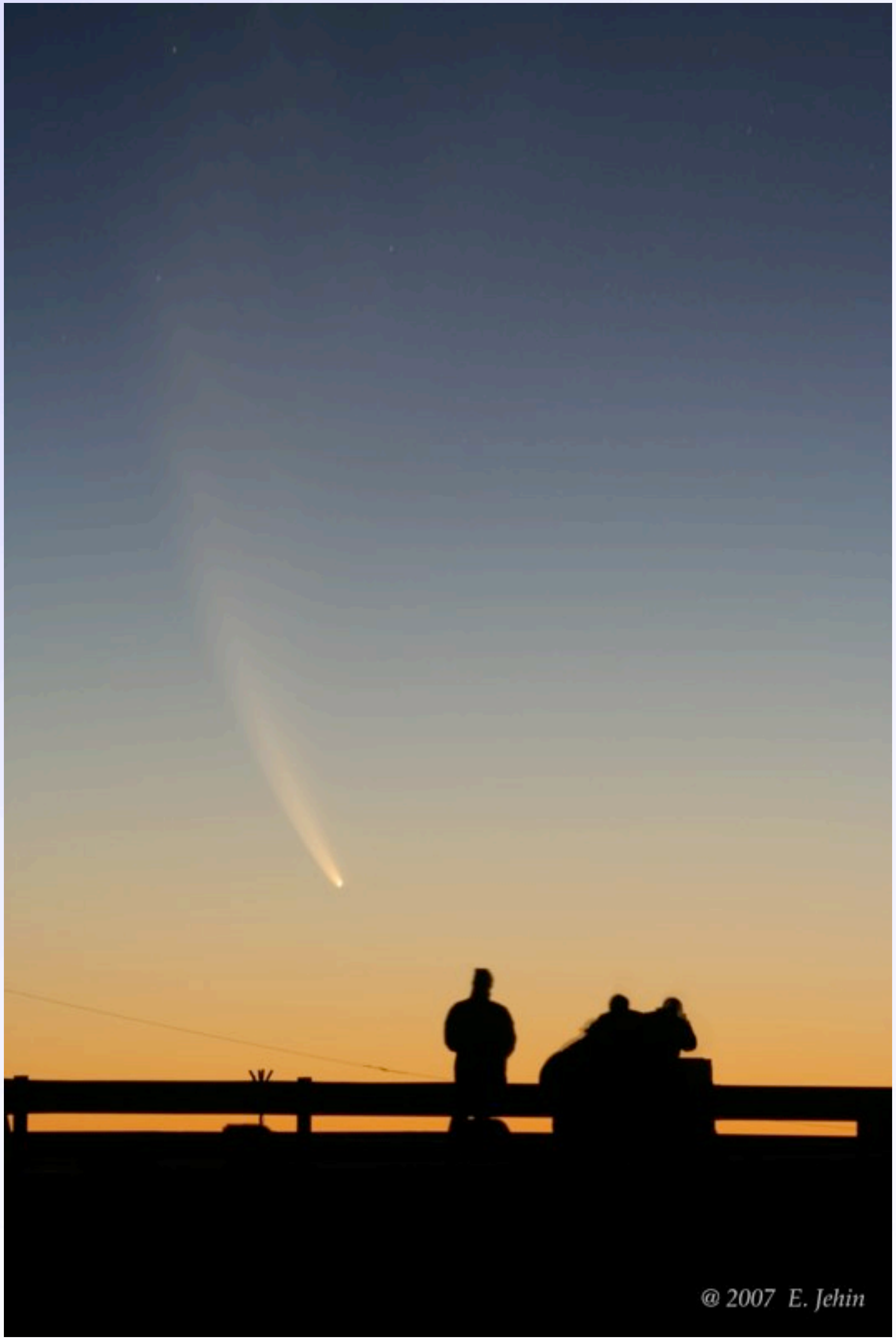
10ème cours de Mécanique Analytique (1/12/2016)

Comète Mc Naught 2006

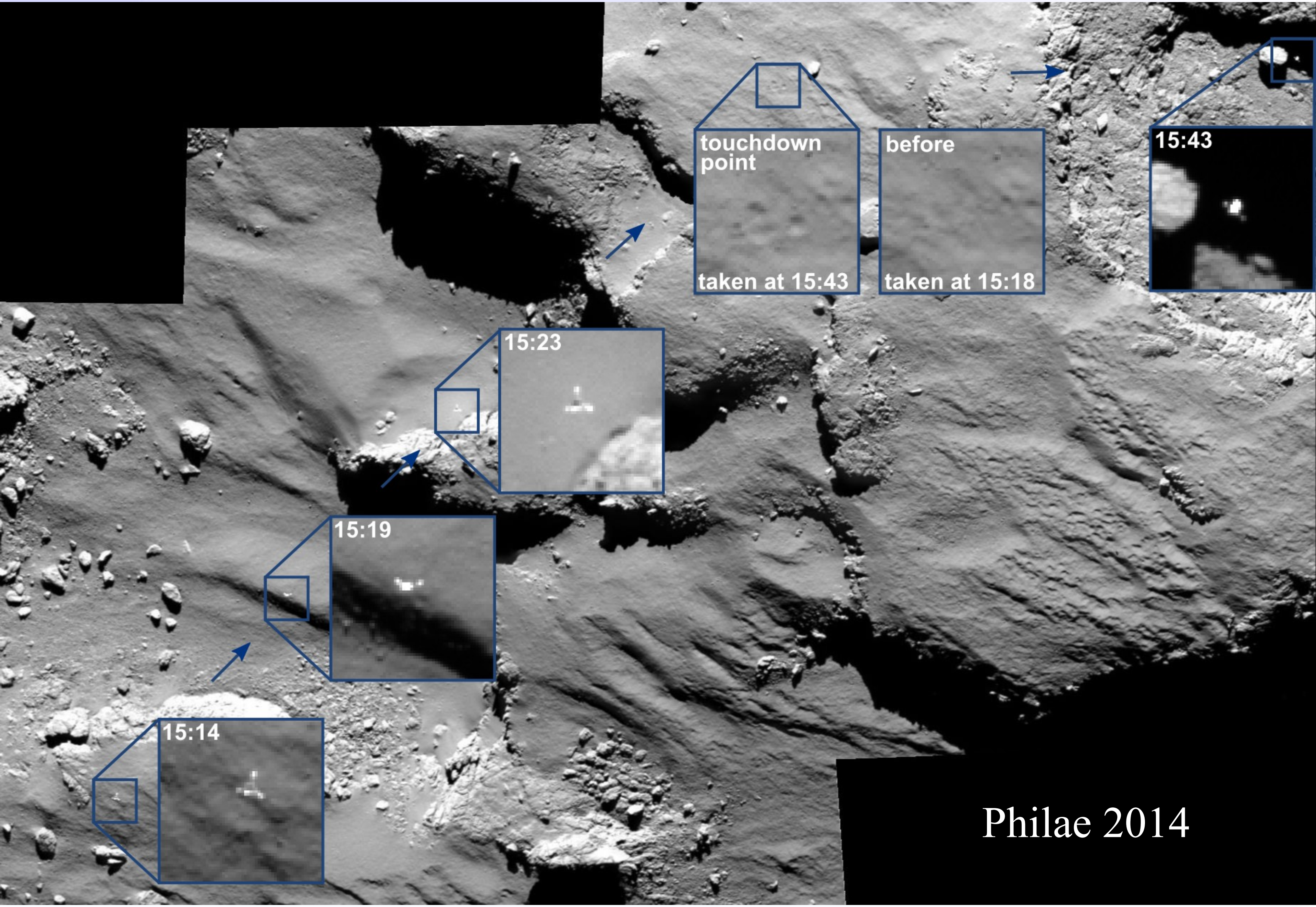

@ 2007 E. Jehin


@ 2007 E. Jehin


@ 2007 E. Jehin


@ 2007 E. Jehin

Philae 2014



touchdown point
taken at 15:43
before
taken at 15:18
15:43
15:23
15:19
15:14

Philae 2014

- 2.3 Le principe variationnel d'Hamilton modifié

$$L(q,\dot{q}(q,p,t),t) = p_i\dot{q}_i(q,p,t) - H(q,p,t) \quad (2.11)$$

$$\delta \int_{t_1}^{t_2} (p_i\dot{q}_i - H(q,p,t))\, dt = 0 \quad (2.12)$$

$$\frac{d}{dt}\left(\frac{\partial(p_i\dot{q}_i - H)}{\partial \dot{q}_k}\right) = \frac{\partial(p_i\dot{q}_i - H)}{\partial q_k} \Rightarrow \boxed{\dot{p}_k = -\frac{\partial H}{\partial q_k}} \quad (2.13)$$

$$\frac{d}{dt}\left(\frac{\partial(p_i\dot{q}_i - H)}{\partial \dot{p}_k}\right) = \frac{\partial(p_i\dot{q}_i - H)}{\partial p_k} \Rightarrow \boxed{0 = \dot{q}_k - \frac{\partial H}{\partial p_k}} \quad (2.14)$$

• 2.4 Transformations canoniques

$$\dot{p}_k = -\frac{\partial H}{\partial q_k} = 0$$ (2.15)

$$H(q_1, ..., q_{k-1}, q_{k+1}, ...q_f, p_1 ..., p_{k-1}, \alpha_k, p_{k+1}, ..., p_f, t)$$

si $H = H(p_1, ..., p_f, t)$

$\dot{p}_i = 0$ $p = \alpha_i =$ constante $(i = 1, ..., f)$

$$\dot{q}_i = \left.\frac{\partial H}{\partial p_i}\right|_{p_i=\alpha_i} = \nu_i(t)$$ (2.16) $\nu_i \; (i = 1, ..., f)$

- 2.4 Transformations canoniques

$$\dot{q}_i = \left.\frac{\partial H}{\partial p_i}\right|_{p_i=\alpha_i} = \nu_i(t) \quad (2.16)$$

$\nu_i \, (i = 1, ..., f)$

$$q_i = \int_{t_0}^{t} \nu_i(t')\, dt' + \beta_i \quad (2.17)$$

si le hamiltonien n'est pas une fonction explicite du temps

$$q_i = \nu_i t + \beta_i.$$

$$Q_i = Q_i(q, t) \quad (2.18)$$

$q = \{q_1, ..., q_f \,;\, i = 1, ..., f\}$ $\quad Q = \{Q_1, ..., Q_f \,;\, i = 1, ..., f\}$

• 2.4 Transformations canoniques

$$\begin{cases} Q_i = Q_i(q, p, t) \\ P_i = P_i(q, p, t) \end{cases} \qquad (2.19)$$

$$\boxed{\dot{Q}_i = \frac{\partial K}{\partial P_i} \qquad \dot{P}_i = -\frac{\partial K}{\partial Q_i} \qquad (i = 1, \ldots, f)} \qquad (2.20)$$

$$\delta \int_{t_1}^{t_2} \left(p_i \dot{q}_i - H(q, p, t)\right) dt = 0$$

$$\delta \int_{t_1}^{t_2} \left(P_i \dot{Q}_i - K(Q, P, t)\right) dt = 0 \qquad (2.21)$$

$$p_i \dot{q}_i - H = P_i \dot{Q}_i - K + \frac{dF}{dt} \qquad (2.22)$$

• 2.4 Transformations canoniques

a) Le premier choix possible est :

$$F = F_1(q, Q, t) \quad (2.23)$$

$$\begin{aligned} p_i\dot{q}_i - H &= P_i\dot{Q}_i - K + \frac{dF_1}{dt} \\ &= P_i\dot{Q}_i - K + \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial q_i}\dot{q}_i + \frac{\partial F_1}{\partial Q_i}\dot{Q}_i \end{aligned} \quad (2.24)$$

$$p_i = \frac{\partial F_1}{\partial q_i} \quad (2.25a)$$

$$P_i = -\frac{\partial F_1}{\partial Q_i} \quad (2.25b)$$

$$K = H + \frac{\partial F_1}{\partial t} \quad (2.25c)$$

$$\begin{cases} Q_i = Q_i(q, p, t) \\ P_i = P_i(q, p, t) \end{cases} \quad (2.19)$$

• 2.4 Transformations canoniques

b) Un autre choix intéressant possible pour la fonction F est :

$$F = F_2(q, P, t) - Q_i P_i \qquad (2.26)$$

$$p_i \dot{q}_i - H = P_i \dot{Q}_i - K + \frac{dF_2}{dt} \qquad (2.27)$$

$$p_i = \frac{\partial F_2}{\partial q_i} \qquad (2.28a)$$

$$Q_i = \frac{\partial F_2}{\partial P_i} \qquad (2.28b)$$

$$K = H + \frac{\partial F_2}{\partial t} \qquad (2.28c)$$

$$\begin{cases} Q_i = Q_i(q, p, t) \\ P_i = P_i(q, p, t) \end{cases} \qquad (2.19)$$

• 2.4 Transformations canoniques

c) $$F = q_i p_i + F_3(p, Q, t) \qquad (2.29)$$

$$q_i = -\frac{\partial F_3}{\partial p_i} \qquad (2.30a)$$

$$P_i = -\frac{\partial F_3}{\partial Q_i} \qquad (2.30b)$$

$$K = H + \frac{\partial F_3}{\partial t} \qquad (2.30c)$$

• 2.4 Transformations canoniques

d) $F = q_i p_i - Q_i P_i + F_4(p, P, t)$ (2.31)

$$q_i = -\frac{\partial F_4}{\partial p_i} \quad (2.32a)$$

$$Q_i = \frac{\partial F_4}{\partial P_i} \quad (2.32b)$$

$$K = H + \frac{\partial F_4}{\partial t} \quad (2.32c)$$

- 2.4 Transformations canoniques

$$\begin{cases} Q_i = Q_i(q,p,t) \\ P_i = P_i(q,p,t) \end{cases} \quad (2.19)$$

$$\boxed{\dot{Q}_i = \frac{\partial K}{\partial P_i} \qquad \dot{P}_i = -\frac{\partial K}{\partial Q_i} \qquad (i = 1, \ldots, f)} \quad (2.20)$$

$$\delta \int_{t_1}^{t_2} (p_i \dot{q}_i - H(q,p,t))\, dt = 0$$

$$\delta \int_{t_1}^{t_2} (P_i \dot{Q}_i - K(Q,P,t))\, dt = 0 \quad (2.21)$$

$$p_i \dot{q}_i - H = P_i \dot{Q}_i - K + \frac{dF}{dt} \quad (2.22)$$

$$F = F_1(q, Q, t)$$

$$F = F_2(q, P, t) - Q_i P_i$$

$$F = q_i p_i + F_3(p, Q, t)$$

$$F = q_i p_i - Q_i P_i + F_4(p, P, t)$$

• 2.5 Exemples de transformations canoniques

a) $F_2(q, P) = q_i P_i$ (2.33)

$$p_i = \frac{\partial F_2}{\partial q_i} = P_i \quad (2.34a)$$

$$Q_i = \frac{\partial F_2}{\partial P_i} = q_i \quad (2.34b)$$

$$K = H \quad (2.34c)$$

b) $F_1 = q_k Q_k$ (2.35)

$$p_i = \frac{\partial F_1}{\partial q_i} = Q_i \quad (2.36a)$$

$$P_i = -\frac{\partial F_1}{\partial Q_i} = -q_i \quad (2.36b)$$

$$K(Q, P) = H(-P, Q) \quad (2.36c)$$

• 2.5 Exemples de transformations canoniques

c) Considérons le problème classique de l'*oscillateur harmonique*

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2 \qquad (f = 1) \qquad (2.37)$$

$$F_1 = \frac{m\omega q^2}{2} \,\mathrm{cotg} Q \qquad (2.38)$$

$$p = \frac{\partial F_1}{\partial q} = m\omega q \,\mathrm{cotg} Q \qquad (2.39a)$$

$$P = -\frac{\partial F_1}{\partial Q} = \frac{m\omega q^2}{2\sin^2 Q} \qquad (2.39b)$$

$$q = \sqrt{\frac{2P}{m\omega}} \sin Q \qquad (2.40a)$$

$$p = \sqrt{2m\omega P} \cos Q \qquad (2.40b)$$

$$K = \omega P \qquad (2.41)$$

• 2.5 Exemples de transformations canoniques

$$K = \omega P \quad (2.41)$$

$$\dot{P} = -\frac{\partial K}{\partial Q} = 0 \quad (2.42a)$$

$$\dot{Q} = \frac{\partial K}{\partial P} = \omega \quad (2.42b)$$

$$P = \alpha = \text{constante} \qquad \text{et :} \qquad Q = \omega t + \beta \quad (2.43)$$

$$q = \sqrt{\frac{2P}{m\omega}} \sin Q \;\; + \beta) \quad (2.44)$$

$$p = \sqrt{2m\omega P} \cos Q$$

• 2.6 L'approche symplectique des transformations canoniques

vecteur colonne η à $2f$ éléments

$$\eta_i = q_i \quad , \quad n_{i+f} = p_i \quad i \leq f$$

(2.45)

le vecteur colonne $\frac{\partial H}{\partial \eta}$

$$\left(\frac{\partial H}{\partial \eta}\right)_i = \frac{\partial H}{\partial q_i} \quad , \quad \left(\frac{\partial H}{\partial \eta}\right)_{i+f} = \frac{\partial H}{\partial p_i} \quad i \leq f$$

(2.46)

J la matrice carrée $2f \times 2f$

$$J = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}$$

(2.47)

$$\dot{\eta} = J \frac{\partial H}{\partial \eta}$$

(2.48)

• 2.6 L'approche symplectique des transf. canoniques

J la matrice carrée $2f \times 2f$

$$J = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} \quad (2.47)$$

$$J^2 = -\mathbf{1} \quad (2.49a)$$

$$J\tilde{J} = \mathbf{1} \quad (2.49b)$$

$$\tilde{J} = -J = J^{-1} \quad (2.49c)$$

$$dtm\,(J) = +1 \quad (2.49d)$$

• 2.6 L'approche symplectique des transf. canoniques

$$\mathbf{J}_{ij} = 1\ \delta_{i,j-f}\ \Pi_{[1\leq i\leq f, f+1\leq j\leq 2f]} - 1\ \delta_{i-f,j}\ \Pi_{[f+1\leq i\leq 2f, 1\leq j\leq f]}$$

$$\tilde{\mathbf{J}}_{ij} = \mathbf{J}_{ji}$$

$$= 1\ \delta_{j,i-f}\ \Pi_{[1\leq j\leq f, f+1\leq i\leq 2f]} - 1\ \delta_{j-f,i}\ \Pi_{[f+1\leq j\leq 2f, 1\leq i\leq f]}$$

$$= -\left(1\ \delta_{j-f,i}\ \Pi_{[f+1\leq j\leq 2f, 1\leq i\leq f]} - 1\ \delta_{j,i-f}\ \Pi_{[1\leq j\leq f, f+1\leq i\leq 2f]}\right)$$

$$= -\left(1\ \delta_{i,j-f}\ \Pi_{[1\leq i\leq f, f+1\leq j\leq 2f]} - 1\ \delta_{i-f,j}\ \Pi_{[f+1\leq i\leq 2f, 1\leq j\leq f]}\right)$$

$$= -\mathbf{J}_{ij}$$

$$\Rightarrow \quad \tilde{\mathbf{J}} = -\mathbf{J}$$

• 2.6 L'approche symplectique des transf. canoniques

$$\mathbf{J}_{ij} = 1\ \delta_{i,j-f}\ \Pi_{[1\leq i\leq f, f+1\leq j\leq 2f]} - 1\ \delta_{i-f,j}\ \Pi_{[f+1\leq i\leq 2f, 1\leq j\leq f]}$$

$$(\mathbf{J}\ \tilde{\mathbf{J}})_{ij} = \mathbf{J}_{ik}\ \mathbf{J}_{jk} =$$

$$(\delta_{i,k-f}\ \Pi_{[1\leq i\leq f, f+1\leq k\leq 2f]} - \delta_{i-f,k}\ \Pi_{[f+1\leq i\leq 2f, 1\leq k\leq f]}) \cdot$$

$$(\delta_{j,k-f}\ \Pi_{[1\leq j\leq f, f+1\leq k\leq 2f]} - \delta_{j-f,k}\ \Pi_{[f+1\leq j\leq 2f, 1\leq k\leq f]})$$

$$= (\delta_{i,k-f}\ \delta_{j,k-f}\ \Pi_{[1\leq i\leq f, f+1\leq k\leq 2f]}\ \Pi_{[1\leq j\leq f, f+1\leq k\leq 2f]}$$

$$+ \delta_{i-f,k}\ \delta_{j-f,k}\ \Pi_{[f+1\leq i\leq 2f, 1\leq k\leq f]}\ \Pi_{[f+1\leq j\leq 2f, 1\leq k\leq f]}$$

$$= \delta_{ij} = \mathbf{1}_{ij}$$

$$\Rightarrow \quad \mathbf{J}\ \tilde{\mathbf{J}} = \mathbf{1}$$

• 2.6 L'approche symplectique des transf. canoniques

$$\mathbf{J}\,\tilde{\mathbf{J}} = \mathbf{1}, \quad \tilde{\mathbf{J}}\,\mathbf{J} = \mathbf{1} \;\Rightarrow\; \tilde{\mathbf{J}} = -\mathbf{J} = \mathbf{J}^{-1}$$

$$\Downarrow$$

$$\mathrm{dtm}(\mathbf{J}) = +1 \;\Leftarrow\; \mathbf{J}^2 = \mathbf{J}\,\mathbf{J} = -\tilde{\mathbf{J}}\,\mathbf{J} = -\mathbf{1}$$

$$\Downarrow$$

$\Rightarrow$ c.q.f.d.

$$J^2 = -\mathbf{1} \quad (2.49a)$$

$$J\tilde{J} = \mathbf{1} \quad (2.49b)$$

$$\tilde{J} = -J = J^{-1} \quad (2.49c)$$

$$dtm\,(J) = +1 \quad (2.49d)$$

• 2.6 L'approche symplectique des transf. canoniques

vecteur colonne ζ à $2f$ éléments

nouvelles variables canoniques $Q_i,\ P_i$ $\qquad \zeta = \zeta(\eta)$ (2.50)

$$\dot{\zeta}_i = \frac{\partial \zeta_i}{\partial \eta_j}\dot{\eta}_j \qquad (i, j = 1, \ldots, 2f)$$

$$\dot{\zeta} = M\dot{\eta} \quad (2.51) \qquad M_{ij} = \frac{\partial \zeta_i}{\partial \eta_j} \quad (2.52)$$

$$\dot{\zeta} = MJ\frac{\partial H}{\partial \eta} \quad (2.53)$$

$$\frac{\partial H}{\partial \eta_i} = \frac{\partial H}{\partial \zeta_j}\frac{\partial \zeta_j}{\partial \eta_i}$$

$$\frac{\partial H}{\partial \eta} = \tilde{M}\frac{\partial H}{\partial \zeta} \quad (2.54)$$

$$\dot{\zeta} = MJ\tilde{M}\frac{\partial H}{\partial \zeta} \quad (2.55)$$

• 2.6 L'approche symplectique des transf. canoniques

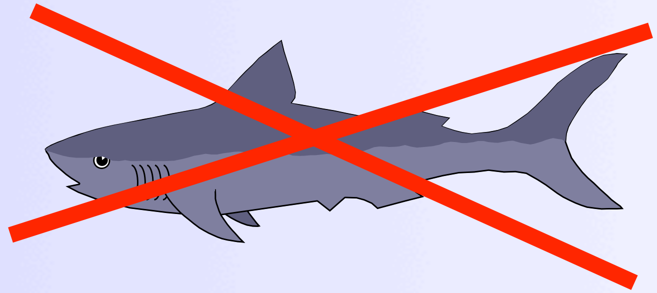
$$\dot{\zeta} = MJ\tilde{M}\frac{\partial H}{\partial \zeta} \quad (2.55)$$

$$\dot{\zeta} = J\frac{\partial H}{\partial \zeta} \quad (2.56)$$

$$MJ\tilde{M} = J \quad (2.57a)$$

$$\tilde{M}JM = J \quad (2.57b)$$

• 2.7 Les crochets de Poisson

[]

$$\boxed{[u,v]_{q,p} = \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}} \qquad (2.58)$$

$$[u,v]_{\eta} = \frac{\widetilde{\partial u}}{\partial \eta} J \frac{\partial v}{\partial \eta} \qquad (2.59)$$

Si $u_i = \frac{\widetilde{\partial u}}{\partial \eta_i}$ *et* $v_k = \frac{\partial v}{\partial \eta_k}$

$$= u_i \, J_{ik} \, v_k$$

$$= u_i \, [\delta_{i,k-f} \, \Pi_{[1 \le i \le f, f+1 \le k \le 2f]} - \delta_{i-f,k} \, \Pi_{[f+1 \le i \le 2f, 1 \le k \le f]}] v_k$$

• 2.7 Les crochets de Poisson

$$= u_i \left[\delta_{i,k-f} \, \Pi_{[1 \le i \le f, f+1 \le k \le 2f]} - \delta_{i-f,k} \, \Pi_{[f+1 \le i \le 2f, 1 \le k \le f]}\right] v_k$$

$$= \frac{\partial u}{\partial q_i} \cdot \frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i} \frac{\partial v}{\partial q_i}$$

$$\boxed{[u, v]_{q,p} = \frac{\partial u}{\partial q_i} \frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i} \frac{\partial v}{\partial q_i}} \qquad (2.58)$$

$$\begin{cases} [q_j, q_k]_{q,p} = 0 \\ [p_j, p_k]_{q,p} = 0 \\ [q_j, p_k]_{q,p} = -[p_j, q_k]_{q,p} = \delta_{jk} \end{cases} \qquad (2.60)$$

• 2.7 Les crochets de Poisson

matrice crochet de Poisson

désignée par $[\eta, \eta]$ dont l'élément (ℓm) est $[\eta_\ell, \eta_m]$

$$[\eta, \eta]_\eta = J \qquad (2.61)$$

$$\zeta = \zeta(\eta, t)$$

$$[\zeta, \zeta]_\eta = \frac{\partial \zeta}{\partial \eta} J \widetilde{\frac{\partial \zeta}{\partial \eta}} \qquad (2.62)$$

$$[\zeta, \zeta]_\eta = M J \tilde{M} \qquad (2.63)$$

$$[\zeta, \zeta]_\eta = J \qquad (2.64)$$

$$[\zeta, \zeta]_\zeta = J \qquad (2.65)$$

- 2.7 Les crochets de Poisson

$$[u,v]_\eta = \widetilde{\frac{\partial u}{\partial \eta}} J \frac{\partial v}{\partial \eta} \quad (2.59)$$

$$\frac{\partial v}{\partial \eta} = \tilde{M} \frac{\partial v}{\partial \zeta} \quad (2.66)$$

$$\widetilde{\frac{\partial u}{\partial \eta}} = \widetilde{\tilde{M} \frac{\partial u}{\partial \zeta}} = \widetilde{\frac{\partial u}{\partial \zeta}} M \quad (2.67)$$

$$[u,v]_\eta = \widetilde{\frac{\partial u}{\partial \eta}} J \frac{\partial v}{\partial \eta} = \widetilde{\frac{\partial u}{\partial \zeta}} M J \tilde{M} \frac{\partial v}{\partial \zeta} \quad (2.68)$$

$$\boxed{[u,v]_\eta = \widetilde{\frac{\partial u}{\partial \zeta}} J \frac{\partial v}{\partial \zeta} = [u,v]_\zeta} \quad (2.69)$$