Group Theoretical Analysis of the Wave Function of the $[\mathbf{70}, 1^-]$ Nonstrange Baryons in the $1/N_c$ Expansion

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Abstract. Using standard group theoretical techniques we construct the exact wave function of the $[\mathbf{70}, 1^-]$ multiplet in the orbital, spin and flavor space. This symmetric wave function is compared to that customarily used in the $1/N_c$ expansion, which is asymmetric. The comparison is made by analyzing the matrix elements of various operators entering the mass formula. These matrix elements are calculated by the help of isoscalar factors of the permutation group, specially derived for this purpose as a function of N_c. We also compare two distinct methods used in the study of the $[\mathbf{70}, 1^-]$ multiplet. In the first method the generators are divided into two parts, one part acting on a subsystem of $N_c - 1$ quarks called core and another on the separated quark. In the second method the system is treated as a whole. We show that the latter is simpler and allows to clearly reveal the physically important operators in the mass formula.

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1. INTRODUCTION

For fifteen years the $1/N_c$ expansion of QCD, where N_c is the number of colors [1, 2] has revealed itself to be an interesting and powerful approach for studying baryon spectroscopy. The method is based on an exact contracted $\mathrm{SU}_c(2N_f)$ symmetry appearing in the large N_c limit, N_f being the number of flavors [3, 4]. For large N_c this algebra becomes the $\mathrm{SU}(2N_f)$ of the constituent quark model. Much work has been devoted to the ground state baryons where the operator reduction rules simplifies the expansion [5, 6]. Usually higher order corrections of order $1/N_c^2$ are neglected.

For excited states the problem is more complicated. To include orbital excitations, by analogy to the quark model, one can classify the large N_c baryons according to an extended symmetry given by the direct product $\mathrm{SU}(2N_f)\times\mathrm{O}(3)$. The group O(3) implies the introduction of a spin-orbit and a tensor interaction. It is a phenomenological fact that these contributions are small so that the breaking of this symmetry is also small. An open problem is to investigate the validity of the $1/N_c$ expansion in this extended symmetry.

In the language of the quark model the excited states can be grouped into excitation bands N. In the $1/N_c$ expansion, the baryon masses have been calculated for the lowest multiplets of all excited bands from $N = 1$ to 4. In these bands the multiplets belong either to the symmetric [**56**] or to the mixed symmetric [**70**] representation of SU(6). The symmetry of the wave function of excited baryons belonging to [**56**] representation allows a similar treatment as that of the ground state. The spin-flavor part being symmetric, the introduction of a symmetric orbital part does not modify the procedure. In the [**70**] representation (mainly the $[\mathbf{70}, 1^-]$), the situation turned out to be more complicated. There is a standard scheme [7] where the wave function is written as a product of a written on spa6.log. [stancu@ins symmetric ground state core composed of $N_c - 1$ quarks and an excited quark. In this approach, based on a Hartree picture, the $s^{N_c-1}p$ orbital part is not properly symmetrized and the excited quark is always the last quark. The flavor-spin part is also asymmetric and corresponds to a single term of the exact wave function.

Recently, a new scheme, which avoids the separation into a core and an excited quark has been suggested [8]. In that case the system is treated as a whole and the orbital-flavor-spin wave function is symmetric under any permutation of N_c quarks. Some convincing quantitative arguments in favor of this procedure can be found in Ref. [9] The exact $[\mathbf{70}, 1^-]$ wave function was written as a product of a core and a separated quark. Its orbital and the spin-flavor parts are mixed symmetric such as to recover the exact symmetric orbital-spin-flavor wave function. The procedure is described in Sec. 4.1.

Using group theoretical arguments here we examine the relation between the exact and the customarily used asymmetric wave function. We argue that the description of the system is unsatisfactory when the spin operator S^2 and the isospin T^2 operators are separated into independent parts in terms of operators acting separately on the core and on the excited quark. Much better results are obtained when we directly consider the operators S^2 and T^2 acting on the whole system. In addition we examine the role of an operator constructed from the product of S, T and G generators of SU(4).

2. SU(4) GENERATORS AS TENSOR OPERATORS

The SU(4) generators S_i, T_a and G_{ia}, globally denoted by E_{ia} [10], are components of an irreducible tensor operator which transforms according to the adjoint representation [211] of dimension **15** of SU(4). We recall that the SU(4) algebra is

$$\begin{gathered}
[S_i, T_a] = 0, \qquad [S_i, G_{ja}] = i\varepsilon_{ijk} G_{ka}, \\
[T_a, G_{ib}] = i\varepsilon_{abc} G_{ic}, \\
[S_i, S_j] = i\varepsilon_{ijk} S_k, \qquad [T_a, T_b] = i\varepsilon_{abc} T_c, \\
[G_{ia}, G_{jb}] = \tfrac{i}{4}\delta_{ij}\varepsilon_{abc} T_c + \tfrac{i}{4}\delta_{ab}\varepsilon_{ijk} S_k.
\end{gathered} \tag{1}$$

As one can see, the tensor operators E_{ia} are of three types: E_i (i = 1,2,3) which form the subalgebra of SU(2)-spin, E_a (a = 1,2,3) which form the subalgebra of SU(2)-isospin and E_{ia} which act both in the spin and the isospin spaces. They are related to S_i, T_a and G_{ia} ($i=1,2,3;\ a=1,2,3$) by

$$E_i = \frac{S_i}{\sqrt{2}}; \quad E_a = \frac{T_a}{\sqrt{2}}; \quad E_{ia} = \sqrt{2} G_{ia}. \tag{2}$$

The matrix elements of every E_{ia} between states belonging to the representation $[N_c-1,1]$ are given by

$$\begin{aligned} \langle [N_c-1,1] I' I'_3 S' S'_3 | E_{ia} | [N_c-1,1] I I_3 S S_3 \rangle &= \sqrt{C^{[N_c-1,1]}(\mathrm{SU}(4))} \\ \times \left(\begin{array}{cc|c} S & S^i & S' \\ S_3 & S^i_3 & S'_3 \end{array} \right) \left(\begin{array}{cc|c} I & I^a & I' \\ I_3 & I^a_3 & I'_3 \end{array} \right) & \left(\begin{array}{cc||c} [N_c-1,1] & [211] & [N_c-1,1] \\ SI & S^i I^a & S' I' \end{array} \right)_{\rho=1}, \end{aligned} \tag{3}$$

where $C^{[N_c-1,1]}(\mathrm{SU}(4)) = N_c(3N_c+4)/8$ is the eigenvalue of the SU(4) Casimir operator for the representation $[N_c-1,1]$. The three factors in the second line are respectively an SU(2)-spin Clebsch-Gordan coefficient (CG), an SU(2)-isospin CG and an SU(4) isoscalar factor. The necessary iscocalar factors for the derivation of the matrix elements of E_{ia} have been calculated by Hecht and Pang [10]. Here, the phases and notations have been adapted to our problem.

3. THE MASS OPERATOR

The mass operator M is defined as a linear combination of independent operators O_i

$$M = \sum_i c_i O_i, \tag{4}$$

where the coefficients c_i are reduced matrix elements that encode the QCD dynamics and are determined from a fit to the existing data. Here we are concerned with nonstrange baryons only. The building blocks of the operators O_i are the $\mathrm{SU}(2N_f)$ generators S_i, T_a and G_{ia} and the SO(3) generators ℓ_i. Their general form is

$$O_i = \frac{1}{N_c^{n-1}} O_\ell^{(k)} \cdot O_{SF}^{(k)}, \tag{5}$$

where $O_\ell^{(k)}$ is a k-rank tensor in SO(3) and $O_{SF}^{(k)}$ a k-rank tensor in SU(2)-spin, but invariant in $\mathrm{SU}(N_f)$. Thus O_i are rotational invariant. For the ground state one has $k=0$. The excited states also require $k=1$ and $k=2$ terms.

The spin-flavor (SF) operators $O_{SF}^{(k)}$ are combinations of $\mathrm{SU}(2N_f)$ generators, the lower index i in the left hand side of (5) representing a specific combination. Each n-body operator is multiplied by an explicit factor of $1/N_c^{n-1}$ resulting from the power counting rules. Some compensating N_c factors may arise in the matrix elements when O_i contains a coherent operator such as G^{ia} or T^a.

3.1. The symmetric core plus excited quark procedure

So far, for the baryons belonging to the $[\mathbf{70},\ell]$ multiplet the general practice was to consider that they consist of one distinguishable excited quark moving in the collective potential generated by N_c-1 ground state quarks [7], the latter subsystem being called core. The wave function of the core is symmetric both in the orbital (O) and flavor-spin (FS) spaces, which makes the treatment of the core analogous to that of ground state baryons. This description is known as the Hartree picture.

To proceed, one defines separate $\mathrm{SU}(2N_f)$ generators that act on the excited quark s^i, t^a and g^{ia} and S_c^i, T_c^a and G_c^{ia} that act on the core. Thus one has

$$S^i = s^i + S_c^i, \quad T^a = t^a + T_c^a, \quad G^{ia} = g^{ia} + G_c^{ia}. \tag{6}$$

As a consequence, the number of linearly independent operators O_i increases tremendously and the number of coefficients c_i, to be determined, becomes much larger than the experimental data available. For example, for the $[\mathbf{70},1^-]$ multiplet with $N_f=2$ one has 12 linearly independent operators up to order $1/N_c$ included [7]. For example, there is one operator of order N_c^1: $N_c\,\mathbb{1}$, three operators of order N_c^0: $\ell\cdot s$, $1/N_c\,\ell\cdot t\cdot G_c$, $1/N_c\,\ell^{(2)}\cdot g\cdot G_c$ and 8 operators of order N_c^{-1}: $1/N_c\,t\cdot T_c$, $1/N_c\,\ell\cdot S_c$, $1/N_c\,\ell\cdot g\cdot T_c$, $1/N_c\,S_c^2$, $1/N_c\,s\cdot S_c$, $1/N_c\,\ell^{(2)}\cdot s\cdot S_c$, $1/N_c^2\,\ell^{(2)}\cdot t\cdot\{S_c,G_c\}$ and $1/N_c^2\,\ell\cdot g\cdot\{S_c,G_c\}$. Then, in making the fit to the data, one faces the difficult situation of selecting among them the physically most dominant operators. We recall that there are only 7 nonstrange resonances belonging to this band. So one must select 7 out of 12 operators. Consequently, in selecting the operators one risks to make an arbitrary choice [7]. A much simpler method can be found, as shown below.

3.2. A simpler procedure

A simpler procedure is to avoid the splitting of the generators and the decoupling of the wave function and to consider instead only the global generators S^i, T^a and G^{ia} acting on the whole system of N_c quarks. However, the approach is not free of difficulties as the derivation of the matrix elements of the operators is more involved for a mixed symmetric wave function. Presently, the study of strange baryons is not possible. In the case of three flavors, one needs the analogue of Eq. (3) containing the corresponding SU(6) isoscalar. These factors have not been calculated yet.

In addition to the fact that it uses an exact wave function, this approach implies only seven independent operators up to order $\mathcal{O}(1/N_c)$ appearing in the mass operator: the order N_c operator $N_c\mathbb{1}$, three operators of order 1, $\ell\cdot s$, $1/N_c\,\ell^{(2)}\cdot G\cdot G$ and $1/N_c\,\ell\cdot T\cdot G$ and three operators of order $\mathcal{O}(1/N_c)$, namely $1/N_c\,S^2$, $1/N_c\,T^2$ and $1/N_c^2\,S\cdot T\cdot\,G$.

4. THE EXACT WAVE FUNCTION

The total wave function is the product of the orbital (O), the spin (S), the flavor (F) and the color (C) parts. The color part being always antisymmetric, in order to fulfill the Fermi statistics, the orbital-spin-flavor must be symmetric. As the mass operator does not involve color operators, the color being integrated out, we are concerned with the orbital-spin-flavor part only.

Here, as we are interested in the multiplet $[\mathbf{70},1^-]$, the orbital and the spin-flavor parts must have both the mixed symmetry $[N_c-1,1]$. In terms of inner products of the permutation group S_{N_c}, the wave function takes the form

$$|[N_c]1\rangle = \frac{1}{\sqrt{N_c-1}}\sum_Y |[N_c-1,1]Y\rangle_O |[N_c-1,1]Y\rangle_{FS}, \tag{7}$$

where Y is the corresponding Young tableau. Here we sum over the N_c-1 possible standard Young tableaux. The factor $1/\sqrt{N_c-1}$ represents the CG coefficient of S_{N_c} needed to construct a symmetric wave function $[N_c]$ from its mixed symmetric parts.

4.1. The decoupled wave function

In order to calculate matrix elements of the operators listed in the Sec. 3.1 one must decouple the wave function as well. Using the Racah's *factorization lemma*, it is possible to decouple N_cth quark from the rest. The S_{N_c} CG coefficients can be factorized into an isoscalar factor times a CG coefficient of S_{N_c-1}. In the following, we need to know the position of the N_cth quark inside a given Young tableau. In that purpose, one introduces the integer p which denotes the row where is the N_cth quark is located inside the Young tableau.

The exact $[\mathbf{70},1^-]$, but decoupled, wave function reads

$$\begin{aligned}
&|\ell S J J_3; I I_3\rangle = \\
&\quad \sum_{\substack{p,p',p'',\ell_c,\ell_q,m_\ell,m_q,\\ m_c,m_s,m_1,m_2,i_1,i_2}} a(p,\ell_c,\ell_q) \left(\begin{array}{cc|c} \ell_c & \ell_q & \ell \\ m_c & m_q & m_\ell \end{array}\right) \left(\begin{array}{cc|c} \ell & S & J \\ m_\ell & m_s & J_3 \end{array}\right) \\
&\quad \times K([f']p'[f'']p''|[N_c-1,1]p) \left(\begin{array}{cc|c} S_c & \frac{1}{2} & S \\ m_1 & m_2 & m_s \end{array}\right) \left(\begin{array}{cc|c} I_c & \frac{1}{2} & I \\ i_1 & i_2 & I_3 \end{array}\right) \\
&\quad \times |\ell_c m_c\rangle |S_c m_1\rangle |I_c i_1\rangle |\ell_q m_q\rangle |1/2 m_2\rangle |1/2 i_2\rangle,
\end{aligned} \tag{8}$$

where ℓ_c and ℓ_q represent the angular momenta of the core and of the decoupled quark respectively and where $a(p,\ell_c,\ell_q)$ are the one-body fractional parentage coefficients to decouple the N_cth quark from the rest in the orbital part. These are given by [9]

$$a(2,\ell_c=0,\ell_q=1) \quad = \quad \sqrt{\frac{N_c-1}{N_c}}, \tag{9}$$

$$a(2, \ell_c=1, \ell_q=0) = -\sqrt{\frac{1}{N_c}}, \quad (10)$$

$$a(1, \ell_c=1, \ell_q=0) = 1. \quad (11)$$

The isoscalar factors $K([f']p'[f'']p''|[N_c-1,1]p)$ used in Eq. (8) are given in Appendix A (Tables 10, 11 and 12). The columns corresponding to $p=1$ have been derived in Ref. [9]. If we compare Eq. (8) with Eq. (3.4) of Ref. [7] one can notice that in the latter only the terms with $p=2$ have been taken into account. Furthemore, as the core was assumed to be in the ground state, the authors had considered $a(2, \ell_c=1, \ell_q=0)=0$ and $a(2, \ell_c=0, \ell_q=1)=1$. Thus the wave function of Ref. [7] breaks S_{N_c} symmetry. As it represents only one part from the exact wave function we shall call it approximate or asymmetric.

Tables 1 and 2 show the matrix elements for some spin and the isospin operators respectively calculated with the exact and with the approximate wave function. One can notice that the analytic expressions are different. Consequently, one expects the c_i coefficients determined from the fit to the data to be different if we use the exact or the approximate wave function.

TABLE 1. Matrix elements of the spin operators calculated with the approximate (Ref. [7]) and the exact, Eq. (8), wave functions, with s and S_c defined by Eq. (6).

	$\langle s \cdot S_c \rangle$		$\langle S_c^2 \rangle$	
	Approx. w.f.	Exact w.f.	Approx. w.f.	Exact w.f.
$^2 8$	$-\frac{N_c+3}{4N_c}$	$-\frac{3(N_c-1)}{4N_c}$	$\frac{N_c+3}{2N_c}$	$\frac{3(N_c-1)}{2N_c}$
$^4 8$	$\frac{1}{2}$	$-\frac{3(N_c-5)}{4N_c}$	2	$\frac{3(3N_c-5)}{2N_c}$
$^2 10$	-1	$-\frac{3(N_c-1)}{4N_c}$	2	$\frac{3(N_c-1)}{2N_c}$

TABLE 2. Matrix elements of the isospin operators calculated with the approximate (Ref. [7]) and the exact ,Eq. (8), wave functions, with t and T_c defined by Eq. (6).

	$\langle t \cdot T_c \rangle$		$\langle T_c^2 \rangle$	
	Approx. w.f.	Exact w.f.	Approx. w.f.	Exact w.f.
$^2 8$	$-\frac{N_c+3}{4N_c}$	$-\frac{3(N_c-1)}{4N_c}$	$\frac{N_c+3}{2N_c}$	$\frac{3(N_c-1)}{2N_c}$
$^4 8$	-1	$-\frac{3(N_c-1)}{4N_c}$	2	$\frac{3(N_c-1)}{2N_c}$
$^2 10$	$\frac{1}{2}$	$-\frac{3(N_c-5)}{4N_c}$	2	$\frac{3(3N_c-5)}{2N_c}$

4.2. The global wave function

As already mentioned above, one can write the exact $[\mathbf{70}, 1^-]$ states without decoupling them into a core and an excited quark. If there is no decoupling, there is no need to specify Y, the matrix elements being identical for all Y's, due to

Weyl's duality between a linear group and a symmetric group in a given tensor space[1]. Then the explicit form of a wave function of total angular momentum $\vec{J}=\vec{\ell}+\vec{S}$ and isospin I is

$$|\ell S J J_3; I I_3\rangle = \sum_{m_\ell,m_s} \left(\begin{array}{cc|c} \ell & S & J \\ m_\ell & m_s & J_3 \end{array} \right) |[N_c-1,1]\ell m_\ell\rangle |[N_c-1,1] S m_s I I_3\rangle, \quad (12)$$

each term containing an SU(2) CG coefficient, an orbital part $|[N_c-1,1]\ell m_\ell\rangle$ an a spin-flavor part $|[N_c-1,1]Sm_sII_3\rangle$.

5. RESULTS

Here we present the results obtained from different fits to the experimental data. In the fits, the seven nonstrange resonances have been taken into account: ${}^2N_{1/2}(1538\pm18)$, ${}^4N_{1/2}(1660\pm20)$, ${}^2N_{3/2}(1523\pm8)$, ${}^4N_{3/2}(1700\pm50)$, ${}^4N_{5/2}(1678\pm 8)$, ${}^2\Delta_{1/2}(1645\pm30)$ and ${}^2\Delta_{3/2}(1720\pm50)$.

In a first stage, we describe the fits obtained when we use the exact decoupled wave function. Afterwards, the results obtained with the global wave function are presented.

In each case, we follow the spirit of the Hartree picture which leads to a one-body spin-orbit operator $\ell\cdot s$. Its matrix elements are naturally of order N_c^0.

5.1. With the decoupled wave function

Tables 3–6 show the four different fits considered. Each time, the results obtained with the exact decoupled wave function are compared to the ones obtained with the approximate wave function.

In Table 3, we decouple the spin and the isospin operators. The operator $1/N_c\, T_c\cdot T_c$ is not present because its matrix elements are identical to of those $1/N_c\, S_c\cdot S_c$ for the approximate wave function (see Tables 1 and 2). This is apparently a practical advantage in the decoupling scheme but it has considerable physical disadvantages. One can notice that even if the $\chi^2_{\rm dof}$ is satisfactory, the fit is very bad. Indeed, the value of c_1 is under-evaluated with respect to the commonly found value of around 500 MeV and the values of c_3 and c_4 are exceedingly large and of opposite signs, which suggest some compensation.

The fits presented in Tables 4 and 5 seem better. Here the linear combinations $2s\cdot S_c+S_c\cdot S_c+3/4=S^2$ or $2t\cdot T_c+T_c\cdot T_c+3/4=T^2$ have been introduced. The coefficient c_1 has recovered its common value and the coefficients c_3' or c_5' have reasonable sizes, being about 70 MeV smaller for the exact wave function than for the approximate one. The quark-core operators $1/N_c\, s\cdot S_c$ or $1/N_c\, t\cdot T_c$ are still

[1] see Ref. [11], Sec 4.5.

problematic because the value of their respective coefficients are too high, of order 500 MeV.

The last fit shown in Table 6 correct these problems. All the coefficients have their natural sizes. This shows the necessity to consider the isospin-isospin operator on the same footing as the spin-spin operator. The values obtained with the exact wave function and the approximate one are identical in this case because the matrix elements of the operators considered are the same for the two wave functions. By construction, in both cases they are eigenfunctions of the total spin and isospin operators.

TABLE 3. List of operators O_i and coefficients c_i obtained in the numerical fit to the 7 known experimental masses of the lowest negative parity resonances (see text). For the operators O_3, O_4 and O_5 we use the matrix elements from Tables 1 and 2.

O_i	c_i(MeV) with approx. w.f.	c_i(MeV) with w.f. Eq. (8)
$O_1 = N_c \mathbb{1}$	211 ± 23	299 ± 20
$O_2 = \ell^i s^i$	3 ± 15	3 ± 15
$O_3 = \frac{1}{N_c} s^i S_c^i$	-1486 ± 141	-1096 ± 125
$O_4 = \frac{1}{N_c} S_c^i S_c^i$	1182 ± 74	1545 ± 122
$O_5 = \frac{1}{N_c} t^a T_c^a$	-1508 ± 149	417 ± 79
$\chi^2_{\rm dof}$	1.56	1.56

TABLE 4. Same as Table 3 but for O_3', which combines O_3 and O_4 instead of using them separately.

O_i	c_i(MeV) with approx. w.f.	c_i(MeV) with w.f. Eq. (8)
$O_1 = N_c \mathbb{1}$	513 ± 4	519 ± 5
$O_2 = \ell^i s^i$	3 ± 15	3 ± 15
$O_3' = \frac{1}{N_c} \left(2 s^i S_c^i + S_c^i S_c^i + \frac{3}{4}\right)$	219 ± 19	150 ± 11
$O_5 = \frac{1}{N_c} t^a T_c^a$	417 ± 80	417 ± 80
$\chi^2_{\rm dof}$	1.04	1.04

TABLE 5. Same as Table 4 but combining isospin operators instead of spin operators.

O_i	c_i(MeV) with approx. w.f.	c_i(MeV) with w.f. Eq. (8)
$O_1 = N_c \mathbb{1}$	516 ± 3	522 ± 3
$O_2 = \ell^i s^i$	3 ± 15	3 ± 15
$O_3 = \frac{1}{N_c} s^i S_c^i$	450 ± 33	450 ± 33
$O_5' = \frac{1}{N_c} \left(2 t^a T_c^a + T_c^a T_c^a + \frac{3}{4}\right)$	214 ± 28	139 ± 27
$\chi^2_{\rm dof}$	1.04	1.04

TABLE 6. Fit with global operators proportional to the SU(2)-spin and SU(2)-isospin Casimir operators acting on the whole system (see text).

O_i	c_i(MeV) with approx. w.f.	c_i(MeV) with w.f. Eq. (8)
$O_1 = N_c \mathbb{1}$	484 ± 4	484 ± 4
$O_2 = \ell^i s^i$	3 ± 15	3 ± 15
$O'_3 = \frac{1}{N_c}\left(2s^i S_c^i + S_c^i S_c^i + \frac{3}{4}\right)$	150 ± 11	150 ± 11
$O'_5 = \frac{1}{N_c}\left(2t^a T_c^a + T_c^a T_c^a + \frac{3}{4}\right)$	139 ± 27	139 ± 27
χ^2_{dof}	1.04	1.04

5.2. With the global wave function

With the simplified procedure described in Sec. 3.2 we can analyse the role of every of the seven independent operators introduced there. Table 7 shows six different fits to the experiment. The operators O_5, O_6 and O_7 are normalized to allow their coefficients c_i to have a natural size. As already emphasized, the Fits 1–4 indicate that the coefficients of O_3 and O_4 have similar values.

The partial contributions and the theoretical masses obtained from the Fits 1 and 6 are presented in Tables 8 and 9 respectively. From Table 8 one can notice that the isospin-isospin operator in Δ masses plays a comparable role to the spin-spin operator in N^* resonances. This was impossible to observe in the symmetric core + excited quark procedure where the isospin-isospin operators were always ignored, for reasons explained above. From Table 9 one can see that the operator O_7, never included before, is dominant in all resonances except ${}^2N_J$. This is a new finding, to be algebraically understood.

The operators O_5 and O_6 do not seem to play an important role because, in addition to the fact that their coefficients are small and have an error bar comparable to their central values, their removal from the fit does not deteriorate it too badly. This justifies the previous choice presented in Section 5.1 where O_5 and O_6 were neglected. Of course, Fit 4 is identical to the one shown in Table 6.

Table 7 does not include a fit with O_3, O_4 and O_7 together. In our calculations we found that the simultaneous presence of O_3, O_4 and O_7 leads to a $\chi^2_{\text{dof}} \approx 2$. In this case the coefficients c_3 and c_4 become a bit higher, of the order 270 MeV and c_7 becomes negative suggesting a possible compensation with the contributions of O_3 and O_4. This suggests that, by construction, O_7 contains part of the contribution of the spin-spin and isospin-isospin interactions. As mentioned above, the role of O_7 needs more investigation.

6. CONCLUSIONS

In principle, both the core + excited quark (Sec. 3.1) or the global (Sec. 3.2) procedures are legitimate as long as they are combined with adequately constructed wave functions. The core + excited quark procedure was the first attempt to study

TABLE 7. List of operators and the coefficients resulting from numerical fits using the global wave function. The values of c_i are indicated under the headings Fit n, in each case.

Operator	Fit 1 (MeV)	Fit 2 (MeV)	Fit 3 (Mev)	Fit 4 (MeV)	Fit 5 (MeV)	Fit 6 (MeV)
$O_1 = N_c \mathbb{1}$	481 ± 5	482 ± 5	484 ± 4	484 ± 4	498 ± 3	495 ± 3
$O_2 = \ell^i s^i$	-31 ± 26	-20 ± 23	-12 ± 20	3 ± 15	38 ± 34	-30 ± 25
$O_3 = \frac{1}{N_c} S^i S^i$	161 ± 16	149 ± 11	163 ± 16	150 ± 11	156 ± 16	
$O_4 = \frac{1}{N_c} T^a T^a$	169 ± 36	170 ± 36	141 ± 27	139 ± 27		
$O_5 = \frac{15}{N_c} \ell^{(2)ij} G^{ia} G^{ja}$	-29 ± 31		-34 ± 30		-34 ± 31	-32 ± 29
$O_6 = \frac{3}{N_c} \ell^i T^a G^{ia}$	32 ± 26	35 ± 26			-67 ± 30	28 ± 20
$O_7 = \frac{3}{N_c^2} S^i T^a G^{ia}$						649 ± 61
χ^2_{dof}	0.43	0.68	0.94	1.04	11.5	0.24

excited states in the $1/N_c$ expansion and naturally it has been proposed to make the problem tractable at that time, by reducing it to the knowledge of ground state matrix elements of the operators from the mass formula. Presently it seems obsolete. We have shown that the global procedure is much more advantageous. It involves a smaller number of independent operators which allow to clearly identify

TABLE 8. The partial contribution and the total mass (MeV) predicted by the $1/N_c$ expansion using Fit 1 and the global wave function. The last two columns give the empirically known masses, name and status.

	Part. contrib. (MeV)						Total (MeV)	Exp. (MeV)	Name, status
	c_1O_1	c_2O_2	c_3O_3	c_4O_4	c_5O_5	c_6O_6			
$^2N_{\frac{1}{2}}$	1444	10	40	42	0	-8	1529 ± 11	1538 ± 18	$S_{11}(1535)$****
$^4N_{\frac{1}{2}}$	1444	26	201	42	-31	-20	1663 ± 20	1660 ± 20	$S_{11}(1650)$****
$^2N_{\frac{3}{2}}$	1444	-5	40	42	0	4	1525 ± 8	1523 ± 8	$D_{13}(1520)$****
$^4N_{\frac{3}{2}}$	1444	10	201	42	25	-8	1714 ± 45	1700 ± 50	$D_{13}(1700)$***
$^4N_{\frac{5}{2}}$	1444	-16	201	42	-6	12	1677 ± 8	1678 ± 8	$D_{15}(1675)$****
$^2\Delta_{\frac{1}{2}}$	1444	-10	40	211	0	-40	1645 ± 30	1645 ± 30	$S_{31}(1620)$****
$^2\Delta_{\frac{3}{2}}$	1444	5	40	211	0	20	1720 ± 50	1720 ± 50	$D_{33}(1700)$****

the physically dominant operators in the mass formula. Our conclusion is that the spin operator $1/N_c\, S \cdot S$ is dominant in N^* resonances, that the isospin operator $1/N_c\, T \cdot T$ is equally important in Δ resonances and that $1/N_c\, S \cdot T \cdot G$ plays about the same dominant role both in N^* and Δ resonances except for ^2N_J, the contribution to the mass being of the order of 200 MeV in all cases. More work should be done algebraically in order to understand the role of $1/N_c\, S \cdot T \cdot G$.

Moreover, we found that all operators containing the O(3) generators bring only small contributions to the mass, from 4 MeV to 42 MeV. This finding is consistent with the constituent quark model assumptions about the feebleness of the spin-

TABLE 9. The partial contribution and the total mass (MeV) predicted by the $1/N_c$ expansion using Fit 6 and the global wave function. The last two columns give the empirically known masses, name and status.

	Part. contrib. (MeV)					Total (MeV)	Exp. (MeV)	Name, status
	c_1O_1	c_2O_2	c_5O_5	c_6O_6	c_7O_7			
$^2N_{\frac{1}{2}}$	1486	10	0	-7	41	1529 ± 11	1538 ± 18	$S_{11}(1535)$****
$^4N_{\frac{1}{2}}$	1486	25	-33	-18	203	1663 ± 20	1660 ± 20	$S_{11}(1650)$****
$^2N_{\frac{3}{2}}$	1486	-5	0	4	41	1525 ± 7	1523 ± 8	$D_{13}(1520)$****
$^4N_{\frac{3}{2}}$	1486	10	26	-7	203	1718 ± 41	1700 ± 50	$D_{13}(1700)$***
$^4N_{\frac{5}{2}}$	1486	-15	7	11	203	1677 ± 8	1678 ± 8	$D_{15}(1675)$****
$^2\Delta_{\frac{1}{2}}$	1486	-10	0	-35	203	1643 ± 29	1645 ± 30	$S_{31}(1620)$****
$^2\Delta_{\frac{3}{2}}$	1486	5	0	18	203	1711 ± 24	1720 ± 50	$D_{33}(1700)$****

orbit and the smallness of the tensor interaction.

We have shown that the separation core + quark procedure fails to emphasize the role of the isospin operator. This is due to the inherent structure of the asymmetric ground state core + excited quark wave function [7] which leads to equal matrix elements for S_c^2 and T_c^2. Then the remaining part of the isospin interaction $1/N_c\, t \cdot T_c$ becomes exceedingly large if included in the fit (Table 3) and it is not surprising that in all previous studies (see Ref. [9] for a review) it has been totally ignored.

In conclusion, the simple procedure we advocate here brings much more physical insight into the study of the nonstrange $[\mathbf{70}, 1^-]$ baryons in the $1/N_c$ expansion. It is an urgent need to determine the isoscalar factors of SU(6) for mixed symmetric representations $[N_c - 1, 1]$ in order to extend Eq. (3) to SU(6) and apply it to strange baryons.

A. ISOSCALAR FACTORS

Here we reproduce the isoscalar factors needed to construct the exact decoupled wave function (see Eq. (8)). Detailed information can be found in Refs. [9, 11, 12]

TABLE 10. Isoscalar factors $K([f']p'[f'']p''|[f]p)$ for $S=I=1/2$, corresponding to $^{2}8$ when $N_c=3$. The second column gives results for $p=1$ and the third for $p=2$.

$[f']p'[f'']p''$	$[N_c-1,1]1$	$[N_c-1,1]2$
$\left[\frac{N_c+1}{2},\frac{N_c-1}{2}\right]1\left[\frac{N_c+1}{2},\frac{N_c-1}{2}\right]1$	0	$-\sqrt{\frac{3(N_c-1)}{4N_c}}$
$\left[\frac{N_c+1}{2},\frac{N_c-1}{2}\right]2\left[\frac{N_c+1}{2},\frac{N_c-1}{2}\right]2$	$\sqrt{\frac{N_c-3}{2(N_c-2)}}$	$\sqrt{\frac{N_c+3}{4N_c}}$
$\left[\frac{N_c+1}{2},\frac{N_c-1}{2}\right]2\left[\frac{N_c+1}{2},\frac{N_c-1}{2}\right]1$	$-\frac{1}{2}\sqrt{\frac{N_c-1}{N_c-2}}$	0
$\left[\frac{N_c+1}{2},\frac{N_c-1}{2}\right]1\left[\frac{N_c+1}{2},\frac{N_c-1}{2}\right]2$	$-\frac{1}{2}\sqrt{\frac{N_c-1}{N_c-2}}$	0

TABLE 11. Isoscalar factors $K([f']p'[f'']p''|[f]p)$ for $S=3/2,\ I=1/2$, corresponding to $^{4}8$ when $N_c=3$. The second column gives results for $p=1$ and the third for $p=2$.

$[f']p'[f'']p''$	$[N_c-1,1]1$	$[N_c-1,1]2$
$\left[\frac{N_c+3}{2},\frac{N_c-3}{2}\right]1\left[\frac{N_c+1}{2},\frac{N_c-1}{2}\right]1$	$\frac{1}{2}\sqrt{\frac{(N_c-1)(N_c+3)}{N_c(N_c-2)}}$	0
$\left[\frac{N_c+3}{2},\frac{N_c-3}{2}\right]2\left[\frac{N_c+1}{2},\frac{N_c-1}{2}\right]2$	$\frac{1}{2}\sqrt{\frac{5(N_c-1)(N_c-3)}{2N_c(N_c-2)}}$	0
$\left[\frac{N_c+3}{2},\frac{N_c-3}{2}\right]1\left[\frac{N_c+1}{2},\frac{N_c-1}{2}\right]2$	$\frac{1}{2}\sqrt{\frac{(N_c-3)(N_c+3)}{2N_c(N_c-2)}}$	1
$\left[\frac{N_c+3}{2},\frac{N_c-3}{2}\right]2\left[\frac{N_c+1}{2},\frac{N_c-1}{2}\right]1$	0	0

TABLE 12. Isoscalar factors $K([f']p'[f'']p''|[f]p)$ for $S=1/2,\ I=3/2$, corresponding to $^{2}10$ when $N_c=3$. The second column gives results for $p=1$ and the third for $p=2$.

$[f']p'[f'']p''$	$[N_c-1,1]1$	$[N_c-1,1]2$
$\left[\frac{N_c+1}{2},\frac{N_c-1}{2}\right]1\left[\frac{N_c+3}{2},\frac{N_c-3}{2}\right]1$	$\frac{1}{2}\sqrt{\frac{(N_c-1)(N_c+3)}{N_c(N_c-2)}}$	0
$\left[\frac{N_c+1}{2},\frac{N_c-1}{2}\right]2\left[\frac{N_c+3}{2},\frac{N_c-3}{2}\right]2$	$\frac{1}{2}\sqrt{\frac{5(N_c-1)(N_c-3)}{2N_c(N_c-2)}}$	0
$\left[\frac{N_c+1}{2},\frac{N_c-1}{2}\right]2\left[\frac{N_c+3}{2},\frac{N_c-3}{2}\right]1$	$\frac{1}{2}\sqrt{\frac{(N_c-3)(N_c+3)}{2N_c(N_c-2)}}$	1
$\left[\frac{N_c+1}{2},\frac{N_c-1}{2}\right]1\left[\frac{N_c+3}{2},\frac{N_c-3}{2}\right]2$	0	0

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