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## VARIATIONAL PRINCIPLES IN FLUID MECHANICS

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### Introduction

There has been in recent years renewed interest for the powerful variational methods in fluid mechanics. Surveys of known applications to transonic and supersonic flow have been published by W. FISZDON [10, 11].

The use of a lagrangean description of the flow, coupled with lagrangean variations of particle displacements, provides the most natural formalism for a variational principle. It is a natural extension of the powerful methods introduced in mechanics by LAGRANGE, HAMILTON and JACOBI. The only difficulty is the introduction of the side condition of conservation of particle mass when passing from a discrete system to the continuous medium. Most of the older work on variational principles is in this direction [2, 4, 5]; an excellent account of this approach is to be found in a more recent work of ECKART [8].

For practical applications however variational principles based on the eulerian description of fields, coupled with eulerian, or local, variations are more useful. Such principles have first been constructed, rather artificially, to yield the known equations of conservation in eulerian form [3]. The first attempt to derive them logically from Hamilton's principle seems to be due to HERIVEL [6]. As reported by SERRIN [7], Herivel's principle, which yields only irrotational flow, was completed by LIN.

The present paper attempts a completely logical derivation from Hamilton's principle to a purely eulerian principle. It is an extension of an earlier work [9] where the same difficulty of removing the restriction to irrotational flow was dealt with rather artificially. A logical step, which departs from the procedure adopted by LIN, has been taken. Since independant variations are to be taken on the velocity field, it is only natural to remove the constraints implying that the velocity is the derivative of the particle's position vector. This introduces a new vector multiplier enjoying a property of constant circulation along any segment fixed in the flow. This procedure yields almost immediatly a general variational principle that can easily be specialized to the principles given by BATEMAN.

### 1. Lagrangean and Eulerian Variations

Let

$$(1.1) \quad x_i = X_i(a_j, t, \varepsilon), \quad i = 1, 2, 3, \quad j = 1, 2, 3$$

denote the cartesian coordinates of a particle at time  $t$ ; each particle is identified

by its parameters, or lagrangean coordinates,  $a_j$ . Equations (1.1) describe a family of flows; the true flow corresponding to  $\varepsilon = 0$ , varied flows to  $\varepsilon \neq 0$ .

The displacement during the interval  $dt$  in true motion is

$$(1.2) \quad d_a x_i = \left( \frac{\partial X_i}{\partial t} \right)_{\varepsilon=0} dt, \quad i = 1, 2, 3.$$

Similarly, the lagrangean variation of displacement, is defined as

$$(1.3) \quad \delta_a x_i = \left( \frac{\partial X_i}{\partial \varepsilon} \right)_{\varepsilon=0} d\varepsilon, \quad i = 1, 2, 3.$$

Both notations suggest that the lagrangean coordinates are kept fixed so that both displacements refer to the same particle.

More generally, let

$$(1.4) \quad f = F(a_j, t, \varepsilon)$$

denote an intensive variable of the field, then

$$(1.5) \quad d_a f = \left( \frac{\partial F}{\partial t} \right)_{\varepsilon=0} dt$$

is the material or total change due to true motion:

$$(1.6) \quad \delta_a f = \left( \frac{\partial F}{\partial \varepsilon} \right)_{\varepsilon=0} d\varepsilon$$

is the *lagrangean variation*.

Obviously the operators  $d_a$  and  $\delta_a$  commute:

$$(1.7) \quad \delta_a d_a f = d_a \delta_a f = \left( \frac{\partial^2 F}{\partial \varepsilon \partial t} \right)_{\varepsilon=0} d\varepsilon dt$$

and commute with partial derivatives with respect to the lagrangean coordinates. Expression (1.4) is the lagrangean description of the field of the intensive variable  $f$ . By solving Eqs. (1.1) for the lagrangean coordinates

$$(1.8) \quad a_j = a_j(x_i, t, \varepsilon)$$

and substituting in (1.4) one obtains in principle the eulerian description of the field:

$$(1.9) \quad f = F(a_j(x_i, t, \varepsilon), t, \varepsilon) = f(x_i, t, \varepsilon).$$

This naturally suggests corresponding definitions for the changes in  $f$  when the cartesian coordinates are kept fixed:

$$(1.10) \quad d_x f = \left( \frac{\partial f}{\partial t} \right)_{\varepsilon=0} dt$$

is the local change in  $f$  due to true motion,

$$(1.11) \quad \delta_x f = \left( \frac{\partial f}{\partial \varepsilon} \right)_{\varepsilon=0} d\varepsilon$$

is the local or *eulerian variation* of the field.

The operators  $d_x$  and  $\delta_x$  commute with each other and with the partial derivatives with respect to the cartesian coordinates.

In subsequent calculations general use will be made of the eulerian description of fields. The connection between the lagrangean and eulerian changes will therefore be explicitated as follows:

$$(1.12) \quad d_a f = d_x f + \sum_1^3 \frac{\partial f}{\partial x_i} d_a x_i,$$

$$(1.13) \quad \delta_a f = \delta_x f + \sum_1^3 \frac{\partial f}{\partial x_i} \delta_a x_i.$$

When the more common notations

$$\frac{df}{dt} = \frac{d_a f}{dt}, \quad \frac{\partial f}{\partial t} = \frac{d_x f}{dt}$$

are used for the material and local time-derivatives, (1.12) reduces to the known expression

$$(1.14) \quad \frac{df}{dt} = \frac{\partial f}{\partial t} + \mathbf{u} \cdot \text{grad } f,$$

where  $u_i = \frac{d_a x_i}{dt}$  are the velocity-field components.

Similarly, (1.13) can be written

$$(1.15) \quad \delta_a f = \delta_x f + \delta_a \mathbf{x} \cdot \text{grad } f.$$

Observe that, by their very definitions

$$\frac{da_j}{dt} = \frac{\partial a_j}{\partial t} + \mathbf{u} \cdot \text{grad } a_j = 0,$$

$$j = 1, 2, 3.$$

$$\delta_a a_j = \delta_x a_j + \delta_a \mathbf{x} \cdot \text{grad } a_j = 0,$$

$$\frac{\partial x_i}{\partial t} = u_i - \mathbf{u} \cdot \text{grad } x_i = 0,$$

$$i = 1, 2, 3.$$

$$\delta_x x_i = \delta_a x_i - \delta_a \mathbf{x} \cdot \text{grad } x_i = 0,$$

We need corresponding results for the volume element

$$dv = J da_1 da_2 da_3, \quad J = \frac{D(X_1, X_2, X_3)}{D(a_1, a_2, a_3)}.$$

The change in volume for the same aggregate of particles will be

$$\frac{d}{dt} dv = \left( \frac{d}{dt} J \right) da_1 da_2 da_3.$$

In calculating the derivative of the jacobian determinant use is made of the following expansions

$$\frac{d_a}{dt} \frac{\partial X_j}{\partial a_i} = \frac{\partial}{\partial a_i} \frac{d_a x_j}{dt} = \frac{\partial u_j}{\partial a_i} = \sum_{m=1}^3 \frac{\partial u_j}{\partial x_m} \frac{\partial X_m}{\partial a_i}.$$

Rearrangement of the terms produces the well known result

$$\frac{d}{dt} J = J \operatorname{div} \mathbf{u}$$

so that

$$(1.16) \quad \frac{d}{dt} dv = dv \operatorname{div} \mathbf{u}.$$

Similarly for the lagrangean variation

$$(1.17) \quad \delta_a dv = dv \operatorname{div} \delta_a \mathbf{x}.$$

Again, by definition,

$$(1.18) \quad \frac{\partial}{\partial t} dv = 0, \quad \delta_x dv = 0.$$

We are now in a position to evaluate the changes occurring in any extensive quantity  $f dv$ :

$$(1.19) \quad \frac{d}{dt} (f dv) = \frac{df}{dt} dv + f \frac{d}{dt} dv = \left( \frac{df}{dt} + f \operatorname{div} \mathbf{u} \right) dv.$$

But also, in view of (1.14)

$$(1.20) \quad \frac{d}{dt} (f dv) = \left( \frac{\partial f}{\partial t} + \operatorname{div} (f \mathbf{u}) \right) dv.$$

Hence, integrating in a volume  $D$ , bounded by a surface  $S$  of outward normal  $\mathbf{n}$ ,

$$(1.21) \quad \frac{d}{dt} \int_D f dv = \frac{\partial}{\partial t} \int_D f dv + \int_S f (\mathbf{u} \cdot \mathbf{n}) dS.$$

This gives the rate of change of the volume integral when the particles are followed in their true motion.

The perfectly similar calculation for variations gives

$$(1.22) \quad \begin{aligned} \delta_a(fdv) &= (\delta_a f + f \operatorname{div} \delta_a \mathbf{x}) dv = (\delta_x f + \operatorname{div}(f \delta_a \mathbf{x})) dv \\ \delta_a \int_D f dv &= \delta_x \int_D f dv + \int_S f(\delta_a \mathbf{x} \cdot \mathbf{n}) dS. \end{aligned}$$

From this basic formula, a variational principle involving lagrangean variations can be transformed in one involving only local variations, provided the additional surface integral can be cancelled or suitably interpreted. The variational principles of dynamics also involve time integrals. A useful transformation is then obtained from (1.21) by integration:

$$(1.23) \quad \int_{t_1}^{t_2} \int_D \frac{\partial f}{\partial t} dv dt = - \int_{t_1}^{t_2} \int_S f(\mathbf{u} \cdot \mathbf{n}) dS dt + \left| \int_D f dv \right|_{t_1}^{t_2}.$$

## 2. Hamilton's Principle

We consider the case of isentropic flow of a perfect fluid.

The specific internal energy  $E$  is a function of specific mass

$$(2.1) \quad E = E(\rho) \quad \text{with} \quad \frac{dE}{d\rho} = \frac{p}{\rho^2}.$$

The specific potential energy  $\Omega$  of external forces is a function of cartesian coordinates only, so that

$$(2.2) \quad \delta_x \Omega = 0, \quad \delta_a \Omega = \operatorname{grad} \Omega \cdot \delta_a \mathbf{x}.$$

The Lagrangian per unit mass will be

$$(2.3) \quad L = \frac{1}{2}(\mathbf{u} \cdot \mathbf{u}) - (E + \Omega).$$

Hamilton's principle applies to a system that is closed with respect to mass exchange; the particles must be followed in their motion, so that the basic formulation is lagrangean in nature:

$$(2.4) \quad \delta_a \int_{t_1}^{t_2} \int_D \rho L dv dt - \int_{t_1}^{t_2} \int_{S^*} p^*(\delta_a \mathbf{x} \cdot \mathbf{n}) dS dt = 0.$$

The boundary conditions envisaged are as follows:

1) on a portion  $S^*$  of the bounding surface the pressure is given ( $p = p^*$ ) and the corresponding virtual work due to variation of particle displacements is included in the principle,

2) on the remainder  $S-S^*$  of the surface the fluid is gliding along flexible moving walls. At any time the variation of particle displacement must verify the geometrical constraint

$$(2.5) \quad \delta_a \mathbf{x} \cdot \mathbf{n} = 0 \quad \text{on} \quad S-S^*.$$

The only independent variation in the principle is  $\delta_a \mathbf{x}$ .

The corresponding variation on the specific mass  $\varrho$  must be obtained from a separate statement of conservation of mass of a particle:

$$(2.6) \quad \delta_a (\varrho dv) = 0.$$

As a result of this the variational principle (2.4) assumes the equivalent form

$$(2.7) \quad \int_{t_1}^{t_2} \int_D \varrho \delta_a L dv dt - \int_{t_1}^{t_2} \int_{S^*} p^* (\delta_a \mathbf{x} \cdot \mathbf{n}) dS dt = 0.$$

Expanding the variation of the Lagrangian:

$$(2.8) \quad \delta_a L = \mathbf{u} \cdot \delta_a \mathbf{u} - \frac{p}{\varrho^2} \delta_a \varrho - \text{grad } \Omega \cdot \delta_a \mathbf{x},$$

where

$$(2.9) \quad \delta_a \mathbf{u} = \delta_a \frac{d\mathbf{x}}{dt} = \frac{d}{dt} \delta_a \mathbf{x}$$

and, by virtue of (2.6) and (1.17)

$$(2.10) \quad \delta_a \varrho = -\varrho \text{div} (\delta_a \mathbf{x}).$$

In preparation for the required integrations by parts we write finally

$$\varrho \delta_a L dv = \frac{d}{dt} (\delta_a \mathbf{x} \cdot \varrho \mathbf{u} dv) + \text{div} (p \delta_a \mathbf{x}) dv - \delta_a \mathbf{x} \cdot \mathbf{M} dv,$$

where

$$(2.11) \quad \mathbf{M} dv = \frac{d}{dt} (\varrho \mathbf{u} dv) + (\text{grad } p + \varrho \text{grad } \Omega) dv.$$

Consequently, the Euler equation of the variational principle is  $\mathbf{M} dv = 0$ , a statement of conservation of momentum of a particle. It will be observed that it assumes the newtonian form of the equations of motion

$$(2.12) \quad \varrho \frac{d\mathbf{u}}{dt} + \text{grad } p + \varrho \text{grad } \Omega = 0$$

after an independent statement of conservation of mass during true motion

$$(2.13) \quad \frac{d}{dt} (\varrho dv) = 0.$$

The boundary conditions obtained are

$$\int_{t_1}^{t_2} \int_S p (\delta_a \mathbf{x} \cdot \mathbf{n}) dS dt - \int_{t_1}^{t_2} \int_{S^*} p^* (\delta_a \mathbf{x} \cdot \mathbf{n}) dS dt = 0.$$

By virtue of the geometrical constraint (2.5), they reduce correctly to

$$(2.14) \quad p - p^* = 0 \quad \text{on} \quad S^*.$$

Finally there are initial conditions

$$\int_D \delta_a \mathbf{x} \cdot \rho \mathbf{u} \, dv = 0 \quad \text{for} \quad t = t_1 \quad \text{and} \quad t = t_2.$$

They are satisfied by the usual rule stating that the varied configurations must coincide with the true configuration in the initial and final state.

$$(2.15) \quad \delta_a \mathbf{x} = 0 \quad \text{for} \quad t = t_1 \quad \text{and} \quad t = t_2.$$

### 3. Eulerian form of Hamilton's Principle

A direct application of formula (1.22) brings the Hamilton principle into the eulerian form

$$(3.1) \quad \delta_x \int_{t_1}^{t_2} \int_D \rho L \, dv \, dt + \int_{t_1}^{t_2} \int_S \rho L (\delta_a \mathbf{x} \cdot \mathbf{n}) \, dS \, dt - \int_{t_1}^{t_2} \int_{S^*} p^* (\delta_a \mathbf{x} \cdot \mathbf{n}) \, dS \, dt = 0.$$

Introducing

$$(3.2) \quad E + \rho \frac{dE}{d\rho} = E + \frac{p}{\rho} = I$$

the thermodynamical enthalpy, and remembering (1.18) and (2.2), we calculate

$$(3.3) \quad \delta_x(\rho L) = \left( \frac{1}{2} (\mathbf{u} \cdot \mathbf{u}) - I - \Omega \right) \delta_x \rho + \rho \mathbf{u} \cdot \delta_x \mathbf{u},$$

wherein we must substitute

$$\delta_x \rho = -\text{div}(\rho \delta_a \mathbf{x})$$

$$\delta_x \mathbf{u} = \delta_a \mathbf{u} - (\delta_a \mathbf{x} \cdot \text{grad}) \mathbf{u} = \frac{d}{dt} \delta_a \mathbf{x} - (\delta_a \mathbf{x} \cdot \text{grad}) \mathbf{u}.$$

Hence

$$\delta_x(\rho L) = (I + \Omega) \text{div}(\rho \delta_a \mathbf{x}) + \rho \mathbf{u} \cdot \frac{d}{dt} \delta_a \mathbf{x} - \frac{1}{2} (\mathbf{u} \cdot \mathbf{u}) \text{div}(\rho \delta_a \mathbf{x}) - \rho \mathbf{u} \cdot (\delta_a \mathbf{x} \cdot \text{grad}) \mathbf{u}.$$

The two last terms on the last line combine into

$$-\text{div} \left( \rho \frac{\mathbf{u} \cdot \mathbf{u}}{2} \delta_a \mathbf{x} \right)$$

and finally

$$\delta_x(\rho L) \, dv = \text{div} \left[ \rho \left( I + \Omega - \frac{\mathbf{u} \cdot \mathbf{u}}{2} \right) \delta_a \mathbf{x} \right] \, dv + \frac{d}{dt} (\rho \mathbf{u} \cdot \delta_a \mathbf{x} \, dv) - \delta_a \mathbf{x} \cdot \mathbf{M} \, dv,$$

where

$$\mathbf{M} \, dv = \frac{d}{dt} (\rho \mathbf{u} \, dv) + \rho \, \text{grad} (I + \Omega) \, dv.$$

After use of (2.13), the Euler equation appears in the equivalent form

$$(3.4) \quad \frac{d\mathbf{u}}{dt} + \text{grad}(I + \Omega) = 0$$

showing the existence of an acceleration potential.

Because

$$\varrho \left( I + \Omega - \frac{\mathbf{u} \cdot \mathbf{u}}{2} \right) = p - \varrho L$$

the boundary conditions of the eulerian form are identical to those of the lagrangean form.

#### 4. Generalization of the Eulerian Principle

By incorporating the constraint on the conservation of mass:

$$(4.1) \quad \frac{\partial \varrho}{\partial t} + \text{div}(\varrho \mathbf{u}) = 0$$

with a lagrangean multiplier, the principle can be made self-supporting. This step alone, however, is known to be insufficient. In transferring to variations on the velocity field it restricts the flow to the irrotational case. If variations on the velocity field are to replace variations on particle coordinates, it is necessary to remove also the constraints

$$(4.2) \quad \frac{d\mathbf{x}}{dt} - \mathbf{u} = 0.$$

This second step departs from the procedure adopted by Lin to amplify the eulerian principle formulated by HERIVEL [6]. It seems more satisfactory from the viewpoint of proceeding without preconceived knowledge of the results to obtain. The generalized principle is then

$$(4.3) \quad \delta_{\mathbf{x}} \int_{t_1}^{t_2} \int_D \left[ \varrho L + \phi \left( \frac{\partial \varrho}{\partial t} + \text{div}(\varrho \mathbf{u}) \right) + \varrho \Psi \cdot \left( \frac{d\mathbf{x}}{dt} - \mathbf{u} \right) \right] dv dt \\ + \int_{t_1}^{t_2} \int_S \varrho L (\delta_a \mathbf{x} \cdot \mathbf{n}) dS dt - \int_{t_1}^{t_2} \int_{S^*} p^* (\delta_a \mathbf{x} \cdot \mathbf{n}) dS dt = 0.$$

Variation of the scalar multiplier  $\phi$  yields (4.1) as Euler-Lagrange equation, variation of the vector multiplier  $\Psi$  yields the constraints (4.2).

The independant variations  $\delta_{\mathbf{x}} \mathbf{u}$  produce as Euler equation

$$(4.4) \quad \mathbf{u} - \text{grad} \phi - \Psi = 0$$



while the independent variation  $\delta_x \rho$  yields

$$(4.5) \quad \frac{\mathbf{u} \cdot \mathbf{u}}{2} - I - \Omega - \frac{d\phi}{dt} = 0.$$

To establish the Euler equation for  $\delta_a \mathbf{x}$  we must manipulate

$$\delta_x \frac{d\mathbf{x}}{dt} = \delta_a \frac{d\mathbf{x}}{dt} - (\delta_a \mathbf{x} \cdot \text{grad}) \mathbf{u} = \frac{d}{dt} \delta_a \mathbf{x} - (\delta_a \mathbf{x} \cdot \text{grad}) \mathbf{u}.$$

And consequently

$$(4.6) \quad \rho \Psi \cdot \delta_x \frac{d\mathbf{x}}{dt} dv = \frac{d}{dt} (\delta_a \mathbf{x} \cdot \rho \Psi dv) - \delta_a \mathbf{x} \cdot \frac{d}{dt} (\rho \Psi dv) - \rho dv \Psi \cdot (\delta_a \mathbf{x} \cdot \text{grad}) \mathbf{u}.$$

Using the property (2.13), furnished by the principle in the equivalent form (4.1), it turns out that the components of the vector multiplier verify, according to (4.6), the differential equations

$$(4.7) \quad \frac{d\psi_i}{dt} + \sum_{j=1}^3 \psi_j \frac{\partial u_j}{\partial x_i} = 0, \quad i = 1, 2, 3.$$

An equivalent vectorial equation is

$$(4.8) \quad \frac{\partial \Psi}{\partial t} + \text{grad}(\Psi \cdot \mathbf{u}) + \text{rot} \Psi \times \mathbf{u} = 0.$$

The independent boundary conditions turn out to be:

$$(4.9) \quad \int_{t_1}^{t_2} \int_S \phi \rho (\delta_x \mathbf{u} \cdot \mathbf{n}) dS dt = 0$$

and, because of (4.5),

$$(4.10) \quad \int_{t_1}^{t_2} \int_S \left( p + \rho \frac{d\phi}{dt} \right) (\delta_a \mathbf{x} \cdot \mathbf{n}) dS dt - \int_{t_1}^{t_2} \int_{S^*} p^* (\delta_a \mathbf{x} \cdot \mathbf{n}) dS dt = 0.$$

A necessary constraint on the velocity field is

$$(4.11) \quad (\delta_x \mathbf{u} \cdot \mathbf{n}) = 0 \quad \text{on} \quad S - S^*,$$

if the compared motions are to remain tangent to the moving walls. On the surface  $S^*$ , the variations  $\delta_x \mathbf{u}$  are free and consequently (4.9) imposes a natural condition

$$(4.12) \quad \phi = 0 \quad \text{on} \quad S^*.$$

Since particles on the free surface will remain on it, (4.12) will entail

$$(4.13) \quad \frac{d\phi}{dt} = 0 \quad \text{on} \quad S^*.$$

Thus, while along the moving walls the constraint (2.5) prevails, (4.10) is again satisfied along  $S^*$  by the pressure condition (2.14).

Initial conditions for the principle are

$$(4.14) \quad \int_D (\rho \psi \cdot \delta_a \mathbf{x}) dv = 0 \quad \text{for } t = t_1 \text{ and } t = t_2,$$

which is satisfied by the rule (2.15), and

$$(4.15) \quad \int_D \phi \delta_x \rho dv = 0 \quad \text{for } t = t_1 \text{ and } t = t_2.$$

Indeed, to have (2.6) satisfied at all times, we need to have it satisfied initially. After that, Eq. (2.13) which is now catered for by the principle, will keep it true at later times. Thus we need to have

$$\delta_x \rho = -\rho \operatorname{div}(\delta_a \mathbf{x}) \quad \text{for } t = t_1$$

and by virtue of (2.15) this reduces to the condition

$$(4.16) \quad \delta_x \rho = 0 \quad \text{for } t = t_1.$$

The similar condition for  $t = t_2$  will then be automatically satisfied. To achieve the consistency proof of the generalized principle it remains to show that it furnishes the newtonian equations of motion. Indeed, taking  $\psi$  from (4.4) and  $d\phi/dt$  from (4.5), (4.8) becomes

$$(4.17) \quad \frac{\partial \mathbf{u}}{\partial t} + \operatorname{grad} \left( \frac{\mathbf{u} \cdot \mathbf{u}}{2} + I + \Omega \right) + \operatorname{rot} \mathbf{u} \times \mathbf{u} = 0.$$

##### 5. $\psi$ As a Constant Circulation Field

Since the final aim is to remove entirely the necessity for variations of particle displacements, the corresponding Euler-Lagrange Eq. (4.8) must somehow be solved and the solution substituted into the principle.

The nature of the solution can be recognized by the property of the  $\psi$  field to keep a constant circulation along any segment carried with the particles in their motion.

If  $d\mathbf{x}$  ( $dx_1, dx_2, dx_3$ ) is such a segment

$$(5.1) \quad \frac{d}{dt} \left( \sum_1^3 \psi_i dx_i \right) = \sum_1^3 \left( \frac{d\psi_i}{dt} dx_i + \psi_j du_j \right) = 0$$

by virtue of the differential Eqs. (4.7). Hence the differential form

$$\sum_1^3 \psi_i dx_i = \sum_1^3 \psi_i(a_j, t) dX_i(a_j, t) = \sum_{j=1}^3 \left( \sum_{i=1}^3 \psi_i \frac{\partial X_i}{\partial a_j} \right) da_j$$

expressed in lagrangean coordinates, is explicitly independent of time and can be written

$$(5.2) \quad \sum_1^3 \psi_i dx_i = \sum_1^3 A_j(a_k) da_j,$$

where the functions  $A_j$  depend only on the chosen set of lagrangean variables. It will be convenient to think of the Pfaffian form (5.2) in the lagrangean variables as reduced to any canonical form:

$$\sum_1^3 A_j(a_k) da_j = d\Gamma + A dB.$$

Then, returning to the eulerian representation,

$$(5.3) \quad \sum_1^3 \psi_i dx_i = d\gamma(x_i, t) + \alpha(x_i, t) d\beta(x_i, t),$$

where

$$(5.4) \quad \frac{d\alpha}{dt} = 0, \quad \frac{d\beta}{dt} = 0, \quad \frac{d\gamma}{dt} = 0.$$

It appears finally from (5.3) that the general solution of (4.8) is

$$(5.5) \quad \Psi = \text{grad } \gamma + \alpha \text{ grad } \beta,$$

where  $\alpha$ ,  $\beta$  and  $\gamma$  are lagrangean variables.

#### 6. Elimination of the Variations $\delta_a \mathbf{x}$

The solution (5.4) and (5.5) of the vector multipliers is substituted into the generalized principle as follows.

From (5.4)

$$\frac{d\beta}{dt} = \frac{\partial\beta}{\partial t} + \frac{d\mathbf{x}}{dt} \cdot \text{grad } \beta = 0, \quad \frac{d\gamma}{dt} = \frac{\partial\gamma}{\partial t} + \frac{d\mathbf{x}}{dt} \cdot \text{grad } \gamma = 0.$$

This allows us to write

$$\Psi \cdot \frac{d\mathbf{x}}{dt} = (\text{grad } \gamma + \alpha \text{ grad } \beta) \cdot \frac{d\mathbf{x}}{dt} = -\left( \frac{\partial\gamma}{\partial t} + \alpha \frac{\partial\beta}{\partial t} \right)$$

and the kernel of the variational principle becomes

$$K = \rho L + \phi \left( \frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{u}) \right) - \rho \left( \frac{\partial \gamma}{\partial t} + \mathbf{u} \cdot \text{grad } \gamma \right) - \rho \alpha \left( \frac{\partial \beta}{\partial t} + \mathbf{u} \cdot \text{grad } \beta \right).$$

There are no  $\delta_a \mathbf{x}$  terms stemming any more from variation of the kernel. Accordingly all boundary terms in  $\delta_a \mathbf{x}$  in the variational principle are also removed.

which will entail that any pressure condition will have to be satisfied independently. In expanding the principle

$$(6.1) \quad \delta_x \int_{t_1}^{t_2} \int_D K dv dt = 0$$

it is easily seen that  $d\gamma/dt = 0$  is lost as an Euler equation but, simultaneously, that variations on  $\phi$  and  $\gamma$  play a similar role. As a matter of fact we can write the second term of  $K dv$  as follows:

$$\phi \left( \frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{u}) \right) dv = \frac{d}{dt} (\phi \rho dv) - \rho \left( \frac{\partial \phi}{\partial t} + \mathbf{u} \cdot \text{grad} \phi \right) dv.$$

Removal of the total time derivative will only modify the conditions at  $t = t_1$  and  $t = t_2$ , while it is seen that  $\phi$  and  $\gamma$  combine into a single variable  $\theta = \phi + \gamma$ . Hence we consider the principle (6.1) with the simplified kernel

$$K = \rho \left\{ \frac{\mathbf{u} \cdot \mathbf{u}}{2} - E - \Omega - \left( \frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \text{grad} \theta \right) - \alpha \left( \frac{\partial \beta}{\partial t} + \mathbf{u} \cdot \text{grad} \beta \right) \right\}.$$

The following Euler equations are obtained

$$(6.2) \quad \delta_x \mathbf{u} \quad \mathbf{u} = \text{grad} \theta + \alpha \text{grad} \beta,$$

$$(6.3) \quad \delta_x \rho \quad \frac{\mathbf{u} \cdot \mathbf{u}}{2} = I + \Omega + \frac{d\theta}{dt} + \alpha \frac{d\beta}{dt},$$

$$(6.4) \quad \delta_x \theta \quad \frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{u}) = 0,$$

$$(6.5) \quad \delta_x \alpha \quad \frac{d\beta}{dt} = 0,$$

$$(6.6) \quad \delta_x \beta \quad \frac{\partial}{\partial t} (\rho \alpha) + \text{div}(\rho \alpha \mathbf{u}) = 0$$

together with initial conditions requiring  $\delta_x \theta$  and  $\delta_x \beta$  to vanish for  $t = t_1$  and  $t = t_2$ .

From (6.4) and (6.6) the result  $d\alpha/dt = 0$  is retrieved and (6.2) is then the formulation of the velocity field originally given by Clebsch [1].

The vorticity is represented by

$$(6.7) \quad \text{rot} \mathbf{u} = \text{grad} \alpha \times \text{grad} \beta$$

so that the vortex lines are intersections of the surfaces  $\alpha = \text{constant}$  and  $\beta = \text{constant}$  which move with the particles.

Combining the result

$$\mathbf{u} \cdot \mathbf{u} = \mathbf{u} \cdot (\text{grad} \theta + \alpha \text{grad} \beta) = \frac{d\theta}{dt} + \alpha \frac{d\beta}{dt} - \left( \frac{\partial \theta}{\partial t} + \alpha \frac{\partial \beta}{\partial t} \right)$$

obtained from (6.2) with (6.3), we find the energy integral

$$(6.8) \quad I + \Omega + \frac{\mathbf{u} \cdot \mathbf{u}}{2} + \frac{\partial \theta}{\partial t} + \alpha \frac{\partial \beta}{\partial t} = 0.$$

The newtonian equations of motions can be derived by taking the material derivative of (6.2).

The absence of natural surface boundary conditions is a matter of closer scrutiny of the technique of integration by parts. In the stationary case there are surface boundary terms; in the non stationary case those terms are cancelled by application of formula (1.23). A natural way to correct this situation is to replace (1.21) by the statement

$$(6.9) \quad \frac{d}{dt} \int_D f dv = \frac{\partial}{\partial t} \int_D f dv + \int_S f(\mathbf{w} \cdot \mathbf{n}) dS,$$

where  $\mathbf{w}$  is equal to the fluid velocity  $\mathbf{u}$  on the free surface  $S^*$ , but elsewhere equal to the local velocity of the wall. If the fluid is gliding along the wall there is no flux across it and (6.9) holds true as well as (1.21). Replacing (1.23) by the new formula

$$(6.10) \quad \int_{t_1}^{t_2} \int_D \frac{\partial f}{\partial t} dv dt = - \int_{t_1}^{t_2} \int_S f(\mathbf{w} \cdot \mathbf{n}) dS dt + \left| \int_D f dv \right|_{t_1}^{t_2}$$

the surface boundary terms obtained are

$$\int_{t_1}^{t_2} \int_S \varrho (\delta_x \theta + \alpha \delta_x \beta) (\mathbf{w} - \mathbf{u} \cdot \mathbf{n}) dS dt = 0.$$

They require correctly that

$$(6.11) \quad \mathbf{u} \cdot \mathbf{n} = \mathbf{w} \cdot \mathbf{n} \quad \text{on} \quad S.$$

### 7. The Variational Principles of Bateman

There are two ways of modifying the presentation of the variational principle (6.1) with the simplified kernel.

When (6.3) is used *a priori*, the variations on  $\varrho$  should disappear. This is achieved by substitution of (6.3) into the kernel, producing:

$$K = \varrho(I - E) = p.$$

In this presentation  $p$  should be considered as a function of the enthalpy, the enthalpy itself being taken from (6.3). The procedure is very similar to the transformation of the total energy principle of elasticity into the complementary energy principle. One first looks after the variable conjugate to  $\varrho$  with respect to the energy per unit volume, which turns out to be the enthalpy:

$$\frac{d}{d\varrho} (\varrho E) = E + \varrho \frac{dE}{d\varrho} = I.$$

The complementary energy is then built up by a contact transformation and turns out to be the pressure

$$\varrho \frac{d}{d\varrho} (\varrho E) - \varrho E = \varrho (I - E) = p.$$

This complementary energy is to be considered as a function of the conjugate variable  $I$ . In fact, from the involutive property of contact transformations:

$$(7.1) \quad \frac{dp}{dI} = \varrho.$$

Bateman's first principle is precisely

$$(7.2) \quad \delta_x \int_{t_1}^{t_2} \int_D p \, dv \, dt = 0,$$

where  $p = p(I)$  and one considers *a priori* that

$$(7.3) \quad I = \frac{\mathbf{u} \cdot \mathbf{u}}{2} - \Omega - \left( \frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \text{grad } \theta \right) - \alpha \left( \frac{\partial \beta}{\partial t} + \mathbf{u} \cdot \text{grad } \beta \right).$$

In view of (7.1) the Euler equations are again (6.2), (6.4), (6.5) and (6.6).

Suppose now that (6.2) is used *a priori*; then  $\mathbf{u}$  and its variation should disappear. Replacing (6.2) into the kernel, it takes the form

$$(7.4) \quad K = -\varrho \left( E + \Omega + \frac{\partial \theta}{\partial t} + \alpha \frac{\partial \beta}{\partial t} + \frac{1}{2} (\text{grad } \theta + \alpha \text{grad } \beta) \cdot (\text{grad } \theta + \alpha \text{grad } \beta) \right).$$

The Euler equations of this principle are (6.3), (6.4), (6.5) and (6.6), where  $\mathbf{u}$  stands for the expression given by (6.2).

With this understanding, Bateman's second principle for the stationary case is sometimes loosely written as

$$(7.5) \quad \delta_x \int_{t_1}^{t_2} \int_D \varrho \left( E + \Omega + \frac{\mathbf{u} \cdot \mathbf{u}}{2} \right) dv \, dt = 0.$$

In most practical applications both modifications are used. In applying (7.2), the pressure is expressed as a function of the enthalpy and the enthalpy directly expressed in terms of the functions  $(\vartheta, \alpha, \beta)$

$$(7.6) \quad I = -\Omega - \frac{\partial \theta}{\partial t} - \alpha \frac{\partial \beta}{\partial t} - \frac{1}{2} (\text{grad } \theta + \alpha \text{grad } \beta) \cdot (\text{grad } \theta + \alpha \text{grad } \beta).$$

### 8. Isoenergetic Flows

The restriction to isentropic flow is easily removed to treat the more general case of isoenergetic flow, characterized by

$$(8.1) \quad \frac{dS}{dt} = 0.$$

This law of conservation of entropy for each particle, shows that  $S$  is a function of lagrangean variables only. Instead of introducing it as a side condition with a lagrangean multiplier, a procedure followed by HERIVEL [6] and ECKART [8], we prefer to keep the velocity field in the form (6.2) and consider the entropy as a function

$$(8.2) \quad S = S(\theta, \alpha, \beta).$$

With this understanding, and remembering that:

$$(8.3) \quad \frac{\partial E(\varrho, S)}{\partial \varrho} = p/\varrho^2 \quad \text{and} \quad \frac{\partial E(\varrho, S)}{\partial S} = T,$$

where  $T$  is the absolute temperature, the Euler equations of principle (6.1) are modified as follows: (6.2) and (6.3) remain true, (6.4) remains true provided:

$$(8.4) \quad \frac{\partial S}{\partial \theta} = 0 \quad \text{i.e.} \quad S = S(\alpha, \beta)$$

(6.5) and (6.6) are replaced by:

$$(8.5) \quad \frac{d\beta}{dt} = -T \frac{\partial S}{\partial \alpha}, \quad \frac{d\alpha}{dt} = T \frac{\partial S}{\partial \beta}.$$

Observe that from these equations

$$\frac{dS}{dt} = \frac{\partial S}{\partial \alpha} \frac{d\alpha}{dt} + \frac{\partial S}{\partial \beta} \frac{d\beta}{dt} = 0$$

as required.

The vortex lines are still given by the intersections of the surfaces  $\alpha = \text{constant}$  and  $\beta = \text{constant}$  but they are no more fixed in the fluid. Another consequence of (8.5) is

$$(8.6) \quad \frac{\partial \mathbf{u}}{\partial t} + \text{rot } \mathbf{u} \times \mathbf{u} = T \text{ grad } S$$

so that, by CROCCO's theorem

$$(8.7) \quad \text{grad} \left( I + \Omega + \frac{\mathbf{u} \cdot \mathbf{u}}{2} \right) = 0.$$

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