[1]}^q(p) - \sqrt{\frac{1}{N_c}} R_{[N_c-1]}^c(s^{N_c-2}p) R_{[1]}^q(s). \tag{B1}$$

In the first term the core is in the ground state, *i.e.* $\ell_c = 0$, and is described by the symmetric orbital wave function $R_{[N_c-1]}^c(s^{N_c-1})$. The quark is excited and carries the angular momentum $\ell_q = 1$. In the second term, the orbital wave function is still symmetric but it contains a unit of angular momentum and the quark is unexcited.

For $p = 1$, irrespective of q and y one has

$$|[N_c - 1, 1]1\rangle_O = R_{[N_c-2,1]}^c(s^{N_c-2}p) R_{[1]}^q(s) \tag{B2}$$

i.e. the core is always excited, thus it is described by a mixed symmetric state, and the quark is in the ground state. Denoting the one-body cfp by $a(p, \ell_c, \ell_q)$ we get

$$a(2, \ell_c = 0, \ell_q = 1) = \sqrt{\frac{N_c - 1}{N_c}}, \quad (\text{B3})$$

$$a(2, \ell_c = 1, \ell_q = 0) = -\sqrt{\frac{1}{N_c}}, \quad (\text{B4})$$

$$a(1, \ell_c = 1, \ell_q = 0) = 1. \quad (\text{B5})$$

to be used in Eq. (6).

APPENDIX C

This appendix contains the matrix elements of some operators used to derive the isoscalar factors $K([f']p'[f'']p''|[N_c - 1, 1]1)$ presented in Tables I, II and III.

Let us first recall the general formula of the matrix elements of the generator G^{ia} of SU(4) for a given irrep $[f]$. This is [33]

$$\begin{aligned} \langle [f]; I' I'_3; S' S'_3 | G^{ia} | [f]; I I_3; S S_3 \rangle = \\ \sqrt{\frac{C^{[f]}(\text{SU}(4))}{2}} \begin{pmatrix} [f] & [21^2] \\ IS & 11 \end{pmatrix} \Bigg|_{\rho=1} \begin{pmatrix} [f] \\ I' S' \end{pmatrix} \begin{pmatrix} S & 1 \\ S_3 & S_3^i \end{pmatrix} \begin{pmatrix} S' \\ S'_3 \end{pmatrix} \begin{pmatrix} I & 1 \\ I_3 & I_3^a \end{pmatrix} \begin{pmatrix} I' \\ I'_3 \end{pmatrix}, \end{aligned} \quad (\text{C1})$$

where $C^{[f]}$ is the eigenvalue of the SU(4) Casimir operator for the irrep $[f]$, the second factor is an isoscalar factor of SU(4), the third a CG coefficient of SU(2)-spin and the last a CG coefficient of SU(2)-isospin. The isoscalar factors used here are those of Table A4.2 and A4.5 of Hecht and Pang [33] adapted to our notations for $[f] = [N_c]$ and $[f] = [N_c - 1, 1]$ respectively. We recall that

$$C^{[N_c]}(\text{SU}(4)) = \frac{3N_c(N_c + 4)}{8}, \quad (\text{C2})$$

$$C^{[N_c - 1, 1]}(\text{SU}(4)) = \frac{N_c(3N_c + 4)}{8}. \quad (\text{C3})$$

From Eqs. (8)–(10) and (C1), one can derive the following matrix elements:

$$\begin{aligned} \langle [N_c - 1, 1]p; I' I'_3; S' S'_3 | G_c^{ia} | [N_c - 1, 1]p; I I_3; S S_3 \rangle = \\ (-1)^{S+I+1} \sqrt{(2S+1)(2I+1)} \sqrt{\frac{C^{[f]}(\text{SU}(4))}{2}} \begin{pmatrix} S & 1 \\ S_3 & S_3^i \end{pmatrix} \begin{pmatrix} S \\ S'_3 \end{pmatrix} \begin{pmatrix} I & 1 \\ I_3 & I_3^a \end{pmatrix} \begin{pmatrix} I \\ I'_3 \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
& \times \sum_{p', p'', q', q''} (-1)^{S'_c + I'_c} K([f']p'[f'']p''|[N_c - 1, 1]p) K([f']q'[f'']q''|[N_c - 1, 1]p) \\
& \times \sqrt{(2S'_c + 1)(2I'_c + 1)} \left(\begin{array}{c|c} [f] & [21^2] \\ \hline I_c S_c & 11 \end{array} \left\| \begin{array}{c} [f] \\ I'_c S'_c \end{array} \right. \right)_{\rho=1} \left\{ \begin{array}{ccc} S & 1 & S \\ S'_c & 1/2 & S_c \end{array} \right\} \left\{ \begin{array}{ccc} I & 1 & I \\ I'_c & 1/2 & I_c \end{array} \right\}, \quad (\text{C4})
\end{aligned}$$

$$\begin{aligned}
& \langle [N_c - 1, 1]p; II'_3; SS'_3 | g^{ia} | [N_c - 1, 1]p; II_3; SS_3 \rangle = \\
& (-1)^{S+I+1} \frac{3}{2} \sqrt{(2S+1)(2I+1)} \left(\begin{array}{c|c} S & 1 \\ \hline S_3 & S_3^i \end{array} \left\| \begin{array}{c} S \\ S'_3 \end{array} \right. \right) \left(\begin{array}{c|c} I & 1 \\ \hline I_3 & I_3^a \end{array} \left\| \begin{array}{c} I \\ I'_3 \end{array} \right. \right) \\
& \times \sum_{p', p''} (-1)^{S_c + I_c} (K([f']p'[f'']p''|[N_c - 1, 1]p))^2 \left\{ \begin{array}{ccc} S & 1 & S \\ 1/2 & S_c & 1/2 \end{array} \right\} \left\{ \begin{array}{ccc} I & 1 & I \\ 1/2 & I_c & 1/2 \end{array} \right\}, \quad (\text{C5})
\end{aligned}$$

$$\begin{aligned}
& \langle [N_c - 1, 1]p; II'_3; SS'_3 | g \cdot G_c | [N_c - 1, 1]p; II_3; SS_3 \rangle = \\
& \delta_{S'_3 S_3} \delta_{I'_3 I_3} (-1)^{S+I+1} \frac{3}{2} \sqrt{\frac{C^{[f]}(\text{SU}(4))}{2}} \\
& \times \sum_{p', p'', q', q''} (-1)^{S_c + I_c} K([f']p'[f'']p''|[N_c - 1, 1]p) K([f']q'[f'']q''|[N_c - 1, 1]p) \\
& \times \sqrt{(2S'_c + 1)(2I'_c + 1)} \left(\begin{array}{c|c} [f] & [21^2] \\ \hline I_c S_c & 11 \end{array} \left\| \begin{array}{c} [f] \\ I'_c S'_c \end{array} \right. \right)_{\rho=1} \left\{ \begin{array}{ccc} 1/2 & S'_c & S \\ S_c & 1/2 & 1 \end{array} \right\} \left\{ \begin{array}{ccc} 1/2 & I'_c & I \\ I_c & 1/2 & 1 \end{array} \right\}, \quad (\text{C6})
\end{aligned}$$

where $[f] = [N_c - 2, 1]$ for $p = 1$ and $[f] = [N_c - 1]$ for $p = 2$. S_c, I_c, S'_c and I'_c are determined by p', p'', q' and q'' .

There are also needed

$$\begin{aligned}
& \langle [N_c - 1, 1]p; II'_3; SS'_3 | s \cdot S_c | [N_c - 1, 1]p; II_3; SS_3 \rangle = \delta_{S'_3 S_3} \delta_{I'_3 I_3} (-1)^{S+1/2} \sqrt{\frac{3}{2}} \\
& \times \sum_{p', p''} (-1)^{S_c} \sqrt{S_c(S_c + 1)(2S_c + 1)} (K([f']p'[f'']p''|[N_c - 1, 1]p))^2 \left\{ \begin{array}{ccc} 1/2 & S_c & S \\ S_c & 1/2 & 1 \end{array} \right\}, \quad (\text{C7})
\end{aligned}$$

and

$$\begin{aligned}
& \langle [N_c - 1, 1]p; II'_3; SS'_3 | t \cdot T_c | [N_c - 1, 1]p; II_3; SS_3 \rangle = \delta_{S'_3 S_3} \delta_{I'_3 I_3} (-1)^{I+1/2} \sqrt{\frac{3}{2}} \\
& \times \sum_{p', p''} (-1)^{I_c} \sqrt{I_c(I_c + 1)(2I_c + 1)} (K([f']p'[f'']p''|[N_c - 1, 1]p))^2 \left\{ \begin{array}{ccc} 1/2 & I_c & I \\ I_c & 1/2 & 1 \end{array} \right\}. \quad (\text{C8})
\end{aligned}$$

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