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DISPLACEMENT AND EQUILIBRIUM MODELS IN THE FINITE ELEMENT METHOD.

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1. Introduction.

The purpose of the finite elements method in matrix structural analysis is

- to define generalized loads and generalized displacement in terms of which the connection between elements is expressible by simple identification of displacements and transmission of loads;
- to relate an internal stress field and, if possible, an internal displacement field to the generalized quantities;
- to establish a stiffness matrix or any other convenient form of relations between generalized displacements describing the elastic properties of the element;
- to solve for the unknown quantities of the connected structure, taking into account the external loading and the appropriate boundary conditions.

The discretization, that necessarily occurs at some stage of a structural analysis when numerical results are to be produced, is here incorporated at the outset in the limited number of degrees of freedom allocated to the element deformation. This, as a rule, implies approximations either to the compatibility conditions or to the equilibrium conditions or to both.
The variety of models that can be set up to approximate the element behaviour can therefore be subdivided into three classes according to their properties of

- satisfying compatibility but not equilibrium,
- satisfying equilibrium but not compatibility,
- violating both equilibrium and compatibility.

Confronted with so many possibilities, it is natural that the question of accuracy of the overall analysis be raised with respect to any particular choice of model. In general the accuracy is established on an experimental basis by checking the analysis of simple structure against a known exact solution or by examining the convergence trends of analyses repeated with a finer network of elements. Much theoretical work remains to be done in order to establish the model requirements ensuring convergence to the exact solution when the size of the elements tends to zero.

There are however two particular types of models that allow a direct estimate of the convergence from energy considerations. For convenience they will be referred to as the displacement models'and the equilibrium models'.

Displacement models are the best known. They are based on parametric displacement fields ensuring compatibility of deformations both internal to the elements and across their boundaries. Under those conditions the whole displacement field of the connected structure is continuous and piecewise differentiable. The examples given are presented with a particular emphasis on the role of the displacement modes attached to the generalized variables as weight functions for the external loads.

Equilibrium models are based on parametric stress fields securing internal equilibrium and continuous stress transmission between elements. The whole stress field is then in equilibrium. (This chapter presents later a general theory for constructing such equilibrium models). The examples illustrate the possibilities of false kinematical deformation modes appearing either in a single element or in a combination of them and methods for dealing with such situations.

Arguments based on two-field variational principles are presented to justify the superiority of the minimal principles in deriving the stiffness properties of the elements. Still more powerful arguments are to be found in the upper and lower bound character of the approximate influence coefficients so obtained. Short proofs of the upper and lower bound properties are presented to make this chapter self contained.

A dual analysis of a given structure, based on the alternate use of displacement and equilibrium models, and submitted to the same external loads and boundary conditions allows a direct quantitative estimate of the convergence to the true solution by comparison of the upper and lower bounds obtained.

Both the direct stiffness method and the self-straining method for solving the connected fields of elements are given a brief review. The last one is the more promising for dealing with equilibrium models.

2. Variational principles applicable to simplify the analysis of finite elements.

The use of selected deformation modes or stress-transmission modes to approximate the elastic behaviour of a finite element must in the end result in a set of equations relating generalized loads to generalized displacements. This set is conveniently derived from energy theorems or variational principles, the most widely used being the principle of variation of displacements and the principle of variation of stresses, also called the complementary energy principle.

More general principles exist which, in theory, allow simultaneous approximations on displacements and stresses. It seems worthwhile to discuss them briefly, if only to show their limited usefulness.

For brevity, only the plane stress case will be considered; the extensions to three dimensional problems being obvious. By the term "field" will be understood a field of stresses and strains related by the energy equations

$$\sigma_x = \frac{\partial W}{\partial \varepsilon_x} \qquad \tau_{xy} = \frac{\partial W}{\partial \gamma_{xy}} \qquad \sigma_y = \frac{\partial W}{\partial \varepsilon_y} \qquad (1)$$

where $\mathcal{W}(\varepsilon_x, \gamma_{xy}, \varepsilon_y)$, more briefly denoted by $\mathcal{W}(\varepsilon)$, is the strain energy density, or energy per unit surface area in the reference condition. Hence $(\sigma_x, \tau_{xy}, \sigma_y)$ are not properly speaking stresses but rather normal loads and shear flows defined per unit thickness of slice. The same stress-strain relations are expressed by

$$\varepsilon_x = \frac{\partial \phi}{\partial \sigma_x} \qquad \gamma_{xy} = \frac{\partial \phi}{\partial \tau_{xy}} \qquad \varepsilon_y = \frac{\partial \phi}{\partial \sigma_y} \tag{2}$$

where $\phi(\sigma_x, \tau_{xy}, \sigma_y)$, more briefly denoted by $\phi(\sigma)$, is the complementary energy density defined by the Legendre transformation

$$\phi = \sigma_x \varepsilon_x + \tau_{xy} \gamma_{xy} + \sigma_y \varepsilon_y - \mathcal{W} \tag{3}$$

A field is "integrable" if single-valued, continuously differentiable displacement functions u(x, y) and v(x, y) exist such that

$$\varepsilon_x = \frac{\partial u}{\partial x} = u_x \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = u_y + v_x \qquad \varepsilon_y = \frac{\partial v}{\partial y} = v_y \tag{4}$$

The use of those equations implies that the analysis is also restricted to small strains and small rotations.

A field is "compatible" if, in addition to equations (4) the displacement functions satisfy prescribed boundary displacements on parts of the boundary denoted by Γ_u

$$u = \bar{u} \qquad v = \bar{v} \qquad \text{on} \quad \Gamma_u \tag{5}$$

A field is an "equilibrium field" if the stresses satisfy the following equilibrium equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \bar{X} = 0 \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \bar{Y} = 0 \tag{6}$$

$$p_x = l\sigma_x + m\tau_{xy} = \bar{p}_x \qquad p_y = l\tau_{xy} + m\sigma_y = \bar{p}_y \quad \text{on } \Gamma_\sigma \tag{7}$$

where (l, m) are the direction cosines of the outward normal to the boundary, $(\bar{X}, \bar{Y})$ are prescribed internal loads and $(\bar{p}_x, \bar{p}_y)$ prescribed boundary loads. The part Γ_σ of the boundary is complementary to Γ_u.

2.1. The general variational principle.

The general variational principle of elasticity can be stated as follows

$$\delta\,(U + P + D) = 0 \tag{8}$$

where : $U = \iint_A W(\varepsilon)\,dxdy$ is the total strain energy, calculated from a field represented by the strains $(\varepsilon_x, \gamma_{xy}, \varepsilon_y)$

$$P = -\iint_A (\bar{X}u + \bar{Y}v)\,dx\,dy - \int_{\Gamma_\sigma} (\bar{p}_x u + \bar{p}_y v)\,ds$$

is the potential energy of prescribed loads, calculated from an integrable field represented by the displacements (u, v);

$$D = \iint_A \left[\sigma_x'(u_x - \varepsilon_x) + \tau_{xy}'(u_y + v_x - \gamma_{xy}) + \sigma_y'(v_y - \varepsilon_y)\right] dx\,dy$$
$$+ \int_{\Gamma_u} \left[p_x'(\bar{u} - u) + p_y'(\bar{v} - v)\right] ds$$

is a dislocation potential introducing a third field represented by the stresses $(\sigma_x', \tau_{xy}', \sigma_y')$. These stresses can be considered as Lagrangean multipliers removing the compatibility constraints (4) and (5) that must be satisfied a priori in the simpler principle of variation of displacements

$$\delta(U+P)=0$$

In this simpler principle only displacements (u, v) are involved and subject to variations. In the general principle all three fields are subject to independant variations. Those on the Lagrangean multipliers restore the compatibility conditions (4) and (5). Those on the $(\varepsilon_x, \gamma_{xy}, \varepsilon_y)$ field require that

$$\sigma_x = \frac{\partial W}{\partial \varepsilon_x} = \sigma_x' \qquad \tau_{xy} = \frac{\partial W}{\partial \gamma_{xy}} = \tau_{xy}' \qquad \sigma_y = \frac{\partial W}{\partial \varepsilon_y} = \sigma_y' \tag{9}$$

In other words they require that the Lagrangean multipliers become identical with the stresses of the $(\varepsilon_x, \gamma_{xy}, \varepsilon_y)$ field. Finally the variations on (u, v) require that the Lagrangean multipliers be an equilibrium field; the stresses $(\sigma_x', \tau_{xy}', \sigma_y')$ should satisfy the equilibrium equations (6) and (7). A functional diagram of this general principle is illustrated on

Fig. 1.

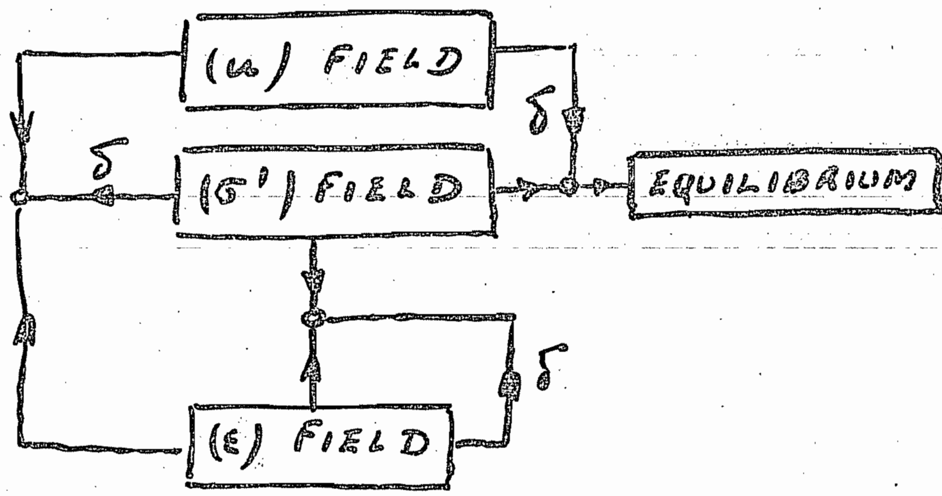

Fig. 1 and is self explanatory. In the end there is complete identification between the three fields and the solution must, as it should, be compatible and in equilibrium.

2.2. E. Reissner's principle.

By making a priori assumptions on the multipliers the general principle can be simplified, still leaving us with two independant fields. The assumption leading to Reissner's principle [3,4] is that equations (9) are satisfied a priori. Then, in view of definition (3), the general theorem reduces to

$$\delta \left\{ \iint_A \left[\sigma_x u_x + \tau_{xy}(u_y + v_x) + \sigma_y v_y - \phi(\sigma) \right] dx\,dy \right.$$

$$\left. + P + \int_{\Gamma_u} \left[p_x(\bar{u} - u) + p_y(\bar{v} - v) \right] ds \right\} = 0 \qquad (10)$$

The functional diagram of this is illustrated on Fig. 2.

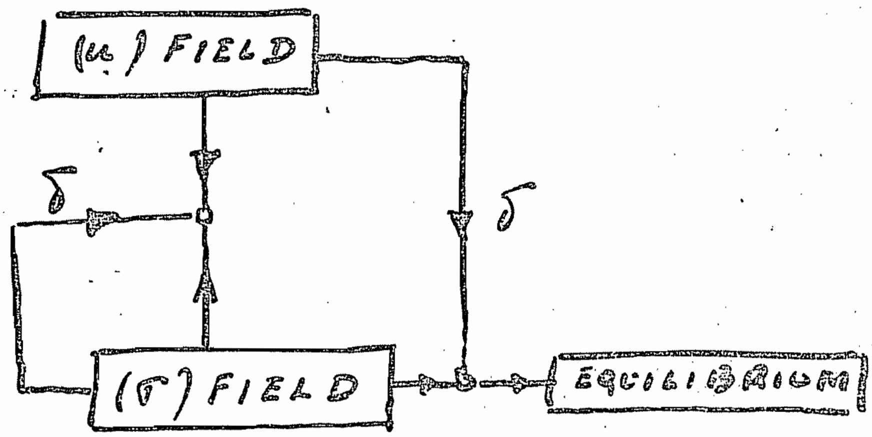

Fig. 2.

The theoretical possibility of using separate approximation on the displacements (u, v) and the stresses $(\sigma_x, \tau_{xy}, \sigma_y)$, with this theorem, raises the following points :

(a) If no restrictive assumptions are made on the stresses, their variation requires

$$u_x = \frac{\partial \phi}{\partial \sigma_x} = \varepsilon_x \qquad u_y + v_x = \frac{\partial \phi}{\partial \tau_{xy}} = \gamma_{xy} \qquad v_y = \frac{\partial \phi}{\partial \sigma_y} = \varepsilon_y \qquad (11)$$

and

$$u = \bar{u} \qquad v = \bar{v} \qquad \text{on } \Gamma_u \qquad (12)$$

Equations (11) show that $(\sigma_x, \tau_{xy}, \sigma_y)$ are the stresses related to the strains $(u_x, u_y + v_x, v_y)$ so that, in view of (3), Reissner's principle (10) reduces to the ordinary displacement principle

$$\delta \left\{ \iint_A W(u_x, u_y + v_x, v_y)\, dx\, dy + P \right\} = 0 \qquad (13)$$

This result can be stated in the form of a limitation principle : If a net of finite elements is analyzed by compatible displacement modes and the stresses left free to be determined by energy considerations, the best stresses are those associated with the strains derived from the displacements and the degrees of freedom in the displacement modes are governed by the ordinary principle of variation of displacements. In other words it is useless to look for a better solution by injecting additional degrees of freedom in the stresses, although the stresses obtained will not, as a rule satisfy the detailed equilibrium conditions.

(b) Conversely, if no restrective assumptions are made on the displacements, variations on those will require the stresses to be solutions of the equilibrium equations. This is precisely the type of approach one wishes to use in the stress model analysis of finite elements. If we then assume a stress field in equilibrium and consider the following integration by parts

$$\iint_A \left[\sigma_x u_x + \tau_{xy}(u_y + v_x) + \sigma_y v_y \right] dx\, dy$$

$$= \int_{\Gamma_u + \Gamma_\sigma} (p_x u + p_y v)\, ds - \iint_A \left[u \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} \right) + v \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} \right) \right] dx\, dy$$

we obtain, in view of equations (6) and (7)

$$\iint_A \left[\sigma_x u_x + \tau_{xy}(u_y + v_x) + \sigma_y v_y \right] dx\, dy = - P + \int_{\Gamma_u} (p_x u + p_y v)\, ds$$

This result, substituted into (10) reduces Reissner's principle to the complementary energy principle

$$\delta \left\{ \iint_A \Phi(\sigma)\, dx\, dy - \int_{\Gamma_u} \left(p_x \bar{u} + p_y \bar{v} \right) ds \right\} = 0 \qquad (14)$$

This can again be stated as another limitation principle : If the stresses $(\sigma_x, \tau_{xy}, \sigma_y)$ form an equilibrium field a priori their degrees of freedom are governed by the ordinary complementary energy principle. in as much at the resulting strains are not, as a rule integrable the principle gives no indication concerning a best associated displacement field.

It would be wrong to conclude from the limitation principles that Reissner's principle is worthless. It can be used to advantage in developing a consistent equilibrium analysis based on partial assumptions on the stresses.
A good example is provided by Reissner's theory of plate bending[5,6]

2.3. Another two-field variational principle.

Another way to simplify the general principle is to endow the multipliers with the property of equilibrium. Then, through an integration by parts similar to that achieved before, the general principle reduces to :

$$\delta \left\{ \iint_A \left[W(\varepsilon) - \left(\sigma_x' \varepsilon_x + \tau_{xy}' \gamma_{xy} + \sigma_y' \varepsilon_y \right) \right] dx\, dy + \int_{\Gamma_u} \left(p_x' \bar{u} + p_y' \bar{v} \right) ds \right\} = 0 \qquad (15)$$

It contains the equilibrium field $(\sigma_x', \tau_{xy}', \sigma_y')$ and the arbitrary field $(\varepsilon_x, \gamma_{xy}, \varepsilon_y)$. Its functional diagram is represented on Fig. 3. The limitations

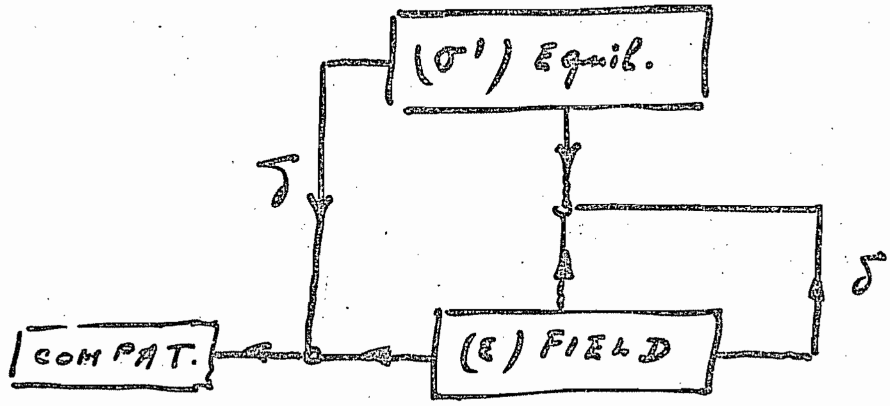

Fig. 3.

involved in the use of simultaneous approximations are very similar to those of Reissner's principle;

(a) A strain field free of restrictive assumptions will result in Euler-Lagrange equations

$$\sigma_x = \frac{\partial W}{\partial \varepsilon_x} = \sigma_x' \qquad \tau_{xy} = \frac{\partial W}{\partial \gamma_{xy}} = \tau_{xy}' \qquad \sigma_y = \frac{\partial W}{\partial \varepsilon_y} = \sigma_y'$$

identifying the best strain field with the one associated to the equilibrium field. As a result (15) again degenerates into the complementary energy principle (14) that will govern the degrees of freedom in the equilibrium field. Unless the strain field associated to this equilibrium field is integrable, no indications are given concerning displacements.

(b) If, in order to obtain displacements, the strain field is assumed a priori to be integrable, substitution of equations (4) followed by an integration by parts and consideration of the equilibrium equations satisfied by the (σ') field, transform the principle back to the displacement principle (13).

2.4. Orthogonality of equilibrium solutions and compatible solutions in function space.

Up to now the stress-strain relations expressed by equations (1) or (2) were kept of general nature; in particular the conclusions reached before are valid for non-linear relations.

In the case of linear stress-strain relations the absence of interaction between compatible fields and equilibrium fields can be presented in a more striking form. Both the strain energy denssity and the complementary energy density are quadratic homogeneous positive definite forms so that, by Euler's theorem, the following numerical equivalence exists

$$W = \frac{1}{2}\left(\sigma_x \varepsilon_x + \tau_{xy}\gamma_{xy} + \sigma_y \varepsilon_y\right) = \phi \tag{16}$$

It is a local expression of Clapeyron's theorem. Furthermore the expansion of the energy density of the difference between two fields can be written

$$W(\varepsilon - \varepsilon') = W(\varepsilon) + W(\varepsilon') - I(\varepsilon, \sigma') \tag{17}$$

with

$$W(\varepsilon - \varepsilon') = \phi(\sigma - \sigma') \qquad W(\varepsilon) = \phi(\sigma) \qquad W(\varepsilon') = \phi(\sigma')$$

where

$$\begin{aligned} I(\varepsilon, \sigma') &= \varepsilon_x \sigma'_x + \gamma_{xy}\tau'_{xy} + \varepsilon_y \sigma'_y \\ &= \varepsilon'_x \sigma_x + \gamma'_{xy}\tau_{xy} + \varepsilon'_y \sigma_y = I(\varepsilon', \sigma) \end{aligned} \tag{18}$$

This is a local form of Betti-Rayleigh's reciprocal theorem. If one considers a field in the domain Δ as a point F in function space, the distance d(F, F') between two fields is conveniently defined by $d(F, F') = \sqrt{d^2(F, F')}$

$$d^2(F, F') = 2\iint_\Delta W(\varepsilon - \varepsilon')\,dx\,dy = 2\iint_\Delta \phi(\sigma - \sigma')\,dx\,dy \tag{19}$$

This definition satisfies the required axioms of distance (see for instance ref. 5).

$$d(F, F') = d(F', F)$$

$$d(F, F') + d(F', F'') \geq d(F, F'')$$

$$d(F, F') = 0 \iff F \equiv F'$$

The function space is thereby provided with a metric[6]; in particular the scalar product (F, F') between two fields is thereby defined. From

$$d^2(F,F') = d^2(F,0) + d^2(F',0) - 2(F,F')$$

where 0 denotes the origin (stresses and strains identically zero), applying the definition of distance

$$(F,F') = \iint_A \left[W(\varepsilon) + W(\varepsilon') - W(\varepsilon - \varepsilon') \right] dx\, dy$$

And finally, in view of equations (17) and (18)

$$(F,F') = \iint_A (\varepsilon_x \sigma_x' + \gamma_{xy} \tau_{xy}' + \varepsilon_y \sigma_y')\, dx\, dy$$

$$= \iint_A (\varepsilon_x' \sigma_x + \gamma_{xy}' \tau_{xy} + \varepsilon_y' \sigma_y)\, dx\, dy = (F',F) \tag{20}$$

Consequently the scalar product is the total interference energy between the two fields.

Of special interest will be the study of the distance between a compatible field, for which equation (4) and (5) are valid and an equilibrium field in which the stresses satisfy equations (6) and (7). Let this be the case respectively for the fields F and F'. Their scalar product is then expressible in terms of the prescribed loads and boundary displacements :

$$(F,F') = \iint_A \left(u_x \sigma_x' + (u_y + v_x) \tau_{xy}' + v_y \sigma_y' \right) dx\, dy$$

becomes after integration by parts and use of the equilibrium equations satisfied by the field F' and equations (5)

$$(F,F') = \iint_\Delta (\bar{X}u + \bar{Y}v)\,dx\,dy$$

$$+ \int_{\Gamma_\sigma} (\bar{p}_x u + \bar{p}_y v)\,ds + \int_{\Gamma_u} (p'_x \bar{u} + p'_y \bar{v})\,ds \qquad (21)$$

It follows then from (17) that the distance between the two fields can be placed into the form

$$\frac{1}{2} d^2(F,F') =$$

$$\iint_\Delta W(\varepsilon)\,dx\,dy - \iint_\Delta (\bar{X}u + \bar{Y}v)\,dx\,dy - \int_{\Gamma_\sigma} (\bar{p}_x u + \bar{p}_y v)\,ds$$

$$+ \iint_\Delta \Phi(\sigma')\,dx\,dy - \int_{\Gamma_u} (p'_x \bar{u} + p'_y \bar{v})\,ds \qquad (22)$$

Since the exact solution must enjoy both qualities of equilibrium and compatibility, the distance between a compatible approach and an equilibrium approach must in the end vanish. Conversely, when the distance vanishes we know that both fields are identical and must then represent the exact solution. Hence a suitable energy principle will be

$$d^2(F,\ F') \text{ minimum} \qquad (23)$$

Inspection of equation (22) reveals however that this principle spontaneously degenerates in two other principles :

$$\iint_\Delta W(u_x, u_y + v_x, v_y)\,dx\,dy + P \quad \text{minimum} \qquad (24)$$

$$\iint_A \phi(\sigma^0)\,dx\,dy - \int_{\Gamma_u} (p_x'\bar{u} + p_y'\bar{v})\,ds \text{ minimum} \qquad (25)$$

The first is the principle of minimum total potential, the second the principle of minimum complementary energy. They are both more precise statements of principles (13) and (14) under the linear stress-strain assumption.

It is clear that approximations on displacements will not interact with approximations on equilibrium stresses and conversely, each will be governed by its own independant principle. It is one aspect of the orthogonality in function space of the subspace of equilibrium solutions and the subspace of compatible solutions.

In the applications, the minimum of total potential will be used to establish the properties of displacement models, the minimum complementary energy for stress models. By securing the continuity of displacements between elements, or the continuity of stress transmission, both principles will also apply to the structure as a whole. It can then be established[9,10] that a lower bound to a direct influence coefficient is produced by the displacement analysis, an upper bound by the stress analysis. Furthermore the results of both analyses allow a similar enclosure to be built for the mutual influence coefficients[11,12,13,14]. This already provides a powerful control on the overall accuracy of the analysis in finite elements. Bounds on local stress values can also be obtained in principle through the use of Green functions[8]; the amount of analytical and numerical work involved seems however too high to be of real practical utility.

3. Displacement model analysis.

3.1. General theory.

Let

$$u = \sum_i \alpha_i U_i(x,y) \qquad v = \sum_i \alpha_i V_i(x,y) \qquad (26)$$

be a displacement field expressed in terms of assumed displacement modes ($U_i(x,y)$, $V_i(x,y)$) with unknown amplitudes α_i. The column matrix of the amplitudes will be denoted by α ; its transpose α^T is the row matrix

$$\alpha^T = (\alpha_1, \alpha_2 \ldots \alpha_n)$$

The following matrix notations are also introduced

$$\varepsilon^T = (\varepsilon_x , \varepsilon_y , \gamma_{xy})$$

$$\sigma^T = (\sigma_x , \sigma_y , \tau_{xy})$$

From the displacement field follows by taking partial derivatives

$$\varepsilon = B \alpha \qquad \varepsilon^T = \alpha^T B^T \tag{27}$$

with the (nx3) matrix

$$B^T = \left(\frac{\partial U_i}{\partial x} , \frac{\partial V_i}{\partial y} , \frac{\partial U_i}{\partial y} + \frac{\partial V_i}{\partial x} \right) \quad \text{row index } i = 1, 2, 3.$$

The linear stress-strain relations are given in matrix form by

$$\sigma = D \varepsilon \tag{28}$$

where, taking as example an isotropic plate in plane stress of thickness $t(x,y)$,

$$D = \frac{E\, t(x,y)}{1-\nu^2} \begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{pmatrix} \tag{29}$$

The strain energy density can now be expressed as a quadratic form in the α_i starting from Clapeyron's theorem

$$W = \frac{1}{2}\,\sigma^T\varepsilon = \frac{1}{2}\,\varepsilon^T \mathcal{D}\,\varepsilon = \frac{1}{2}\,\alpha^T\left(B^T \mathcal{D} B\right)\alpha \qquad (30)$$

Generalized displacements (q_j) are next defined according to certain rules :

- connection of the element with another element across a common boundary is to be expressed by equating, two by two, the generalized displacements pertaining to this boundary,
- the whole displacement field along this boundary must be determined uniquely by the generalized displacements pertaining to the boundary and conversely. In this manner, equating the generalized displacements secures the continuity of displacements along the boundary,
- the number of parameters α_i must be equal or superior to the number of required generalized displacements. If superior, convenient additional generalized displacements can be chosen so that the matrices α and q can be connected by a non singular linear transformation of matrix $T = (T_{ij})$

$$\alpha = T q \qquad q = T^{-1}\alpha \qquad (31)$$

The unrequired components of q can be eliminated later (see example of section 3.4.). From (30) and (31) the total strain energy of the element becomes a quadratic function of the generalied displacements

$$U = \iint_{\Delta} W\,dx\,dy = \frac{1}{2}\,q^T K q \qquad (32)$$

with

$$K = T^T\left\{\iint_{\Delta}\left(B^T \mathcal{D} B\right)dx\,dy\right\}T = K^T \qquad (33)$$

No prescribed boundary displacements are used; to apply the minimum total energy principle there remains to express the potential energy of the prescribed loads. From (26)

$$\mathcal{P} = - \sum_i \alpha_i P_i \tag{34}$$

$$P_i = \iint_A (\bar{X} U_i + \bar{Y} V_i)\, dx\, dy + \int_\Gamma (\bar{p}_x U_i + \bar{p}_y V_i)\, ds \tag{35}$$

With the matrix notation

$$p^T = (P_1, P_2 \ldots P_n)$$

equation (34) assumes the form of a scalar product

$$\mathcal{P} = - p^T \alpha = - g^T q \tag{36}$$

the last result following from transformation (31) and introduction of the row matrix

$$g^T = (Q_1, Q_2 \ldots Q_n) = p^T T \tag{37}$$

As implied by the scalar product form of the potential energy, the elements of the row matrix g^T are the generalized loads conjugate to the generalized displacements. They finally derive from the actual loading of the element by the use of weight functions

$$X_j(x,y) = \sum_i U_i(x,y) T_{ij} \qquad Y_j(x,y) = \sum_i V_i(x,y) T_{ij} \tag{38}$$

$$Q_j = \iint_A (\bar{X} X_j + \bar{Y} Y_j)\, dx\, dy + \int_\Gamma (\bar{p}_x X_j + \bar{p}_y Y_j)\, ds \tag{39}$$

The weight functions depend solely on the assumed displacement modes and the choice of the generalized displacements. Once calculated they can also be used directly to establish the stiffness matrix, for one has

$$u = \sum_j q_j X_j(x,y) \qquad v = \sum_j q_j Y_j(x,y) \tag{40}$$

and $\varepsilon = Nq$ with $N = BT.$ (41)

The elements of the matrix N^T are calculated from partial derivatives of the weight functions as those of B^T were from the originally assumed modes.

The matrix K can then be calculated from the formula

$$K = \iint_\Omega (N^T D N)\, dx\, dy \tag{42}$$

Stated in terms of the generalized quantities, the minimum total energy principle is

$\frac{1}{2} q^T K q - q^T g$ should be a minimum.

It furnishes the condition $\delta q^T(Kq-g) = 0$ for an arbitrary row matrix δq^T and consequently the matrix equation

$$g = Kq. \tag{43}$$

This set of equations gives the generalized loads in terms of the genaralized displacements; the matrix K is known as the "stiffness matrix" of the element. Observe that under a change of generalized displacements $q = J\bar{q}$, the potential energy preserves its scalar product form $-\bar{g}^T\bar{q} = -g^t q$ if g transforms according to the rule $\bar{g} = J^T g$ and equation (43) turns into

$$\bar{g} = \bar{K}\bar{q} \quad \text{with} \quad \bar{K} = J^T K J,$$

q is of course the matrix representation of a contravariant tensor, g that of a covariant tensor and K that of twice covariant tensor.

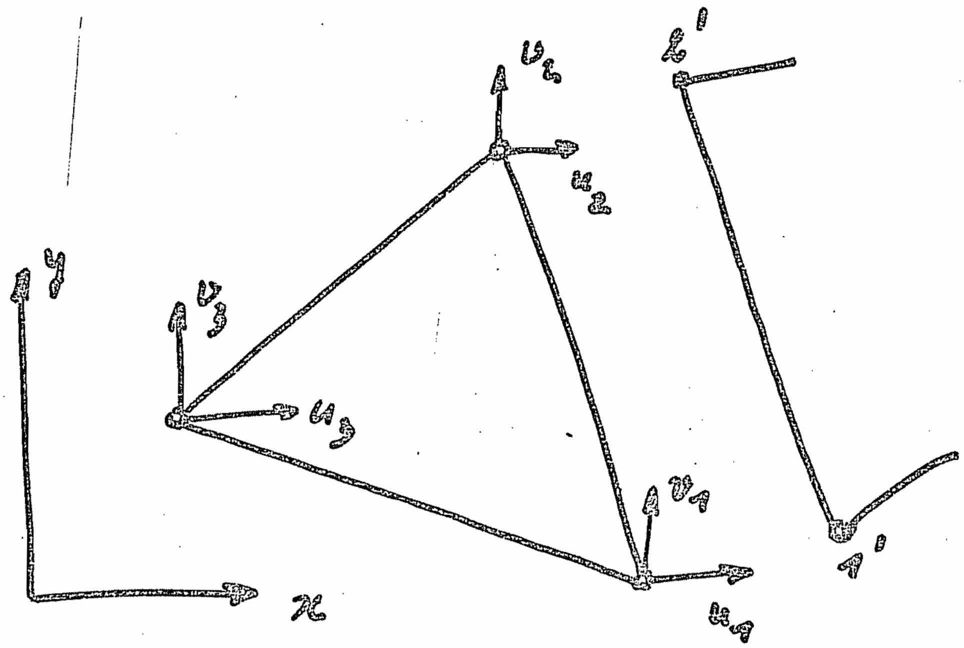

Fig. 4.

3.2. Stiffness of triangular panels and beam segments. Linear displacement analysis.

Those are early examples of displacement models introduced in the literature by Turner, Clough, Martin and Topp[15]. The assumed displacement field is linear and contains the three rigid body modes

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y$$

$$v = \alpha_4 + \alpha_5 x + \alpha_6 y$$

The generalized coordinates are chosen to be the local displacements at the vertices of the triangle (Fig. 4)

$$q^T = (u_1, u_2, u_3, v_1, v_2, v_3).$$

This choice is according to the rules, for along a side of the triangle the displacements vary linearly and are uniquely determined by the displacements at both of its ends. With

$$u_i = \alpha_1 + \alpha_2 x_i + \alpha_3 y_i$$

$$v_i = \alpha_4 + \alpha_5 x_i + \alpha_6 y_i \qquad i = 1, 2, 3.$$

the inverse of the transformation matrix is deduced

$$T^{-1} = \begin{pmatrix} 1 & x_1 & y_1 & 0 & 0 & 0 \\ 1 & x_2 & y_2 & 0 & 0 & 0 \\ 1 & x_3 & y_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_1 & y_1 \\ 0 & 0 & 0 & 1 & x_2 & y_2 \\ 0 & 0 & 0 & 1 & x_3 & y_3 \end{pmatrix}.$$

It can be observed that, provided the vertices are numbered in the anticlockwise sense, the determinant

$$\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = 2\mathcal{A},$$

is equal to twice the area of the triangle. After an elementary inversion to obtain the matrix T, the weight functions are found to be

$$u = u_1 W_1(x,y) + u_2 W_2(x,y) + u_3 W_3(x,y),$$
$$v = v_1 W_1(x,y) + v_2 W_2(x,y) + v_3 W_3(x,y), \qquad (44)$$

$$W_1(x,y) = \frac{1}{2\mathcal{A}}\left[x_2 y_3 - x_3 y_2 + x(y_2 - y_3) + y(x_3 - x_2)\right] \qquad (45)$$

the other two following from a cyclic index permutation.

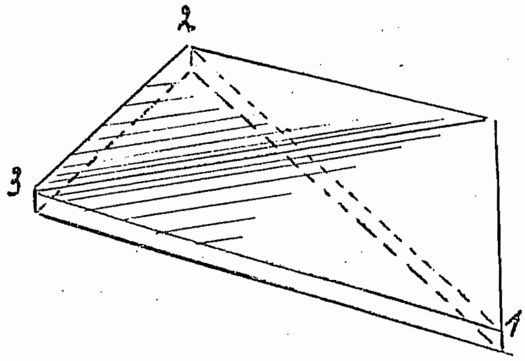

Fig. 5.

The weight function $W_1(x,y)$ is illustrated on Fig. 5. The generalized load conjugate to u_i is

$$\iint_{\Delta} \bar{X}\, W_i(x,y)\, dx\, dy + \int_{\Gamma} \bar{p}_x\, W_i(x,y)\, ds$$

The loads conjugate to v_i follows by replacing $\bar{X}$ with $\bar{Y}$ and $\bar{p}_x$ with $\bar{p}_y$.

From equations (44) and (45) the matrix N of equations (41) turns out to be

$$N = \frac{1}{2\Delta} \begin{bmatrix} y_2-y_3 & y_3-y_1 & y_1-y_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_3-x_2 & x_1-x_3 & x_2-x_1 \\ x_3-x_2 & x_1-x_3 & x_2-x_1 & y_2-y_3 & y_3-y_1 & y_1-y_2 \end{bmatrix} \qquad (46)$$

Finally, denoting by $\bar{t}$ the average thickness of the plate

$$\bar{t} = \frac{1}{\omega} \iint_{\Delta} t(x,y)\, dx\, dy$$

the stiffness matrix of the triangular plate turns out to be

$$K = \frac{E \omega \bar{t}}{1-\nu^2} N^T \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} N \tag{47}$$

In a structure that can be subdivided into such triangular panels and beam segments the only additional information necessary is the stiffness matrix

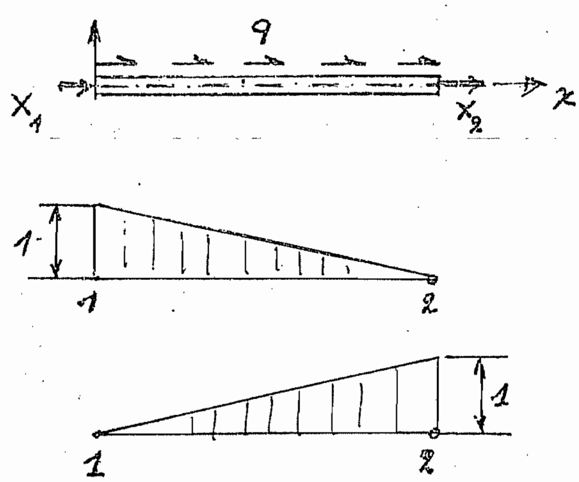

Fig. 6.

of a beam segment (Fig. 6). It is assumed that this element undergoes only uniaxial strain and, for simplicity of calculation, the reference axis Ox is taken parallel to the element. For displacement compatibility with the edges of the triangular panels to which the beam segment is attached, the axial displacement is again taken to be a linear function

$$u = \alpha_1 + \alpha_2 x$$

In terms of the end displacements taken as generalized coordinates

$$u = u_1 W_1(x) + u_2 W_2(x)$$

$$W_1(x) = \frac{x_2 - x}{x_2 - x_1} \qquad W_2(x) = \frac{x - x_1}{x_2 - x_1}$$

If the loads on the segment consist of end loads X_1 and X_2 and a shear flow distribution $q_{12}(x)$, the potential energy

$$-P = \int_{x_1}^{x_2} q_{12}(x) u(x)\, dx + X_1 u_1 + X_2 u_2$$

becomes, after substitution of $u(x)$

$$-P = u_1 Q_1 + u_2 Q_2$$

and the generalized loads conjugate to u_1 and u_2 are

$$Q_i = X_i + \int_{x_1}^{x_2} q_{12}(x) W_i(x)\, dx \qquad (i = 1, 2)$$

The weight functions $W_i(x)$ are illustrated on Fig. 6. The strain in the segment is

$$\varepsilon = u_1 \frac{dW_1}{dx} + u_2 \frac{dW_2}{dx} = \frac{u_2 - u_1}{x_2 - x_1} = n^T q$$

$$n^T = \frac{1}{x_2 - x_1} \begin{pmatrix} -1 & 1 \end{pmatrix}$$

The normal load in the segment is $L=ES(x)\varepsilon$, where $S(x)$ is the cross sectional area of the beam. The strain energy

$$\frac{1}{2}\int_{x_1}^{x_2}\varepsilon L\,dx = \frac{1}{2}\left(\int_{x_1}^{x_2} ES(x)\,dx\right) q^T n n^T q \; .$$

The stiffness matrix

$$K = \left(\int_{x_1}^{x_2} ES(x)\,dx\right) n n^T = \frac{\int_{x_1}^{x_2} ES(x)\,dx}{(x_2 - x_1)^2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

3.3. <u>Stiffness of triangular panels and beam segments. Quadratic displacement analysis.</u>

Both displacements are taken to be general quadratic expressions in the cartesian coordinates. Along an edge they will both vary according to a parabolic law and will be determined by their values in three points along the edge. Accordingly a suitable choice for the generalized displacement coordinates are the twelve local displacement components at the vertices and at the mid points of the edges, Fig. 7.

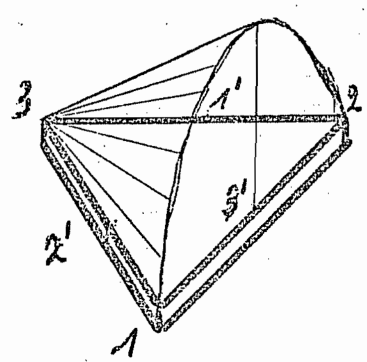

Fig. 7.

The inverse of the matrix T is obtained from the equations

$$u_i = \alpha_1 + \alpha_2 x_i + \alpha_3 y_i + \alpha_4 x_i^2 + \alpha_5 x_i y_i + \alpha_6 y_i^2 \qquad i = 1, 2, 3$$

$$v_i = \alpha_7 + \alpha_8 x_i + \alpha_9 y_i + \alpha_{10} x_i^2 + \alpha_{11} x_i y_i + \alpha_{12} y_i^2 \qquad = 1', 2', 3'.$$

Instead of inverting this matrix analytically, it is much simpler to establish directly the weight functions. Only two different types of weight function need to be calculated:

$W_{3'}(x,y)$ sketched on Fig. 7 and $W_3(x,y)$ sketched on Fig. 8. The other ones follow from a cyclic index permutation. Both must be of the type

$$W = A + Bx + Cy + Dx^2 + Exy + Fy^2$$

In the case of $W_{3'}$ we can express that it is identically zero along the straight lines

$$y = y_3 + \frac{y_{13}}{x_{13}}(x - x_3) \qquad \text{and} \qquad y = y_3 + \frac{y_{23}}{x_{23}}(x - x_3)$$

where in general,

$$x_{mn} = x_m - x_n \qquad y_{mn} = y_m - y_n$$

This yields six equations between the constants (A,B,C,D,E,F) only five of which are independant. The sixth equation expresses that the weight function is equal to unity in the point

$$x = \tfrac{1}{2}(x_1 + x_2) \qquad y = \tfrac{1}{2}(y_1 + y_2).$$

These equations are extremely simple to solve and one obtains

$$\delta^2 W_{3'}(x,y) = -a_{13} a_{23} - x(y_{13} a_{23} + y_{23} a_{13})$$
$$+ y(x_{13} a_{23} + x_{23} a_{13}) - x^2 y_{13} y_{23} + xy(x_{13} y_{23} + x_{23} y_{13}) - y^2 x_{13} x_{23} \qquad (48)$$

with the additional notation $a_{mn} = x_m y_n - x_n y_m$

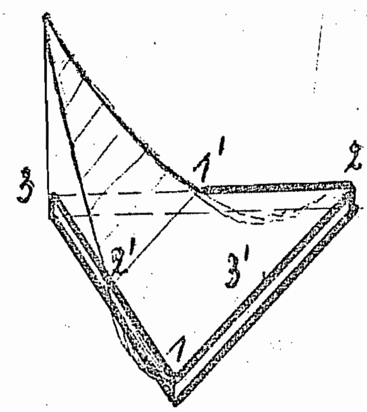

Fig. 8.

In the case of W_3 one expresses its nullity along the lines

$$y = y_{1'} + \frac{y_{21}}{x_{21}}(x - x_{1'}) = \frac{1}{2}(y_2 + y_3) + \frac{y_{21}}{x_{21}}\left[x - \frac{1}{2}(x_2 + x_3)\right]$$

and

$$y = y_1 + \frac{y_{21}}{x_{21}}(x - x_1)$$

and its value of unity at the point $x = x_3$, $y = y_3$. Thus

$$2\Delta^2 W_3(x,y) = a_{21}(a_{21} + \Delta) + x y_{21}(2a_{21} + \Delta)$$

$$- y x_{21}(2a_{21} + \Delta) + x^2 y_{21}^2 - 2xy\, y_{21} x_{21} + y^2 x_{21}^2 \qquad (49)$$

As before the symbol $\mathcal{A}$ denotes the area of the triangle. The displacement field can now be written as

$$u = u_1 W_1 + u_2 W_2 + u_3 W_3 + u_{1'} W_{1'} + u_{2'} W_{2'} + u_{3'} W_{3'}$$

$$v = v_1 W_1 + v_2 W_2 + v_3 W_3 + v_{1'} W_{1'} + v_{2'} W_{2'} + v_{3'} W_{3'}$$

and the matrix N is set up by partial differentiation of the weight functions. For the purpose of numerical calculation it is convenient to expand the N matrix in the form

$$2\mathcal{A}^2 N = N_0 + x N_x + y N_y$$

where the matrices N_0 , N_x and N_y have constant elements. The stiffness matrix calculated from equation (42) is then expanded in the form

$$K = \frac{1}{4\mathcal{A}^2}\Big[N_0^T D_0 N_0 + 2 N_0^T D_x N_x + 2 N_0^T D_y N_y + N_x^T D_{xx} N_x + 2 N_x^T D_{xy} N_y + N_y^T D_{yy} N_y \Big]$$

where

$$D_0 = \iint_\Delta D\,dx\,dy \qquad D_x = \iint_\Delta D\,x\,dx\,dy \qquad D_y = \iint_\Delta D\,y\,dx\,dy$$

$$D_{xx} = \iint_\Delta D x^2\,dx\,dy \qquad D_{xy} = \iint_\Delta D x y\,dx\,dy \qquad D_{yy} = \iint_\Delta y^2 D\,dx\,dy$$

In the case of an isotropic plate of constant thickness D is a matrix of constants and

$$\mathcal{D}_0 = \mathcal{D}\mathcal{A} \qquad \mathcal{D}_x = \mathcal{D}\iint_{\Delta} x\,dx\,dy \qquad \mathcal{D}_y = \mathcal{D}\iint_{\Delta} y\,dx\,dy$$

$$\mathcal{D}_{xx} = \mathcal{D}\iint_{\Delta} x^2\,dx\,dy \qquad \mathcal{D}_{xy} = \mathcal{D}\iint_{\Delta} xy\,dx\,dy \qquad \mathcal{D}_{yy} = \mathcal{D}\iint_{\Delta} y^2\,dx\,dy$$

An appreciable saving of computational time is then obtained by a preliminary shift of the coordinates to the centre of area; then $(x_1 + x_2 + x_3) = 0$, $(y_1 + y_2 + y_3) = 0$, D_x and D_y vanish, while

$$\iint_{\Delta} x^2\,dx\,dy = \frac{\mathcal{A}}{12}\left(x_1^2 + x_2^2 + x_3^2\right)$$

$$\iint_{\Delta} xy\,dx\,dy = \frac{\mathcal{A}}{12}\left(x_1 y_1 + x_2 y_2 + x_3 y_3\right)$$

$$\iint_{\Delta} y^2\,dx\,dy = \frac{\mathcal{A}}{12}\left(y_1^2 + y_2^2 + y_3^2\right)$$

A beam segment attached to an edge of such a triangular plate will have a compatible deformation provided the axial displacement be a parabolic function determined by the local displacements (u_1, u_2, u_3) at the ends and at the central station (Fig.9). Thus

$$u = u_1 W_1(x) + u_2 W_2(x) + u_3 W_3(x)$$

where
$$W_1(x) = \frac{x_3(x_1+x_3) - x(x_1+3x_3) + 2x^2}{(x_3-x_1)^2}$$

$$W_2(x) = 4\,\frac{-x_1x_3 + x(x_1+x_3) - x^2}{(x_3-x_1)^2}$$

$$W_3(x) = \frac{x_1(x_1+x_3) - x(3x_1+x_3) + 2x^2}{(x_3-x_1)^2}$$

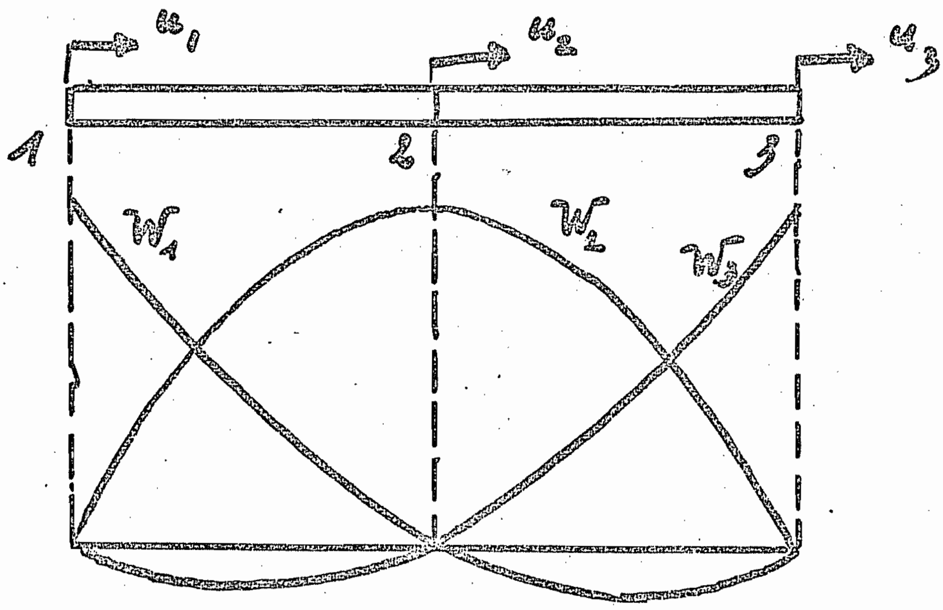

Fig. 9.

The axial strain is $\varepsilon = n^T q$ with

$$n^T = \frac{1}{(x_3-x_1)^2}\Big(4x-(x_1+3x_3) \quad 4(x_1+x_3-2x) \quad 4x-(x_3+3x_1)\Big)$$

The stiffness matrix can then be calculated from

$$K = \int_{x_1}^{x_3} ES(x)\, n n^T dx$$

In the case when cross section S is a constant :

$$K = \frac{ES}{3l} \begin{pmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{pmatrix}$$

l being the length of the segment.

3.4. <u>Stiffness of a spar segment. Displacement model.</u>

To avoid an exaggerated stiffness of a segment of a spar of the type used in aircraft construction, elementary considerations should first be applied to establish the minimum order of polynomial expansions involved. The representation should be able to provide an almost exact solution to the bending of the spar under constant transverse shear or linearly increasing bending moment. This implies that the axial strain be at least a linear function of x , the axial displacement a parabolic function of x. From integration of the elementary curvature equation of the beam it is then also true that the vertical displacement should be a function of the third degree. Accordingly the following assumptions are introduced for spar bending (see Fig. 10)

$$u(x,z) = z\alpha(x) \tag{50}$$

where the cross section rotation is

$$\alpha(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 \tag{51}$$

and

$$w(x,z) = W(x) = \alpha_4 + \alpha_5 x + \alpha_6 x^2 + \alpha_7 x^3 \tag{52}$$

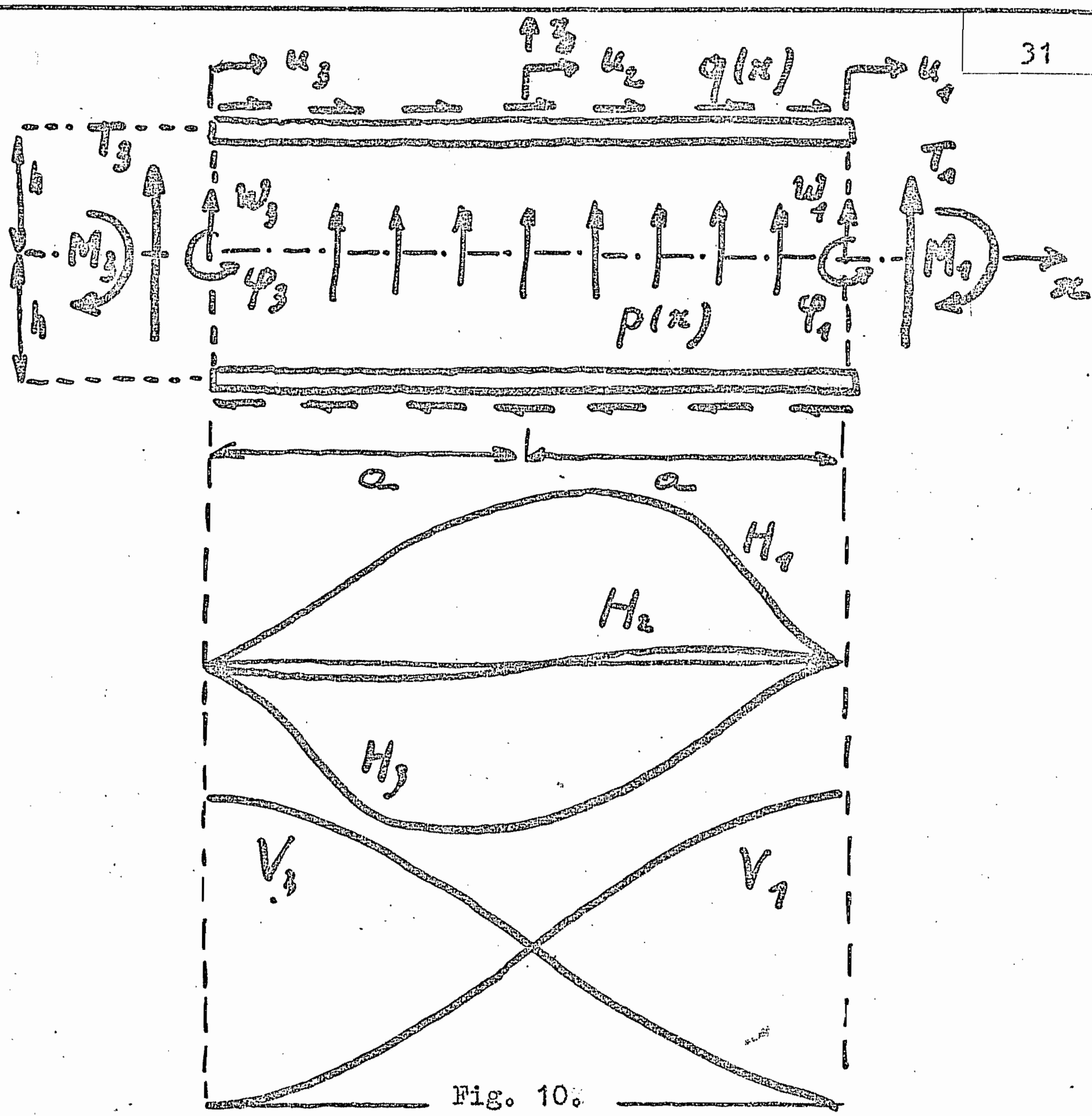

Fig. 10.

Assumption (50) is equivalent to stating that the cross sections remain plane. It also implies that the spar is symmetrical with respect to the Ox axis and, if it is part of a box beam, that the cover sheets are symmetrical.

Assumption (52) implies that the vertical fibers of the web are inextensible; in addition to the reasonable simplification it brings into the calculation of the strain energy it has also the important advantage of introducing transverse loading of the spar only through the resultant transverse loads in each cross section. Hence one needs only to consider a transverse distributed load $p(x)$ and the transverse shears T_1 and T_3 at the end sections. The choice of the generalized coordinates is guided by the following considerations. For full compatibility in the attachment of the spar to cover sheet elements the axial displacements at the ends and in the middle of the upper spar cap are selected :

$$u_1 = h\,\alpha(a) \qquad u_2 = h\,\alpha(0) \qquad u_3 = h\,\alpha(-a) \tag{53}$$

They determine completely the parabolic distribution of the axial displacement along the spar cap; an identical distribution will result along the edge of a triangular panel as analyzed in section 3.3., when the corresponding local displacements are made identical.

The displacement of an end section of the spar is entirely determined by its vertical displacement and its rotation. Since the end rotations are already represented by u_1 and u_3 it is only necessary to add the coordinates

$$w_1 = W(a) \qquad w_3 = W(-a) \tag{54}$$

to have a complete representation of the boundary displacements and to secure full compatibility with respect to the spar connection to neighbouring elements. Since the original field contains seven parameters we are left two internal degrees of freedom. For those we select the end slopes of the $W(x)$ function

$$\varphi_1 = W'(a) \qquad \varphi_3 = W'(-a) \tag{55}$$

In terms of the coordinates introduced by (53-55) the original assumptions (51-52) can now be written

$$h\,\alpha(x) = u_2 + \frac{u_1 - u_3}{2}\,\frac{x}{a} + \frac{u_1 + u_3 - 2u_2}{2}\left(\frac{x}{a}\right)^2 \tag{56}$$

$$\begin{aligned} 4W(x) = {} & 2(w_1 + w_3) - a(\varphi_1 - \varphi_3) + \left[3(w_1 - w_3) - a(\varphi_1 + \varphi_3)\right]\frac{x}{a} \\ & + a(\varphi_1 - \varphi_3)\left(\frac{x}{a}\right)^2 + \left[w_3 - w_1 + a(\varphi_1 + \varphi_3)\right]\left(\frac{x}{a}\right)^3 \end{aligned} \tag{57}$$

For an isotropic web, taking into account the inextensibility in the Oz direction

$$\sigma_z = \nu\,\sigma_x \qquad \sigma_x = \frac{Et}{1-\nu^2}\,z\,\alpha' \qquad \tau_{xy} = Gt(\alpha + W')$$

The normal load in the upper spar cap will be

$$N = ES h \alpha'$$

For simplicity both the thickness t of the web and the area S of the spar caps will be assumed constant. The bending moment and the transverse shear in a cross section becomes

$$M = 2Nh + \int_{-h}^{h} \sigma_x z \, dz = EI\alpha' \qquad I = \frac{2th^3}{3(1-\nu^2)} + 2Sh^2$$

$$T = \int_{-h}^{h} \tau_{xz} \, dz = 2Gth(\alpha + w')$$

The total strain energy can be calculated by the expression

$$\frac{1}{2}\int_{-a}^{a} \left[M\alpha' + T(\alpha + w') \right] dx$$

$$= \frac{1}{2} EI \int_{-a}^{a} \alpha'^2 \, dx + Gth \int_{-a}^{a} (\alpha + w')^2 \, dx$$

The assumptions (56-57) allow the integration to be carried out explicity, whereby the strain energy becomes a quadratic form in the generalized coordinates whose partial derivatives are the conjugate generalized loads. We denote those respectively by $2U_i$ (there are two spar caps), W_i and Φ_i. The resulting stiffness relations (43) are given by the following

$$\begin{pmatrix} 2U_1 \\ 2U_2 \\ 2U_3 \end{pmatrix} = K_{uu} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + K_{uw} \begin{pmatrix} w_1 \\ w_3 \end{pmatrix} + K_{u\varphi} \begin{pmatrix} \varphi_1 \\ \varphi_3 \end{pmatrix} \tag{58}$$

$$\begin{pmatrix} W_1 \\ W_3 \end{pmatrix} = K_{uw}^T \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + K_{ww} \begin{pmatrix} w_1 \\ w_3 \end{pmatrix} + K_{w\varphi} \begin{pmatrix} \varphi_1 \\ \varphi_3 \end{pmatrix} \tag{59}$$

$$\begin{pmatrix} \phi_1 \\ \phi_3 \end{pmatrix} = K^T_{u\varphi} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + K^T_{w\varphi} \begin{pmatrix} w_1 \\ w_3 \end{pmatrix} + K_{\varphi\varphi} \begin{pmatrix} \varphi_1 \\ \varphi_3 \end{pmatrix} \tag{60}$$

where the various stiffness matrices are

$$K_{uu} = \frac{EI}{6ah^2} \begin{pmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{pmatrix} + \frac{Gta}{15h} \begin{pmatrix} 8 & 4 & -2 \\ 4 & 32 & 4 \\ -2 & 4 & 8 \end{pmatrix}$$

$$K_{uw} = \frac{Gt}{5} \begin{pmatrix} 1 & -1 \\ 8 & -8 \\ 1 & -1 \end{pmatrix} \qquad K_{u\varphi} = \frac{Gta}{15} \begin{pmatrix} 7 & -3 \\ -4 & -4 \\ -3 & 7 \end{pmatrix}$$

$$K_{ww} = \frac{6Gth}{5a} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \qquad K_{w\varphi} = \frac{Gth}{5} \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$K_{\varphi\varphi} = \frac{2Gtah}{15} \begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}$$

Turning now to the task of interpreting the generalized loads, we compare the scalar product form of the potential energy and its expression in terms of the applied loads :

$$-P = 2(u_1 U_1 + u_2 U_2 + u_3 U_3) + w_1 W_1 + w_3 W_3 + \varphi_1 \Phi_1 + \varphi_3 \Phi_3$$

$$= 2\int_{-a}^{a} q(x)\, h\alpha(x)\, dx + \int_{-a}^{a} p(x) W(x)\, dx + w_1 T_1 + w_3 T_3$$

$$+ \frac{u_1}{h} M_1 + \frac{u_3}{h} M_3$$

Substitution of equations (56) and (57) and identification of the coefficients of the generalized displacements results in

$$U_1 = \frac{1}{2}\int_{-a}^{a} q(x)\,\frac{x}{a}\left(\frac{x}{a}+1\right)dx + \frac{M_1}{h}$$

$$U_2 = \int_{-a}^{a} q(x)\left[1-\left(\frac{x}{a}\right)^2\right]dx$$

$$U_3 = \frac{1}{2}\int_{-a}^{a} q(x)\,\frac{x}{a}\left(\frac{x}{a}-1\right)dx + \frac{M_3}{h}$$

$$W_1 = \int_{-a}^{a} p(x)\,\frac{1}{4}\left[2+3\,\frac{x}{a}-\left(\frac{x}{a}\right)^3\right]dx + T_1$$

$$W_3 = \int_{-a}^{a} p(x)\,\frac{1}{4}\left[2-3\,\frac{x}{a}+\left(\frac{x}{a}\right)^3\right]dx + T_3$$

$$\Phi_1 = \frac{a}{4}\int_{-a}^{a} p(x)\left(\frac{x}{a}-1\right)\left(\frac{x}{a}+1\right)^2 dx$$

$$\Phi_3 = \frac{a}{4}\int_{-a}^{a} p(x)\left(\frac{x}{a}+1\right)\left(\frac{x}{a}-1\right)^2 dx$$

The auxiliary variables φ_1 and φ_3 can now be taken from equations (60) and substituted into equations (58-59) to give the final stiffness relations :

$$\begin{pmatrix} hM_1 \\ hM_2 \\ hM_3 \end{pmatrix} = K_{hh}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + K_{hv}\begin{pmatrix} w_1 \\ w_3 \end{pmatrix} \qquad (61)$$

$$\begin{pmatrix} V_1 \\ V_3 \end{pmatrix} = K_{hv}^T\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + K_{vv}\begin{pmatrix} w_1 \\ w_3 \end{pmatrix} \qquad (62)$$

$$K_{hh} = \frac{EI}{6ah^2}\begin{pmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{pmatrix} + \frac{Gta}{9h}\begin{pmatrix} 1 & 4 & 1 \\ 4 & 16 & 4 \\ 1 & 4 & 1 \end{pmatrix} \qquad (63)$$

$$K_{hv} = \frac{Gt}{3}\begin{pmatrix} 1 & -1 \\ 4 & -4 \\ 1 & -1 \end{pmatrix} \qquad K_{vv} = \frac{Gth}{a}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \qquad (64)$$

The new generalized loads are

$$H_1 = U_1 - \frac{1}{12h}\left(5\phi_1 - \phi_3\right)$$

$$H_2 = U_2 + \frac{1}{3h}\left(\phi_1 + \phi_3\right)$$

$$H_3 = U_3 - \frac{1}{12h}\left(5\phi_3 - \phi_1\right)$$

$$V_1 = W_1 + \frac{1}{2a}\left(\phi_1 + \phi_3\right)$$

$$V_2 = W_3 - \frac{1}{2a}\left(\phi_1 + \phi_3\right)$$

or, explicity :

$$H_1 = \frac{M_1}{2h} + \frac{1}{2}\int_{-a}^{a} q(x)\,\frac{x}{a}\left(1 + \frac{x}{a}\right)dx$$

$$+ \frac{a}{24h}\int_{-a}^{a}\left(3 + 2\frac{x}{a} - 3\frac{x^2}{a^2} - 2\frac{x^3}{a^3}\right)p(x)\,dx$$

$$H_2 = \int_{-a}^{a} q(x)\left[1-\frac{x^2}{a^2}\right]dx - \frac{a}{6h}\int_{-a}^{a}\frac{x}{a}\left(1-\frac{x^2}{a^2}\right)p(x)\,dx$$

$$H_3 = \frac{M_3}{h} + \frac{1}{2}\int_{-a}^{a} q(x)\,\frac{x}{a}\left(\frac{x}{a}-1\right)$$

$$- \frac{a}{24\,h}\int_{-a}^{a}\left(3-2\frac{x}{a}-3\frac{x^2}{a^2}+2\frac{x^3}{a^3}\right)p(x)\,dx$$

$$V_1 = T_1 + \frac{1}{2}\int_{-a}^{a}\left(1+\frac{x}{a}\right)p(x)\,dx$$

$$V_3 = T_3 + \frac{1}{2}\int_{-a}^{a}\left(1-\frac{x}{a}\right)p(x)\,dx$$

The essential feature of the elimination of the auxiliary variables is this modification of the generalized loads and in particular the appearance of contributions from the external loading $p(x)$ in the loads conjugate to the spar cap displacements. The different weight functions attached to the $p(x)$ distribution are illustrated in Fig. 10. The weight functions attached to $q(x)$ are the same as those encountered for the beam segment in section 3.3.

If the shearing deformation of the spar is neglected, the stiffness problem appears in a different light. The shearing deformation can be made to vanish by letting the shear modulus G go to infinity. Then in equation (61) the terms proportional to G must cancel and this is immediately seen to occur if the following relation is satisfied

$$\frac{a}{3h}\left(u_1 + 4u_2 + u_3\right) + w_1 - w_3 = 0 \qquad (65)$$

This purely geometrical condition is a consequence of the fact that the cross sections remain normal to the parabolic curve of the neutral axis. The general statement of this property is of course the disappearance of the shearing strain :

$$\alpha + w' = 0$$

After substitution of equation (56-57) and identification of the powers of x/a this results into

$$u_1 = -h\varphi_1 \qquad u_3 = -h\varphi_3 \qquad u_2 = \frac{3h}{4a}(w_3 - w_1) + \frac{h}{4}(\varphi_1 + \varphi_3)$$

and the elimination of the slopes again results in equation (65). The only stiffness relations that remain are now

$$\begin{pmatrix} 2H_1 \\ 2H_2 \\ 2H_3 \end{pmatrix} = K_{hh} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

between the equivalent bending moments $2hH_i$ and the spar cap displacements, where K_{hh} is reduced to its part proportional to the bending rigidity. This matrix is singular. In fact if one set $u_1 = u_2 = u_3$ no equivalent bending moments are induced, an obvious result since we then have a regid body (rotation) mode. Corresponding to this we have the equilibrium relation

$$H_1 + H_2 + H_3 = 0$$

The procedure for finding the spar deflexions are then as follows. The spar must be supported against rotation, indeed u_1 or u_3 are generally known from boundary conditions or previous calculations. The stiffness relations determine the two other u values. Inserting those in equation (65) the deflexion increment can be calculated and again either w_1 or w_2 are known from boundary conditions or previous calculations. As was indicated previously the present stiffness matrix was devised to give a correct behaviour

under constant transverse shear load or linearly increasing bending moment. Incorporated in a box beam it requires a parabolic displacement analysis of the cover sheet elements in the direction of the spar flanges. Previous attempts at setting up spar stiffness have generally aimed at a linear displacement along the flanges to be used in conjunction with linear cover sheet elements.
Our analysis can be modified to incorporate this simplification a posteriori by the change (and reduction) in coordinates

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = T \begin{pmatrix} u_1 \\ u_3 \end{pmatrix} \qquad T = \begin{pmatrix} 1 & 0 \\ 1/2 & 1/2 \\ 0 & 1 \end{pmatrix}$$

This states that the middle displacement u_2 is now constrained by the assumption of linearity to become the average of the end displacements. Remembering the types of changes that generalized loads and stiffness matrices undergo as a result of their tensorial character the simplified representation will read

$$T^{T} \begin{pmatrix} 2M_1 \\ 2M_2 \\ 2M_3 \end{pmatrix} = \begin{pmatrix} 2M_1 + M_2 \\ M_2 + 2M_3 \end{pmatrix} = T^{T} K_{uu} T \begin{pmatrix} u_1 \\ u_3 \end{pmatrix} + T^{T} K_{uw} \begin{pmatrix} w_1 \\ w_3 \end{pmatrix}$$

The element is now adapted only to the transmission of a constant bending moment and very much stiffened thereby. It will need a much finer spanwise subdivision to avoid the poor results reported [16,17] for displacement models of similar linear characteristics.

It is parhaps of interest to note that similar simplifications can be introduced in the quadratic displacement analysis of triangular panels to keep them flexible in a preferential direction and linearize them transversally.

4. Equilibrium analysis.

4.1. General theory.

While the general theory of displacement models is reasonably well understood[18] this does not seem to be the case for equilibrium analysis. The reason is probably that it appears less intuitive from a geometrical standpoint. The emphasis is on stress transmission modes rather than deformation modes and the first step is to assume a parametric stress field. In two dimensional analysis, for example

$$\sigma_x = \sum \beta_i S_{xx}^{(i)}(x,y) + \sum P_r T_{xx}^{(r)}(x,y)$$

$$\sigma_y = \sum \beta_i S_{yy}^{(i)}(x,y) + \sum P_r T_{yy}^{(r)}(x,y)$$

$$\tau_{xy} = \sum \beta_i S_{xy}^{(i)}(x,y) + \sum P_r T_{xy}^{(r)}(x,y) \qquad (66)$$

$$\frac{\partial S_{xx}^{(i)}}{\partial x} + \frac{\partial S_{xy}^{(i)}}{\partial y} = 0 \qquad \frac{\partial S_{xy}^{(i)}}{\partial x} + \frac{\partial S_{yy}^{(i)}}{\partial y} = 0 \qquad (67)$$

$$\frac{\partial T_{xx}^{(r)}}{\partial x} + \frac{\partial T_{xy}^{(r)}}{\partial y} = -X_r(x,y) \qquad \frac{\partial T_{xy}^{(r)}}{\partial x} + \frac{\partial T_{yy}^{(r)}}{\partial y} = -Y_r(x,y) \qquad (68)$$

The stress modes of unknown amplitudes β_i are in equilibrium without internal forces, while the modes of amplitudes (P_r) are in equilibrium with assumed body force distributions $(P_r X_r, P_r Y_r)$. In order to extract a maximum of information from the complementary energy principle no loads are prescribed. Instead, displacements are specified along the whole boundary and also the displacements conjugate to the body force distributions. This requires a slight extension in the presentation of the principle :

from virtual work considerations the displacement conjugate to the loading mode (X_r, Y_r) is the weighed average

$$v_r = \iint_A (X_r u + Y_r v)\, dx\, dy \tag{69}$$

If this is specified, the complementary energy principle takes the extended form

$$\iint_A \Phi(\sigma)\, dx\, dy - \int_\Gamma (p_x \bar{u} + p_y \bar{v})\, ds - \Sigma\, \bar{v}_r P_r \quad \text{minimum.} \tag{70}$$

Exactly like (p_x, p_y) are the reactions along the boundary against the specified boundary displacements, the P_r are the reaction amplitudes of stress modes against the prescribed generalized displacements $\bar{v}_r$. In this manner the parameters (β_i) and (P_r) of the equilibrium stress field are arbitrary and independant in the principle.

Generalized boundary loads (Q_j) are next defined according to the following rule : along each connection boundary the stress distribution is uniquely determined by the generalized loads pertaining to this boundary and conversely. In this manner a reciprocity between generalized loads across the boundary entails complete continuity of stress transmission.

The conjugate boundary displacements (q_j) are obtained from the virtual work equation

$$\int (\bar{u} p_x + \bar{v} p_y)\, ds = \Sigma\, q_j Q_j \quad \text{along each boundary.} \tag{71}$$

Let us now express all this in matrix form with the notations

$$b^T = (\beta_1 \cdots \beta_m) \qquad q^T = (\bar{q}_1 \cdots \bar{q}_n)$$

$$p^T = (P_1 \ldots P_t) \qquad v^T = (\bar{v}_1 \ldots \bar{v}_t)$$

$$g^T = (Q_1 \cdots Q_n)$$

The stress field (66) can be written as

$$\sigma = Sb + Tp \tag{72}$$

with the stress modes entered as columns of the (3 x m) matrix S and the (3 x t) matrix T. The complementary strain energy will be

$$\iint_\Delta \Phi(\sigma)\,dx\,dy = \frac{1}{2}\iint_\Delta \sigma^T\varepsilon\,dx\,dy = \frac{1}{2}\iint_\Delta \sigma^T D^{-1}\sigma\,dx\,dy \tag{73}$$

$$= \frac{1}{2} b^T F_{bb} b + b^T F_{bp} p + \frac{1}{2} p^T F_{pp} p$$

with flexibility matrices,

$$F_{bb} = \iint_\Delta S^T D^{-1} S\,dx\,dy \qquad F_{bp} = \iint_\Delta S^T D^{-1} T\,dx\,dy$$

$$F_{pp} = \iint_\Delta T^T D^{-1} T\,dx\,dy \tag{74}$$

From their definitions the generalized loads will be expressed in terms of the field parameters by a matrix equation

$$g = Cb + Ap \tag{75}$$

The (n x m) matrix C and the (n x t) matrix A are load connection matrices. The virtual work of the boundary loads against the prescribed conjugate displacements will be

$$q^T g = q^T Cb + q^T Ap$$

and is now expressed in terms of the field parameters. The principle of minimum complementary energy

$$\frac{1}{2} b^T F_{bb} b + b^T F_{bp} p + \frac{1}{2} p^T F_{pp} p - q^T C b - q^T A p - v^T p \quad \text{minimum}$$

yields the stationary conditions with respect to variations on b and p,

$$F_{bb}\, b + F_{bp}\, p = C^T q$$
$$F_{bp}^T\, b + F_{pp}\, p = A^T q + v$$

Introducing yhe matrices

$$K = C F_{bb}^{-1} C^T \tag{76}$$

$$R = A - C F_{bb}^{-1} F_{bp}$$

$$L = F_{pp} - F_{bp}^T F_{bb}^{-1} F_{bp} \tag{77}$$

the stationary conditions can be rewritten in the form

$$g = K q + R p \tag{78}$$
$$v = L p - R^T q \tag{79}$$

The first expresses the boundary loads as the sum of a part due to the boundary displacements and a part due to a reaction against the body loads. The second expresses the displacements conjugate to the body loads as the sum of a fixed boundary term and a term due to boundary displacement. This form of the equations is the more natural since, instead of considering the p loads as unknown reactions for the purpose of applying the complementary energy principle, these loads are generally given. In many cases there no body forces and equation (78) reduces to the form(43) with the stiffness matrix (76).

It should be observed that the knowledge of the displacements is reduced to that of a certain number of weighed averages. Sometimes the strain field is integrable and a complete internal displacement field available.

4.2. Triangular panels. Equilibrium analysis.

The assumed stress field is the simplest conceivable

$$\sigma_x = \beta_1 \qquad \sigma_y = \beta_2 \qquad \tau_{xy} = \beta_3 \tag{80}$$

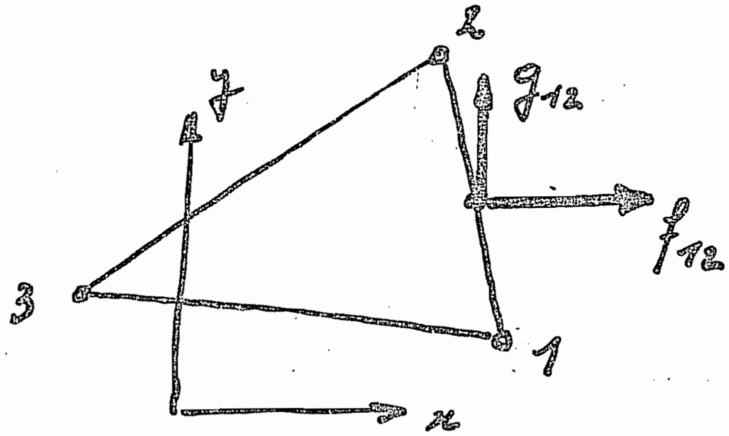

Fig. 11.

There are no body forces. Along the edge 1-2 (Fig. 11) the generalized loads

$$f_{12} = \int_1^2 p_x \, ds = \int_1^2 \sigma_x \, dy - \tau_{xy} \, dx = \beta_1 y_{21} - \beta_3 x_{21}$$

(81)

$$g_{12} = \int_1^2 p_y \, ds = \int_1^2 \tau_{xy} \, dy - \sigma_y \, dx = \beta_3 y_{21} - \beta_2 x_{21}$$

are defined by the edge stresses and conversely the edge tractions are

$$p_x = f_{12} / c_{12} \qquad p_y = g_{12} / c_{12} \qquad c_{12} = \text{length of the edge.}$$

Conjugate edge displacements are obtained from the virtual work equations along this edge

$$\int_1^2 (p_x\bar{u}+p_y\bar{v})\,ds = \frac{f_{12}}{e_{12}}\int_1^2 \bar{u}\,ds + \frac{g_{12}}{e_{12}}\int_1^2 \bar{v}\,ds$$

$$= f_{12}\,u_{12} + g_{12}\,v_{12}$$

Hence (82)

$$u_{12} = \frac{1}{e_{12}}\int_1^2 \bar{u}\,ds \qquad v_{12} = \frac{1}{e_{12}}\int_1^2 \bar{v}\,ds$$

They are the ordinary averages of the displacements along the edge. Results for the other edges follow by cyclic index permutations. Adopting the order

$$g^T = (f_{23}\ f_{31}\ f_{12}\ g_{23}\ g_{31}\ g_{12})$$

$$q^T = (u_{23}\ u_{31}\ u_{12}\ v_{23}\ v_{31}\ v_{12})$$

the connection matrix C condensing in the equation $g = Cb$ all relations of type (81) is

$$C = \begin{pmatrix} y_{32} & 0 & x_{23} \\ y_{13} & 0 & x_{31} \\ y_{21} & 0 & x_{12} \\ 0 & x_{23} & y_{32} \\ 0 & x_{31} & y_{13} \\ 0 & x_{12} & y_{21} \end{pmatrix}$$

Since in the present case we have simply

$$\sigma = b$$

as equation (72), the flexibility matrix is

$$F = \iint_{\Delta} D^{-1} dx\, dy$$

For an isotropic panel

$$F = \frac{\mathcal{A}}{E\tilde{t}} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \qquad \frac{1}{\tilde{t}} = \frac{1}{\mathcal{A}} \iint_{\Delta} \frac{dx\, dy}{t(x,y)}$$

The stiffness matrix is

$$C F^{-1} C^{T} = \frac{E\tilde{t}}{\mathcal{A}(1-\nu^2)}\, C \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} C^{T}$$

When this is compared to the stiffness matrix of the same panel calculated in section 32 we note that since $C = 2\mathcal{A} N^{T}$ the present matrix can be deduced from that of section 3.2. by multiplication with a factor $4(\tilde{t}/t)$.

Under constant thickness conditions the strain field is integrable and the displacements can be expressed by equations of the same type as in section 3.2. i.e. by linear forms. The coefficients (α_i) of the forms are completely determined by the present generalized displacements. In fact the integration of formulae of type (82) yields

$$u_{ij} = \alpha_1 + \alpha_2 \frac{x_i + x_j}{2} + \alpha_3 \frac{y_i + y_j}{2}$$

$$v_{ij} = \alpha_4 + \alpha_5 \frac{x_i + x_j}{2} + \alpha_6 \frac{y_i + y_j}{2}$$

and these equations can be solved for the coefficients. The displacements at the vertices can be deduced and the following relations, obvious in view of the linearity of the field, are valid

$$u_{ij} = \frac{u_i + u_j}{2} \qquad v_{ij} = \frac{v_i + v_j}{2}$$

This was used as starting point for the original derivation of the equilibrium model of the panel[14]. The present derivation is a more correct application of the basic principles. It also applies to the variable thickness case for which the strain field might not be integrable.

The major difference between the direct stiffness analysis and the equilibrium analysis is in the connection properties of the models. For the equilibrium model the stresses are transmitted continuously from one panel to the other but only the average displacements of the edges are coincident. In the other model the edge displacements are everywhere coincident but the stresses suffer discontinuities from one panel to the other.

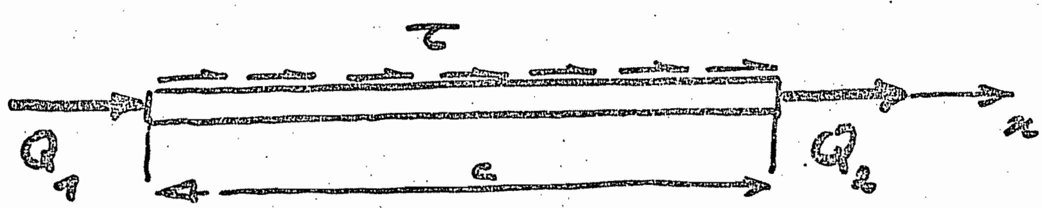

Fig. 12.

4.3. Beam segment equilibrium analysis.

The equilibrium equation governing the axial load L in the beam is (Fig. 12)

$$\frac{dL}{dx} = -\tau \tag{83}$$

We require a solution for a constant shear flow τ. The general solution of equation (83) is then

$$L = \beta - \tau x \tag{84}$$

In accordance with the philosophy of the general theory, β is the only parameter of the stress field in equilibrium without body forces, τ the only amplitude parameter of the field in equilibrium with the assumed body force distribution $\tau = 1$. Let v be the conjugate displacement to this distribution

$$v = \int_0^c u \, dx$$

This is supposed to be specified together with the boundary displacements u(0) and u(c), leading to a virtual work of the reactions equal to

$$- \bar{u}(0) L(0) + \bar{u}(c) L(c) + \bar{v} \tau$$

The generalized loads defined for boundary connections are

$$Q_1 = -L(0) = -\beta \qquad Q_2 = L(c) = \beta - c\tau \tag{85}$$

In this example the body force τ can be generated partially or totally by attachment of the beam to shear panels. It is for this reason that a constant τ solution was indicated; it can produce a continuous stress transmission between the beam and our triangular shear panels of section 4.2. For the same reason the generalized load

$$Q_3 = c\tau \tag{86}$$

that will correspond to a generalized load of the panels must be added to the set of connection loads. Equations (85-86) result in the following connection matrix

$$g = \begin{pmatrix} Q_1 \\ Q_3 \\ Q_2 \end{pmatrix} = C \begin{pmatrix} \beta \\ \tau \end{pmatrix} \qquad C = \begin{pmatrix} -1 & 0 \\ 0 & c \\ 1 & -c \end{pmatrix} \tag{87}$$

The significance of the conjugate generalized displacements is deduced from the virtual work equations for each boundary separately

$$q_1 Q_1 = - \bar{u}(0) L(0) = \bar{u}(0) \beta$$

$$q_3 Q_3 = \tau \int_0^c u \, dx$$

$$q_2 Q_2 = \bar{u}(c) L(c) = \bar{u}(c)(\beta - c\tau)$$

Substitution of equations (85-86) and identification produces

$$q_1 = \bar{u}(0) \qquad q_3 = \frac{1}{c}\int_0^c u \, dx \qquad q_2 = \bar{u}(c) \tag{88}$$

The flexibility matrix F results from the expansion of the complementary strain energy, using equation (84)

$$\frac{1}{2}\int_0^c \frac{L^2}{ES} dx = \frac{1}{2}(\beta \quad \tau) F \begin{pmatrix} \beta \\ \tau \end{pmatrix}$$

$$F = \frac{1}{E} \begin{pmatrix} \int_0^c \frac{dx}{S} & -\int_0^c \frac{x\,dx}{S} \\ -\int_0^c \frac{x\,dx}{S} & \int_0^c \frac{x^2 dx}{S} \end{pmatrix}$$

and we have finally $g = Kq$ with $K = CF^{-1}C^T$.

For the particular case of a beam of constant cross section S, the flexibility matrix turns out to be

$$F = \frac{c}{6ES}\begin{pmatrix} 6 & -3c \\ -3c & 2c^2 \end{pmatrix}$$

and the stiffness matrix,

$$K = \frac{2ES}{c}\begin{pmatrix} 2 & -3 & 1 \\ -3 & 6 & -3 \\ 1 & -3 & 2 \end{pmatrix}$$

The same result was established before[14] in another way.

4.4. <u>Spar bending equilibrium theory.</u>

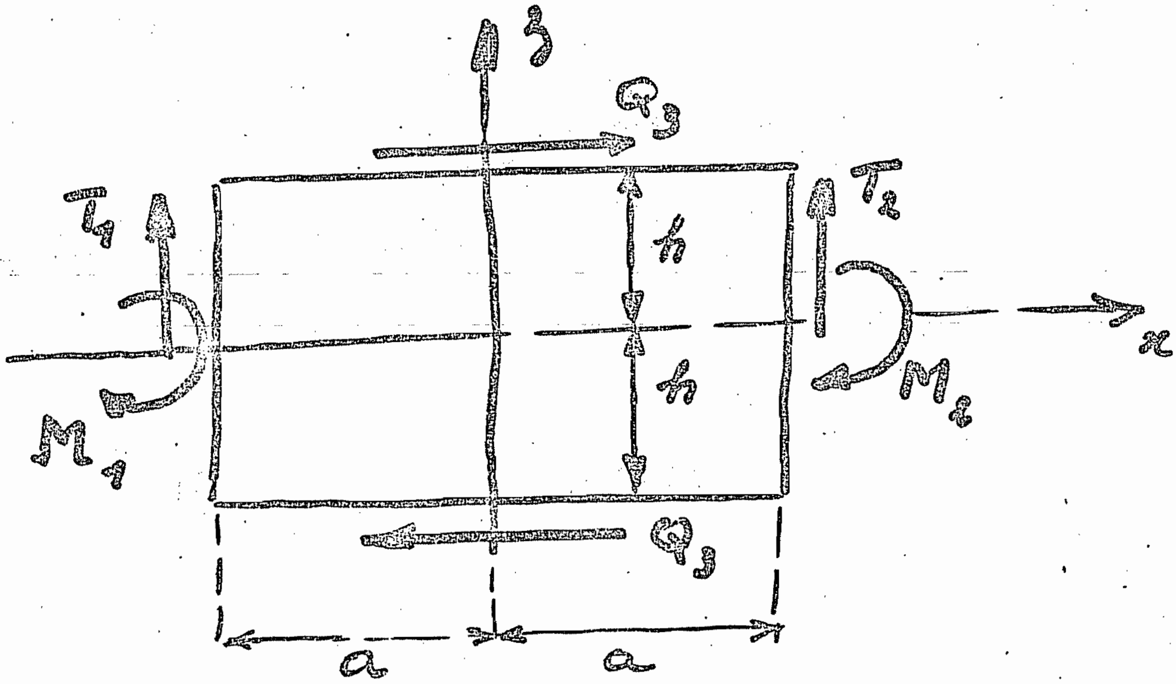

Fig. 13.

Although, as in the case of displacement models, spars could be treated as combinations of triangular plate elements and beam segments, an independant analysis taking into account the symmetry properties of the problem, appears to be more efficient. The following three parameter field is assumed for the web stresses

$$\sigma_x = \frac{6z}{h^2}(\beta_1 + x\beta_2) \qquad \sigma_z = 0 \qquad \tau_{xz} = \beta_3 - 3\frac{z^2}{h^2}\beta_2 \tag{89}$$

It is an equilibrium field without body forces and can easily be derived from an Airy stress function expanded in odd powers of z. The expansion in the x direction is limited so as to give a constant shear flow τ_{xz} along the connection with the spar flanges. In this manner the spar flanges can be added as beam segments treated to the theory of section 4.3. and implementing the equilibrium conditions with the web. Hence it proves adequate to treat the web separately. It is also worth noting that, because the stress transverse to the fibers is zero, the external vertical loading on the spar will have to take place through shears in the sections $x = \pm a$.

Generalized loads giving a complete description of the stress distributions along the connection boundaries are :

(a) $$Q_3 = \int_{-a}^{a} (\tau_{xz})_{z=h}\, dx = -6a\beta_2 + 2a\beta_3 \tag{90}$$

the total load transmitted to the flanges; the shear flow between web and spar flanges is then $Q_3(2a)^{-1}$ and the conjugate displacement u_3 can be deduced from the virtual work equation

$$\int_{-a}^{a} (\tau_{xz}\, u)_{z=h}\, dx = \frac{Q_3}{2a}\int_{-a}^{a} u\, dx = Q_3 u_3 \tag{91}$$

whence

$$u_3 = \frac{1}{2a}\int_{-a}^{a} u\, dx$$

(b) $$M_2 = \int_{-h}^{h} (\sigma_x)_{x=a}\, z\, dz = 4h(\beta_1 + a\beta_2) \tag{92}$$

the web bending moment in the $x = a$ section. The normal load distribution in this section is then $\sigma_x = 3zM_2/(2h^3)$ and the conjugate displacement φ_2 follows from

$$\int_{-h}^{h} (\sigma_x u)_{x=a}\, dz = \frac{3M_2}{2h^3}\int_{-h}^{h}(u)_{x=a}\, z\, dz = M_2 \varphi_2$$

whence

$$\varphi_2 = \frac{3}{2h^3}\int_{-h}^{h}(u)_{x=a}\, z\, dz \qquad (93)$$

Similar definitions

$$M_1 = \int_{-h}^{h}(\sigma_x)_{x=-a}\, z\, dz = 4h(-\beta_0 + a\beta_2) \qquad (94)$$

$$\varphi_1 = \frac{3}{2h^3}\int_{-h}^{h}(u)_{x=-a}\, z\, dz \qquad (95)$$

are used in the section $x = -a$.

(c) $$T_2 = \int_{-h}^{h}(\tau_{xz})_{x=a}\, dz = 2h(\beta_3 - \beta_2) \qquad (96)$$

$$V_2 = \int_{-h}^{h}\left(1 - 3\frac{z^2}{h^2}\right)(\tau_{xz})_{x=a}\, dz = \frac{8}{5} h \beta_2 \qquad (97)$$

These generalized loads describe the shear flow along $x = a$

$$(\tau_{xz})_{x=a} = \frac{1}{2h} T_2 + \frac{5}{8h} V_2 \left(1 - 3\frac{z^2}{h^2}\right)$$

in terms of a constant distribution of total shear T_2 and a parabolic distribution of amplitude V_2 which is statically equivalent to zero. The conjugate displacements follow from the virtual work equation

$$\int_{-h}^{h} (\tau_{xz}\, w)_{x=a}\, dz = \frac{T_2}{2h}\int_{-h}^{h} (w)_{x=a}\, dz + \frac{5V_2}{8h}\int_{-h}^{h} \left(1-3\frac{z^2}{h^2}\right)(w)_{x=a}\, dz$$

whence

$$w_2 = \frac{1}{2h}\int_{-h}^{h} (w)_{x=a}\, dz \qquad v_2 = \frac{5}{8h}\int_{-h}^{h} \left(1-3\frac{z^2}{h^2}\right)(w)_{x=a}\, dz \tag{98}$$

Similar definitions are used in the $x = -a$ section.

$$T_1 = -\int_{-h}^{h} (\tau_{xz})_{x=-a}\, dz = 2h\,(\beta_2 - \beta_3) \tag{99}$$

$$V_1 = -\int_{-h}^{h} \left(1-3\frac{z^2}{h^2}\right)(\tau_{xz})_{x=-a}\, dz = -\frac{8}{5}\, h\, \beta_2$$

$$w_1 = \frac{1}{2h}\int_{-h}^{h} (w)_{x=-a}\, dz \qquad v_1 = \frac{5}{8h}\int_{-h}^{h} \left(1-3\frac{z^2}{h^2}\right)(w)_{x=-a}\, dz \tag{100}$$

Adopting the following sequence of generalized quantities

$$g^T = (T_1,\ V_1,\ M_1,\ T_2,\ V_2,\ M_2,\ 2Q_3),$$

$$q^T = (w_1,\ v_1,\ \varphi_1,\ w_2,\ v_2,\ \varphi_2,\ u_3)$$

the connection matrix, defined by equation (75), follows from equations (90-99) as

$$C^T = \begin{pmatrix} 0 & 0 & -4h & 0 & 0 & 4h & 0 \\ 2h & -(8/5)h & 4ah & -2h & (8/5)h & 4ah & -12a \\ -2h & 0 & 0 & 2h & 0 & 0 & 4a \end{pmatrix} \qquad (101)$$

For an isotropic web, the strain energy can be calculated from

$$\frac{1}{2}\int_{-a}^{a}\int_{-h}^{h}\left(\frac{\sigma_x^2}{Et}+\frac{\tau_{xz}^2}{Gt}\right)dx\,dy$$

After substitution of assumptions (89) the F matrix can be obtained and, in case of a constant thickness t, turns out to be

$$F = \frac{4a}{ht}\begin{pmatrix} \frac{12}{E} & 0 & 0 \\ 0 & \frac{4a^2}{E} + \frac{9h^2}{5G} & -\frac{h^2}{G} \\ 0 & -\frac{h^2}{G} & \frac{h^2}{g} \end{pmatrix} \qquad (102)$$

The web stiffness matrix follows from inversion and matrix multiplications as $K = CF^{-1}C^T$. These operations can be carried out numerically; there is also no difficulty in carrying them out analytically.

5. Properties of element stiffness matrices.

The complete system of equations

$$g = Kq, \qquad K = K^T, \tag{103}$$

between the generalized loads and displacements of an element is always singular, since the corresponding homogeneous system

$$Ku = 0, \tag{104}$$

possesses amongst its non vanishing solutions at least the rigid body displacement modes of the element :

$$Ku_{(i)} = 0 \qquad (i = 1,2 \ldots r). \tag{105}$$

In displacement models this is a consequence of the inclusion a priori of the rigid modes in the original parametric field. In the case of equilibrium models it is a consequence of equilibrium properties of the generalized loads, calculated from an original equilibrium field. Indeed, considering the homogeneous system

$$C^T z = 0, \tag{106}$$

and in view of the structure (76) of the stiffness matrix, it is observed that the non trivial solutions

$$C^T z_{(j)} = 0, \qquad (j = 1,2 \ldots s), \tag{107}$$

are also solutions of equation (104). Moreover, denoting by

$$v_{(j)} = - R^T z_{(j)}, \tag{108}$$

the body load displacements due to the boundary displacements $z_{(j)}$, it is found that, by virtue of equation (78)

$$z^T_{(j)} g + v^T_{(j)} p = 0. \tag{109}$$

These equations express overall equilibrium properties of the generalized loads in the form of vanishing virtual work equations.

The difference between element stiffness matrices derived from displacement models and from equilibrium models is that, in the latter case, there may exist $z_{(j)}$ modes that are not of the rigid body type. It will be remembered that for displacement models it is exactly the opposite : there may be more field parameters than generalized coordinates required for connection purposes. A good example of an equilibrium model where generalized loads exceed in number the stress field parameters by more than the rigid body freedoms is the spar bending model of section 4.4. An independant set of $z_{(j)}$ modes derived from equations (107) is as follows :

$$z^T_{(1)} = (1 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0)$$

$$z^T_{(2)} = (a \quad 0 \quad 1 \quad -a \quad 0 \quad 1 \quad h)$$

$$z^T_{(3)} = (0 \quad b \quad 0 \quad 0 \quad 1 \quad 0 \quad 0)$$

$$z^T_{(4)} = (0 \quad 5a \quad 1 \quad 0 \quad 0 \quad 1 \quad 0)$$

The first is easily recognized as a vertical translation mode, the second as a rotation mode about the origin and this exhausts the rigid body freedoms if due account is taken of the symmetry of the problem. The other two modes are kinematical deformation freedoms that leave the element unstressed. They may seem to be undesirable characteristics of the element representation but represent, admittedly in an extreme form, deficiencies in deformation compatibility that must necessarily be expected from an approximate equilibrium analysis. The equilibrium equations of type (109) associated with these $z_{(j)}$ modes are respectively :

$$T_1 + T_2 = 0, \text{ or vertical equilibrium}$$

$$a(T_1 - T_2) + M_1 + M_2 + 2hQ_3 = 0 \quad \text{or rotational equilibrium}$$

$$\left\{ \begin{array}{ll} V_1 + V_2 = 0, & (110) \\ M_1 + M_2 + 5aV_1 = 0. & (111) \end{array} \right.$$

We conclude this section by showing how, in this particular case, the kinematical deformation freedoms can be removed at the expense of some stress continuity violation at the boundaries but retaining the major benefit of the equilibrium analysis : the proviso of upper bounds to the remaining influence coefficients. The method is generally applicable whenever the strain field is integrable. This is precisely the case in our example, provided the web thickness is constant. For then

$$Et\,u = -\alpha_1 z + 6\beta_1 \frac{xz}{h^2} + \beta_2\left[3\frac{zx^2}{h^2} - (2+\nu)\frac{z^3}{h^2}\right] + 2(1+\nu)\beta_3 z$$

$$(112)$$

$$Et\,w = \alpha_2 + \alpha_1 x - 3\beta_1\left[\frac{x^2}{h^2} + \nu\frac{z^2}{h^2}\right] - \beta_2\left[\frac{x^3}{h^3} + 3\nu\frac{xz^2}{h^2}\right]$$

is a displacement field associated to (89); α_1 and α_2 are integration constants representing the rigid body modes. When this field is substituted into the definitions (91-100) , the seven generalized displacements are correlated with the five parameters $(\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3)$. By eliminating the parameters it turns out that we should have

$$\theta_2 + \theta_1 = \nu\frac{h^2}{6a}(\varphi_2 - \psi_1)$$

$$v_2 - v_1 = \nu\frac{h(\varphi_1 + \varphi_2) - 2u_3}{2\left(\frac{a}{h} + \frac{2+\nu}{5}\frac{h}{a}\right)} \qquad (113)$$

Hence v_1 and v_2 are dependant Poisson ratio effects and, conversely, as can be seen from equations (110-111) the conjugate loads are determined by the bending moments.

In as much as the equilibrium theory considers the seven generalized displacements to be independant, equations (113) are generally unsatisfied. This shows more explicity how the kinematical deformation freedoms are of the nature of compatibility violations. Compatibility can be restored as follows : the original minimum principle can be written

$$\frac{1}{2} b^T F b - \left(w_1 T_1 + w_2 T_2 + M_1 \varphi_1 + M_2 \varphi_2 + 2 Q_3 u_3 \right) - \left(\underline{v_1} V_1 + \underline{v_2} V_2 \right)$$

minimum

where the F matrix is given by expression (102), the genralized loads are to be expressed in terms of the stress parameters and the generalized displacements are to be prescribed. This is now modified by calculating the last bracket in terms of stress parameters; from equations (97-99) one has

$$v_1 V_1 + v_2 \overset{\circ}{V}_2 = \frac{8}{5} h \beta_2 (v_2 - v_1)$$

But also from the displacement field (112) and definitions (98-100)

$$v_2 - v_1 = \nu \frac{2a}{Et} \beta_2$$

so that finally

$$v_1 V_1 + v_2 V_2 = \frac{1}{2} \left(\frac{32}{5} \nu \frac{ah}{Et} \right) \beta_2^2$$

Substitution of this into the original principle gives

$$\frac{1}{2} b^T H b - \left(w_1 T_1 + w_2 T_2 + \varphi_1 M_1 + \varphi_2 M_2 + 2 u_3 Q_3 \right) \quad \text{minimum}$$

with a modified flexibility matrix

$$H = F - \nu \frac{8ah}{5Et}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} =$$

$$\frac{8a}{5Eth}\begin{pmatrix} 6 & 0 & 0 \\ 0 & 8a^2 + \frac{9+5\nu}{5}h^2 & -(1+\nu)h^2 \\ 0 & -(1+\nu)h^2 & (1+\nu)h^2 \end{pmatrix} \qquad (114)$$

The generalized loads matrix is now reduced to

$$g^T = (T_1, M_1, T_2, M_2, 2Q_3),$$

with a connection matrix B^T, deduced from (101) by suppression of the second and fifth columne leading to the new stifness matrix $BH^{-1}B^T$. As the representation still follows from a minimum complementary energy principle, the upper bound character of the influence coefficients is preserved. The advantages of the new representation are : the reduction in size of the stiffness matrix, the elimination of the kinematical deformation freedoms and the existence of a correlated internal displacement field. In counterpart there can exist a violation in stress continuity in the form of different amplitudes of the parabolic shear distributions $(1 - 3z^2/h^2)$ between the edges of vertical web junctions. However equilibrium is restored if one conceives this difference to be artificially absorbed by an external load distribution. Since this local load is statically equivalent to zero, it can be expected, on the basis of de Saint-Venant's principle, that its effect on the deflections and slopes will be small.

6. Properties of connected elements stiffness matrices.

From a kinematic standpoint the connections between elements are defined by relating their generalized displacements to the nodal displacements of the structure. Ordering the various elements by means of a subscript (k), so that $q_{(k)}$ denotes the displacements matrix of the kth element, and denoting simply by q the column matrix of nodal displacements, the kinematic connections can be written in the form

$$q_{(k)} = L_k q. \tag{115}$$

In this equation L_k is the localizing matrix of element k. Often the element displacements are all oriented with respect to a common axes system; the localizing matrix is then simply composed of zero or unit elements and serves to identify each element displacement component with some, similarly defined, nodal displacement. However, in the more general case, the localizing matrix can incorporate the changes in orientation required for conformity with the definitions of the nodal displacements.

The force relationships due to the connections follow directly from consideration of the virtual work of the external loads acting at the level of the nodes. If g denotes the column matrix of these loads, conjugate to the nodal displacements, and $g'_{(k)}$ the generalized loads on the isolated kth element, the following virtual work equality must hold for an arbitrary displacement matrix q,

$$g^T q = \sum_k g_{(k)}^T q_{(k)} = \sum_k g_{(k)}^T L_k q \tag{116}$$

Whence

$$g^T = \sum_k g_{(k)}^T L_k$$

or

$$g = \sum_k L_k^T g_{(k)} \tag{117}$$

This equation shows how the loads on the isolated elements add up to balance the external loads on the nodes. Because of the rule adopted in equilibrium models to attach each generalized quantity to a single boundary, the summation for any row in equation (117) really extends only to two subscript values. If the corresponding external load happens to be zero, we then have a statement of reciprocity of action between two elements.
In view of the identity of stress distributions represented by the reciprocal element loads, this is equivalent to complete continuity in the stress transmission. When the external load components is present the stress continuity is enforced only if the external load possesses the same type of distribution.

In displacement models, while displacement continuity is built in a priori, the summations generally extend to more than two subscript values and stress continuity is not enforced. In this case equations (117) are easily interpreted in terms of weight functions applicable to the arbitrary external loading of the structure for conversion into equivalent nodal forces.

Referring back to equation (78) and writing it for each element

$$g_{(k)} = K_k q_{(k)} + R_k p_{(k)}, \tag{118}$$

We obtain, after substitution into equation (117)

$$g = Kq - p \tag{119}$$

where

$$K = \sum_k L_k^T K_k L_k \tag{120}$$

$$-p = \sum_k L_k^T R_k p_{(k)} \tag{121}$$

In these equations p represents a column matrix of external body loads converted into equivalent nodal forces, K is the complete stiffness matrix of the unsupported structure. It is seen to be built up by properly addressing the element matrices by means of the localizing matrices.

In building the complete stiffness matrix by connecting displacement models, no kinematic deformation modes are generated that do not properly belong to the structure. With equilibrium models however artificial kinematic freedoms can appear. This was already shown to happen within a single element; it can also result from relative motions between elements. Special care must therefore be exercised with respect to the pattern of subdivision into equilibrium models.

It should be observed that a structural representation containing such unreal kinematic freedoms is not necessarily to be rejected. The restrictions put on the external loads to avoid production of virtual work in the kinematic modes can be acceptable. It can happen, for instance, that these modes only displace unloaded nodes. A simple example is provided by the quadrilateral panel when subdivided into triangular elements by the diagonals, each triangle being treated as an equilibrium model according to the analysis of section 4.2.

For the purpose of visualizing the kinematic mode generated by this subdivision (Fig. 14), the triangular panels can be assumed to be pinjointed at the middle of their edges, since for unstressed panels the displacements of these points are the averages of the edge displacements and

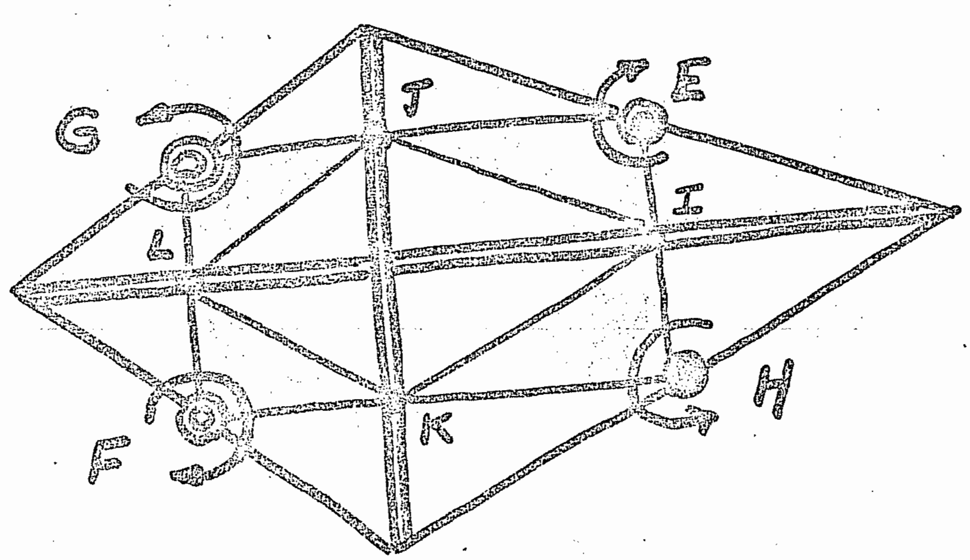

Fig. 14.

comply with the definitions of the generalized displacements. The unstressed quadrilateral can obviously behave as a four bar linkage but it can be proved by geometry[14] that the points E, F, G and H can be taken as instantaneous rotation centers. Consequently the quadrilateral can transmit any self equilibrated load system applied to the nodes E, F, G and H.

The internal nodes I, J, K and L sould either remain unloaded or their load system restricted by a virtual work condition. The impact of this on the mathematical process of elimination of the unloaded nodes is worth investigating. By splitting the nodal displacements and nodal loads matrices in two parts : $q_{(1)}$ and $g_{(1)}$ for the fixed nodes and $q_{(2)}$ and $g_{(2)}$ for the others, the following stiffness equations can be written

$$g_{(1)} = K_{11}q_{(1)} + K_{12}q_{(2)}, \tag{122}$$

$$g_{(2)} = K_{21}q_{(1)} + K_{22}q_{(2)}. \tag{123}$$

By assumption the kinematic mode is of the type

$$q_{(1)} = 0, \qquad q_{(2)} = m$$

and, since it exists without external loads,

$$K_{12}m = 0, \tag{124}$$

$$K_{22}m = 0. \tag{125}$$

If the $g_{(2)}$ loads exist

$$g^T_{(2)}\, m = q^T_{(1)}K^T_{21}m + q^T_{(2)}K^T_{22}m.$$

But, since $K^T_{21} = K_{12}$ and $K^T_{22} = K_{22}$, it follows from equations (124-125) that

$$g^T_{(2)}\, m = 0. \tag{126}$$

This is the virtual work equation restricting these loads. If however the nodes affected by the kinematic mode are unloaded, that is $g_{(2)} = 0$, we shall generally wish to eliminate the conjugate displacements by solving the system

$$K_{22}q_{(2)} = -K_{21}q_{(1)} \tag{127}$$

and substituting for $q_{(2)}$ in equation (122). Although the homogeneous adjoint to equation (127) has the non trivial solution $q_{(2)} = m$, the second member has the orthogonality property

$$q^T_{(1)}K^T_{21}m = 0,$$

required for the existence of a solution. The general solution

$$q_{(2)} = Pq_{(1)} + \alpha m$$

depends on the arbitrary scalar α. However, in view of equation (124) the final result does not depend on it

$$g_{(1)} = (K_{11} + K_{12}P)q_{(1)}, \tag{128}$$

and holds for any particular matrix P satisfying

$$K_{22}P = -K_{21}. \tag{129}$$

A particular matrix can be found by preventing the kinematic deformation to take place, e.g. in the case of the quadrilateral by adding a rigid rod between JK or LI. In this manner a displacement component of node J (or L) is related to one of node K (or I) and system (127) reduced to one with a non singular matrix; the discarded equation being automatically satified.

7. Solution by self-strainings.

The theory of equilibrium models developed in section 4.1) furnishes stiffness matrices and allows in principle a solution of the overall structural problem by the same stiffness methods and the same computer programs as for displacement models. It is however true that the requirements of stress continuity and the resulting definitions of generalized variables increase considerably the number of nodal displacements. As a consequence the size of the complete stiffness matrix can be much larger than in the case of a structure analyzed in the same number of displacement models. Problems of conditioning of the stiffness equations and accuracy of the numerical procedures are raised thereby, which only experience can help to evaluate.

As a matter of fact the topology of the connections between elements is generally such that displacements are better choices for the unknowns in a displacement model analysis and forces in an equilibrium model analysis. When the internal forces themselves are chosen it is better to look for self-equilibrating combinations, whose intensities constitute the generalized unknowns. This section is devoted to a short mathematical description of this procedure, using the matrices previously defined.

We start by expressing the external loads on the nodes in terms of the stress parameters of the elements. This is achieved substituting the relations (75)

$$g_{(k)} = C_k b_{(k)} + A_k p_{(k)}, \qquad (130)$$

into the equilibrium equation (117)

$$p = \sum_k L_k^T g = \sum_k L_k^T C_k b_{(k)} + \sum_k L_k^T A_k p_{(k)} \qquad (131)$$

From equation (121) we also obtain equivalent nodal forces due to the external body loads and transform the expression by means of the definition (77) of the R matrices

$$p = \sum_k L_k^T C_k F_{\ell\ell k}^{-1} F_{\ell p k}\, p_{(k)} - \sum_k L_k^T A_k p_{(k)} \qquad (132)$$

Adding (131) and (133) and denoting by

$$f = g + p, \tag{133}$$

the total nodal external forces, we place the result in the form

$$\sum_k L_k^T C_k b_{(k)} = f - \sum_k L_k^T C_k F_{bbk}^{-1} F_{bpk} p_{(k)} \tag{134}$$

of a linear system to determine the unknowns $b_{(k)}$. The general solution is of the form

$$b_{(k)} = b_{(k)}^{o} + \sum_i x_i b_{(k)}^{i} \tag{135}$$

The particular solution is of the type

$$b_{(k)}^{o} = M_k \left(f - \sum_k L_k^T C_k F_{bbk}^{-1} F_{bpk} p_{(k)} \right) \tag{136}$$

The other terms are the general solution without second member

$$\sum_k L_k^T C_k b_{(k)} = 0 \qquad i = 1, 2 \dots s. \tag{137}$$

They represent the stress parameters of self-equilibrating states of stress, whose unknown intensities x_i are taken to be the generalized quantities to determine in order to satisfy the compatibility requirements. Introducing the matrices

$$B_k = (b^1_{(k)}, b^2_{(k)} \dots b^s_{(k)}),$$

$$x^T = (x_1, x_2 \dots x_s),$$

the solution (135) takes the form

$$b_{(k)} = M_k \left(f - \sum_k L_k^T C_k F_{bbk}^{-1} F_{bpk} p_{(k)} \right) + B_k x \qquad (138)$$

it is a complete description of the possible equilibrium states of the structure. We substitute it into the strain energy (73) of an element

$$\tfrac{1}{2} b_{(k)}^T F_{bbk} b_{(k)} + b_{(k)}^T F_{bpk} p_{(k)} + \tfrac{1}{2} p_{(k)}^T F_{ppk} p_{(k)},$$

add over all the elements and obtain, after cancellation of terms, the following expression of the total strain energy

$$\Psi = \tfrac{1}{2} f^T [\delta_{00}] f + x^T [\delta_{10}] f + \tfrac{1}{2} x^T [\delta_{11}] x + \tfrac{1}{2} p_{(k)}^T F_{ppk} p_{(k)} \qquad (139)$$

where

$$[\delta_{00}] = \sum_k M_k^T F_{bbk} M_k \qquad (140)$$

$$[\delta_{10}] = \sum_k B_k^T F_{bbk} M_k \qquad (141)$$

$$[\delta_{11}] = \sum_k B_k^T F_{bbk} B_k \qquad (142)$$

From equations (119) that can be written $f = Kq$, the nodal displacements matrix q is conjugate to the loads matrix f and the virtual work of the external loads is $f^T q$. We can apply the complementary energy principle in the form

$$\delta \left(\Psi - f^T q \right) = 0 \qquad (143)$$

where the matrix q is prescribed and f considered as a reaction loads matrix.

Taking variations on the self-straining intensities matrix x produces the stationarity condition

$$[\delta_{11}]\,x + [\delta_{10}]\,f = 0 \qquad \text{or}$$

$$x = -[\delta_{11}]^{-1}[\delta_{10}]\,f \tag{144}$$

The stresses are then determined in the whole structure in terms of the external forces. Taking variations on the reaction loads f

$$[\delta_{10}]^{T}x + [\delta_{00}]\,f = q$$

or, in view of equation (144)

$$q = I\,f \tag{145}$$

with the influence coefficients matrix

$$I = [\delta_{00}] - [\delta_{10}]^{T}[\delta_{11}]^{-1}[\delta_{10}] \tag{146}$$

Our last step implies that the elements of f can be varied independantly; this of course is only the case when the complete structure has been supported against rigid body modes by preventing some of the nodal displacements and when the f matrix does not include the reactions against the supports. Those reactions can be determined from overall equilibrium considerations. Should the structure be subjected to additional prescribed displacements (externally hyperstatic) the corresponding unknown reactions can be obtained from the set of equations (145).

As already mentioned the size of the column matrix x is generally much smaller than that of q. The advantage of the self-strainings method is then that it requires only the inversion of the square matrix $[\delta_{11}]$ of the same size as x. The difficulty lies in the determination of the simplest matrices $b^{j}_{(k)}$ or B_k. A great step forward in the automation of the calculations would be achieved if the computer itself could be taught to investigate the topology of the matrices $L_k^T C_k$ and deduce the self-strainings confined to the smallest number of elements.

8. Convergence.

Because the solutions based on displacement models depend on a minimum principle, the lower bounds on the influence coefficients can only be raised by further subdivision of the structure, provided the original system of displacements be contained as a particular case of the new one. This point has been stressed by Melosh[18]. The requirement is however automatically satisfied when the type of element analysis is not altered. For along a new subdivision line the original displacement field behaves with the same polynomial approximation (linear, quadratic or higher) as it will be endowed with in the new analysis. The only difference is that some of the coefficients of the polynomial, which were originally dependant, become independant.

The same considerations apply to the analyses with equilibrium models where further subdivision can only lower the upper bounds on influence coefficients.

This monotonic convergence is however insufficient to guarantee convergence to the true solution and it is obvious that the subdivision pattern must play an important role in the convergence process. The interest of the dual method of analysis is precisely that by comparison between upper and lower bounds the degree of true convergence can be ascertained.

9. Bounds to the direct influence coefficients.

9.1. Lower bounds by the minimum total energy principle.

The displacements are assumed to verify some homogeneous boundary conditions

$$u = 0 \quad \text{and/or} \quad v = 0 \quad \text{on } \Gamma_{66} \tag{147}$$

so that the structure is at least isostatically supported and can resist the application of unit loads.

Since the first variation of the total energy must vanish

$$\iint_A \left(\sigma_x \delta \varepsilon_x + \tau_{xy} \delta \gamma_{xy} + \sigma_y \delta \varepsilon_y \right) dx\, dy$$

$$- \iint_A \left(\bar{X} \delta u + \bar{Y} \delta v \right) dx\, dy$$

$$- \int_{\Gamma_\sigma} \left(\bar{p}_x \delta u + \bar{p}_y \delta v \right) ds = 0 \qquad (148)$$

Consider the special variation ($\delta\varepsilon$ a constant)

$$\delta u = u\, \delta\varepsilon \qquad \delta v = v\, \delta\varepsilon \qquad (149)$$

where u and v are the final displacement components given by application of the principle. This special variation is compatible with (147) and gives also

$$\delta \varepsilon_x = \delta \left(\frac{\partial u}{\partial x} \right) = \frac{\partial}{\partial x} (\delta u) = \delta\varepsilon\; \varepsilon_x$$

and similarly

$$\delta \varepsilon_y = \delta\varepsilon\; \varepsilon_y \qquad \delta \gamma_{xy} = \delta\varepsilon\; \gamma_{xy}$$

Substitution into (148) produces

$$\iint_A \left(\varepsilon_x \sigma_x + \gamma_{xy} \tau_{xy} + \varepsilon_y \sigma_y \right) dx\, dy$$

$$- \iint_A \left(\bar{X} u + \bar{Y} v \right) dx\, dy - \int_{\Gamma_\sigma} \left(\bar{p}_x u + \bar{p}_y v \right) ds = 0$$

In view of equation (16) this is a form of Clapeyron's theorem

$$U = -\frac{1}{2} P = \frac{1}{2} \iint_A (\bar{X} u + \bar{Y} v)\, dx\, dy + \frac{1}{2} \int_{\Gamma_\sigma} (\bar{p}_x u + \bar{p}_y v)\, ds \qquad (150)$$

This special variation certainly exists if no approximations are made; hence for the exact solution

$$U_e = -\tfrac{1}{2} P_e \qquad (151)$$

The special variation also exists for approximate fields of the assumed type (26) as it is possible to take $\delta \alpha_i = \delta \varepsilon\, \alpha_i$. Hence for the approximate solution

$$U_a = -\tfrac{1}{2} P_a \qquad (152)$$

Because the total energy of the exact solution must be an absolute minimum

$$U_a + P_a \geqslant U_e + P_e$$

and then, by virtue of equations (151-152)

$$P_a \geqslant P_e \qquad (153)$$

Let $\bar{F}$ be a prescribed generalized load conjugate to the displacement q. Then $\bar{P}_a = -\bar{F} q_a$ and $P_e = -\bar{F} q_e$, whereby (153) gives

$$\bar{F} q_a \leqslant \bar{F} q_e$$

The proof of the lower bound character of the approximate influence coefficient follows after division by $\bar{F}^2$

$$\frac{q_a}{F} \leqslant \frac{q_e}{F} \; . \tag{154}$$

9.2. Upper bounds by the minimum complementary energy principle.

Since the prescribed displacements on the boundary, where the structure is supported are zero, the principle reduces to

$$\iint_A \phi(\sigma)\,dx\,dy - \bar{q}F = \psi - \bar{q}F \qquad \text{minimum.} \tag{155}$$

The generalized displacement is now prescribed, the conjugate load a reaction. Vanishing of the first variation yields

$$\iint_A \left(\varepsilon_x\,\delta\sigma_x + \gamma_{xy}\,\delta\tau_{xy} + \varepsilon_y\,\delta\sigma_y\right) dx\,dy - \bar{q}\,\delta F = 0 \tag{156}$$

Consider the special variations

$$\delta\sigma_x = \sigma_x \frac{\delta F}{F} \qquad \delta\tau_{xy} = \tau_{xy}\frac{\delta F}{F} \qquad \delta\sigma_y = \sigma_y \frac{\delta F}{F}$$

that certainly exist for the exact solution and also for the approximate stress fields of type (66), where one can set

$$\delta p_i = p_i \frac{\delta F}{F} \qquad \delta P_k = P_k \frac{\delta F}{F}$$

For such special variations, condition (156) gives

$$\frac{1}{F}\iint_A \left(\sigma_x\,\varepsilon_x + \tau_{xy}\,\gamma_{xy} + \sigma_y\,\varepsilon_y\right) dx\,dy - \bar{q} = 0$$

or, in view of equation (16), again a form of Clapeyron's theorem

$$\psi = \frac{1}{2}\bar{q}F$$

From the statement (155) it follows that

$$\psi_a - \bar{q}F_a \geqslant \psi_e - \bar{q}F_e$$

and since

$$\psi_a = \frac{1}{2}\bar{q}F_a \qquad \psi_e = \frac{1}{2}\bar{q}F_e$$

$$\bar{q}F_a \leqslant \bar{q}F_e$$

ψ_a and ψ_e being strain energies are both positive, so that F_a and F_e have both the sign of $\bar{q}$. Division by the positive quantity F_aF_e proves the upper bound character of the approximate influence coefficient :

$$\frac{\bar{q}}{F_a} \geqslant \frac{\bar{q}}{F_e} . \tag{157}$$

The derivation of bounds to the cross influence coefficients is a straightforward algebraic process[11,12,13,14].

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