CLF Based Designs with Robustness to Dynamic Input Uncertainties

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Abstract

The problem of robust stabilization of nonlinear systems in the presence of input uncertainties is of great importance in practical implementation. Stabilizing control laws may not be robust to this type of uncertainty, especially if cancellation of nonlinearities is used in the design. By exploiting a connection between robustness and optimality, "domination redesign" of the control Lyapunov function (CLF) based Sontag's formula has been shown to possess robustness to static and dynamic input uncertainties. In this paper we provide a sufficient condition for the domination redesign to apply. This condition relies on properties of local homogeneous approximations of the system and of the CLF. We show that an inverse optimal control law may not exist when these conditions are violated and illustrate how these conditions may guide the choice of a CLF which is suitable for domination redesign.

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1 Introduction

In this paper we revisit the problem of designing globally stabilizing control laws which minimize a cost function of the form

$$\int_0^\infty l(x) + u^T u \, dt, \quad l(x) \geq 0 \tag{1.1}$$

This is an *inverse* optimal control problem because the state penalty $l(x)$ in not specified in advance. Instead, the minimization is to be achieved for some $l(x) \geq 0$.

Following the recent works [4, 14], our interest in inverse optimality is motivated by the desire to achieve robustness with respect to uncertainties which appear at the plant input, as shown in Figure 1. In the absence of uncertainty, Δ is identity and the feedback loop

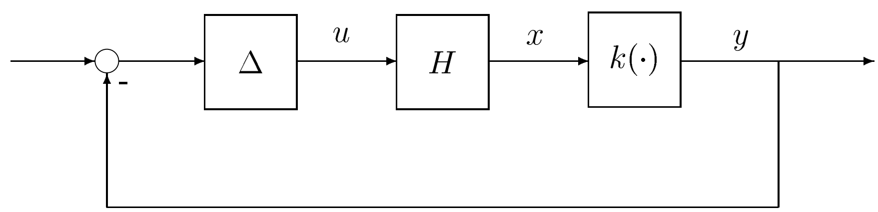

Figure 1: Nonlinear feedback loop with the control law $k(x)$ and input uncertainty Δ.

consists of the nominal plant H given by

$$\dot{x} = f(x) + g(x)u, \qquad x \in \mathbb{R}^n, \; u \in \mathbb{R}^m \tag{1.2}$$

and the nominal control law

$$u = -k(x) =: -y \tag{1.3}$$

The situation in which uncertainty is at the input is common in applications, especially when simplified models of actuators are used for the design.

Early results in optimal control theory [9, 11] have related the optimality of the control law $u = -k(x)$ with respect to a cost (1.1) to a dissipativity property of the system (1.2) with output $y = k(x)$. This dissipativity property implies that the stability of the feedback system in Figure 1 is preserved for a class of uncertainties. This class, which we denote by $\mathcal{I}$, includes all uncertainties that are stable and satisfy the dissipativity inequality

$$\dot{W}_\Delta(z) \leq y_\Delta u_\Delta - \frac{1}{2} u_\Delta^T u_\Delta \tag{1.4}$$

for some positive definite radially unbounded function $W_\Delta(z)$, where $u_\Delta \in \mathbb{R}^m$ is the input, $y_\Delta \in \mathbb{R}^m$ is the output, and $z \in \mathbb{R}^{n_z}$ is a state vector for the system Δ.

An uncertainty in $\mathcal{I}$ may be an unknown gain $\kappa \in (\frac{1}{2}, \infty)$, or a static sector nonlinearity[1] $\varphi(\cdot) \in (\frac{1}{2}, \infty)$, because both satisfy the dissipativity inequality (1.4) with $W_\Delta \equiv 0$. An uncertainty in $\mathcal{I}$ may also be *dynamic* provided that it has relative degree zero. For instance, any stable linear system with the transfer function having the real part greater than $\frac{1}{2}$ belongs to $\mathcal{I}$. For linear optimal systems the robustness to such uncertainties is guaranteed by the fact that the Nyquist curve of the optimal system lies outside a disk in the complex plane [1]. A familiar interpretation is $(\frac{1}{2}, \infty)$ gain margin and $(\pm 60°)$ phase margin of optimal systems. In analogy with this property of linear optimal systems, we have defined in [14] the disk stability margin $\mathrm{D}(\frac{1}{2})$: the system (1.2), (1.3) is said to have a *disk margin* $\mathrm{D}(\frac{1}{2})$ if the closed loop system in Figure 1 remains stable for any Δ in $\mathcal{I}$.

Several recent works [8, 10, 12] deal with the problem of designing control laws for nonlinear systems which are robust to dynamic input uncertainties. These methods, based on the small gain theorem, require that the input-output gain of the system $\Delta - I_m$ (I_m is the $m \times m$ identity matrix) be sufficiently small; in any case, strictly smaller than one. In contrast, our class of uncertainties $\mathcal{I}$ contains systems for which no gain can be defined, like for example

$$\begin{aligned} \dot{z} &= -z + z^2 u_\Delta \\ y_\Delta &= z^3 + u_\Delta \end{aligned} \tag{1.5}$$

The starting point to achieve a disk margin from the knowledge of a CLF is the Sontag's formula [15]. In [14] we have shown that the Sontag's formula achieves optimality with respect to a cost functional of the form

$$J = \int_0^\infty l(x) + u^T R(x) u \; dt \tag{1.6}$$

for some $l(x) \geq 0$ and $R(x) > 0$ which are not chosen by the designer, but directly obtained from the CLF $V(x)$ and the vector fields $f(x)$ and $g(x)$. This inverse optimal result is sufficient to establish robustness to *static* uncertainties in $\mathcal{I}$. However, the presence of the state dependent matrix $R(x)$ in the cost makes it impossible to guarantee any robustness to *dynamic* uncertainties. We have further shown in [14] that, if $R^{-1}(x)$ is bounded on compact sets, a domination redesign achieves optimality for a cost of the form (1.1) and the disk margin $\mathrm{D}(\frac{1}{2})$.

In this paper we provide conditions for the boundedness of $R^{-1}(x)$ on compact sets. As we illustrate by an example, Sontag's formula, in general, does not provide a locally bounded $R^{-1}(x)$. The same is true for the Freeman's formula [4]. A result of this paper shows that in both cases $R^{-1}(x)$ is bounded if the CLF property is retained for some local (homogeneous) approximation of the system and of the CLF. This condition is only sufficient because it implies the stabilizability of the considered local approximation but it may guide the choice of a CLF which is suitable for domination redesign.

In Section 2, we review the derivation of inverse optimal control laws from the knowledge of a CLF. In Section 3 we provide a sufficient condition for a domination redesign to achieve the desired disk stability margin and illustrate different situations with examples.

[1] A function $\varphi(s)$ is said to belong to a sector (a, b) if $as^2 < s\varphi(s) < bs^2$; it is said to belong to a sector $[a, b]$ if $as^2 \leq s\varphi(s) \leq bs^2$.

2 CLF for inverse optimal control

2.1 Inverse optimality

In the direct optimal design we would have to solve the Hamilton-Jacobi-Bellman (HJB) equation which, in general, is not a feasible task. On the other hand, in the inverse approach, a stabilizing feedback is designed first and then shown to be optimal.

A globally stabilizing control law $u(x)$ solves an inverse optimal problem for the system

$$\dot{x} = f(x) + g(x)u \tag{2.1}$$

with the cost functional of the form (1.6), if it can be expressed as

$$u = -k(x) = -\frac{1}{2}R^{-1}(x)(L_gV(x))^T, \quad R(x) > 0, \tag{2.2}$$

where $V(x)$ is a positive definite[2], radially unbounded function, such that the negative semidefiniteness of $\dot{V}$ is achieved with the control $u = -\frac{1}{2}k(x)$, that is

$$\dot{V} = L_fV(x) - \frac{1}{2}L_gV(x)k(x) \leq 0 \tag{2.3}$$

Indeed, if the function $l(x)$ is set to be the negative right-hand side of (2.3)

$$l(x) := -L_fV(x) + \frac{1}{2}L_gV(x)k(x) \geq 0 \tag{2.4}$$

then $V(x)$ is a solution of the Hamilton-Jacobi-Bellman (HJB) equation

$$l(x) + L_fV(x) - \frac{1}{4}(L_gV(x))R^{-1}(x)(L_gV(x))^T = 0 \tag{2.5}$$

Hence, according to the standard results of the optimal control theory (c.f. [7]) $u = -k(x)$ is optimal among all $u(t)$ which achieve $x(t) \to 0$ as $t \to \infty$. What is of interest to us is that $u = -k(x)$ achieves robustness to *static* uncertainties in $\mathcal{I}$. Moreover, if $R(x) = I_m$, $u = -k(x)$ achieves robustness to all uncertainties in $\mathcal{I}$, that is, it possesses a disk margin $D(\frac{1}{2})$ [5, 14, 16].

2.2 Sontag's and Freeman's formulas

The starting point for our design with stability margins for the nonlinear system

$$H: \quad \dot{x} = f(x) + g(x)u \tag{2.6}$$

[2]Positive semidefiniteness is sufficient for global optimality [14]. However, in this paper all Lyapunov and optimal value functions are positive definite because they are CLF's.

is the knowledge of a *control Lyapunov function* (CLF): a smooth, positive definite, radially unbounded function $V(x)$ which satisfies

$$L_g V(x) = 0 \Rightarrow L_f V(x) < 0 \quad \forall x \neq 0 \tag{2.7}$$

A stabilizing optimal control law for the cost of the form (1.6) can be selected such that the CLF V becomes an optimal value function. A particular optimal stabilizing control law is given by the Sontag's formula [15]:

$$u_S(x) = -p_S(x)(L_g V(x))^T \tag{2.8}$$

with

$$p_S(x) := \begin{cases} c_0 + \frac{a(x)+\sqrt{a^2(x)+(b^T(x)b(x))^2}}{b^T(x)b(x)} & , \quad b(x) \neq 0 \\ c_0 & , \quad b(x) = 0 \end{cases} \tag{2.9}$$

where $L_f V(x) = a(x)$, $(L_g V(x))^T = b(x)$. The positive constant c_0 is needed in order for $R(x)$ in the cost functional to be well defined. Indeed, by defining $R_S^{-1}(x) := 2p_S(x)I_m$ the Sontag's formula takes the form

$$u_S(x) = -\frac{1}{2}R_S^{-1}(x)(L_g V(x))^T$$

Because one half of $u_S(x)$ achieves

$$\dot{V}|_{\frac{u_S(x)}{2}} = \frac{1}{2}a(x) - \frac{1}{2}\sqrt{a^2(x) + (b^T(x)b(x))^2} - \frac{1}{2}c_0 b^T(x)b(x) =: -l_S(x) < 0$$

for all $x \neq 0$, we conclude that it is optimal for the cost functional of the form $\int_0^\infty l_S(x) + u^T R_S(x)u \, dt$. Hence, this control law is robust to all static uncertainties in $\mathcal{I}$.

If we strengthen the CLF condition (2.7) to require that a smooth, positive definite function $\alpha(x)$ be known such that

$$L_g V(x) = 0 \Rightarrow L_f V(x) < -\alpha(x) \quad \forall x \neq 0 \tag{2.10}$$

we can construct a formula for an optimal stabilizing control law motivated by the pointwise minimum norm control introduced in [4]. We shall call this control law Freeman's formula and, for notational convenience, the Lyapunov function V that satisfies (2.10) an α-CLF. Because the exact pointwise minimum norm control does not possess the robustness to static uncertainties in $\mathcal{I}$, for the Freeman's formula we double its value:

$$u_F(x) = -p_F(x)(L_g V(x))^T \tag{2.11}$$

with

$$p_F(x) = \begin{cases} c_0 + 2\frac{a(x)+\alpha(x)}{b^T(x)b(x)} & \text{if } \ a(x) + \alpha(x) > 0 \\ c_0 & \text{if } \ a(x) + \alpha(x) \leq 0 \end{cases}$$

As it was the case with the Sontag's formula, the control law (2.11) is of the form

$$u_F(x) = -\frac{1}{2} R_F^{-1}(x)(L_g V(x))^T$$

With $u = \frac{1}{2} u_F(x)$, the time derivative of V along the trajectories of the closed loop system satisfies

$$\dot{V}|_{\frac{u_F(x)}{2}} = \min\{a(x), -\alpha(x)\} - \frac{1}{2} c_0 b^T(x) b(x) =: -l_F(x) < 0$$

for all $x \neq 0$. Hence, Freeman's formula (2.11) provides a control law which is optimal for the cost functional $J = \int_0^\infty l_F(x) + u^T R_F(x) u \; dt$.

Both control laws (2.8) and (2.11) are smooth everywhere except possibly at the origin [4, 15]. The control law (2.8) is continuous at $x = 0$ if the CLF satisfies the *small control property*: for each $\epsilon > 0$, we can find $\delta(\epsilon) > 0$ such that, if $0 < \| x \| < \delta$, there exists u which satisfies $L_f V(x) + (L_g V)^T(x) u < 0$ and $\| u \| < \epsilon$ [15]. The same is true for the Freeman's formula if the α-CLF V has the small control property [4].

In summary, the control laws $u_S(x)$ and $u_F(x)$ are optimal, globally stabilizing, smooth away from the origin, and, under a mild assumption, continuous at the origin. They are robust to *static* uncertainties in $\mathcal{I}$. However, a disk margin is not guaranteed because $R_S(x)$ and $R_F(x)$ are not constant (for a counterexample see Section 3.4 in [14]).

2.3 Domination redesign

Sontag's and Freeman's formula achieve optimality for the cost $J = \int_0^\infty l(x) + u^T R(x) u \; dt$. The functions $l(x)$ and $R(x)$ are uniquely determined by the vector fields $f(x)$ and $g(x)$ and the (α)CLF $V(x)$. If $R^{-1}(x)$ happens to be bounded on compact sets, we can proceed with the domination redesign to obtain a control law with a disk margin [14].

Consider an optimal globally stabilizing control

$$u(x) = -\frac{1}{2} R^{-1}(x)(L_g V(x))^T \tag{2.1}$$

determined from a C^1, positive definite, radially unbounded optimal value function V and assume that $R^{-1}(x)$ is bounded on compact sets. Define a continuous *dominating function* $\gamma : I\!R^+ \to I\!R^+$ which satisfies

$$\gamma(V(x)) I \geq R^{-1}(x), \quad \forall x \in I\!R^n \tag{2.2}$$

$$\lim_{t \to \infty} \int_0^t \gamma(s) \; ds = +\infty \tag{2.3}$$

Such a function always exists. If $R(x)$ is continuous, a possible choice is

$$\gamma(s) = \sup_{\{x : V(x) \leq s\}} \lambda_{\max}(R^{-1}(x)) + d, \quad d > 0$$

with $\lambda_{\max}$ denoting the largest eigenvalue. The redesigned optimal value function

$$\tilde{V}(x) := \int_0^{V(x)} \gamma(s)ds$$

inherits the properties of $V(x)$: it is C^1, positive definite and radially unbounded.

The redesigned control law

$$\tilde{u}(x) = -\frac{1}{2}(L_g\tilde{V}(x))^T = -\frac{1}{2}\gamma(V(x))(L_gV(x))^T \tag{2.4}$$

achieves

$$\begin{aligned}\dot{\tilde{V}} &= \gamma(V)L_fV - \frac{1}{2}\gamma^2(V)L_gV(L_gV)^T \\ &\leq \gamma(V)(-l - \frac{1}{4}L_gV\ R^{-1}(L_gV)^T) \leq 0\end{aligned}$$

Boundedness of solutions follows because V is radially unbounded. To establish GAS, we examine the set E where $\dot{\tilde{V}} = 0$. In E we have $L_gV(x) = 0$ so that $\tilde{u}(x) = 0$ and hence, $u(x) = 0$. Because $u(x)$ is optimal stabilizing, the solutions of $\dot{x} = f(x) + g(x)u(x)$ contained in E converge to the origin. But, since in E the two closed-loop systems corresponding to $u(x)$ and $\tilde{u}(x)$ coincide, we conclude that the redesigned feedback $\tilde{u}(x)$ achieves GAS.

The function $\tilde{l}(x) := -L_f\tilde{V} + \frac{1}{4}(L_g\tilde{V})(L_g\tilde{V})^T$ is positive semidefinite because

$$\begin{aligned}-\tilde{l} &= L_f\tilde{V} - \frac{1}{4}(L_g\tilde{V})(L_g\tilde{V})^T = \gamma(V)(L_fV - \frac{1}{4}\gamma(V)(L_gV)(L_gV)^T) \\ &\leq \gamma(V)(L_fV - \frac{1}{4}(L_gV)R^{-1}(L_gV)^T) = -\gamma(V)l \leq 0\end{aligned}$$

Therefore, by construction, $\tilde{V}$ is a radially unbounded, positive definite solution of the Hamilton-Jacobi-Bellman equation which implies that the control law (2.4) minimizes the modified cost functional

$$\tilde{J} = \int_0^\infty \tilde{l}(x) + u^Tu\ dt \tag{2.5}$$

hence providing a D($\frac{1}{2}$) disk margin.

3 Boundedness of R_S^{-1} and R_F^{-1}

The control laws (2.8) and (2.11) can be used for domination redesign if $R_S^{-1}(x)$ and $R_F^{-1}(x)$ are bounded on compact sets. The boundedness of $R_S^{-1}(x)$ and $R_F^{-1}(x)$, however, is not guaranteed by the present assumptions as illustrated by the next example.

Example 1 For the system

$$\dot{x} = x^3 + x^2u \tag{3.1}$$

the CLF $V(x) = \frac{1}{2}x^2$ provides the control law

$$u_S(x) = \begin{cases} -(c_0 + \frac{x^4+\sqrt{x^8+x^{12}}}{x^6})x^3 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

¿From $R_S(x) = \frac{1}{2}p_S^{-1}(x)$, we obtain that, for $x \neq 0$,

$$R_S(x) = \frac{1}{2}\frac{x^6}{c_0x^6 + x^4 + \sqrt{x^8 + x^{12}}} \to 0$$

as x converges to 0. This means that $R_S^{-1}(x)$ is not bounded around the origin. Hence, the domination redesign fails to provide an optimal stabilizing feedback with a disk margin.

It turns out that the solution to the optimal control problem with $R(x) = 1$ does not exist. Regardless of the choice of $l(x)$ there is no positive semidefinite optimal value function which satisfies the corresponding HJB equation

$$l(x) + x^3\frac{\partial V}{\partial x} - \frac{1}{4}x^4(\frac{\partial V}{\partial x})^2 = 0 \tag{3.2}$$

and results in a stabilizing optimal control law. Solving (3.2) for $\frac{\partial V}{\partial x}$ we obtain

$$\frac{\partial V}{\partial x} = \frac{2}{x} \pm 2\sqrt{\frac{1}{x^2} + \frac{l(x)}{x^4}}$$

The only way to make $\frac{\partial V}{\partial x}$ (and $V(x)$) bounded is to select the negative sign in front of the square root when $x > 0$. Then V is positive semidefinite only if $l(x) \equiv 0$. But this choice of $l(x)$ results in $u \equiv 0$ for $x > 0$ and the closed loop system is unstable. □

In view of Example 1 it is important to establish conditions on the CLF V, and on the vector fields g^i, $i \in \{1, \ldots, m\}$, under which the domination redesign is possible. The following theorem provides a sufficient condition based on the properties of homogeneous local approximations of the system and of the CLF. Basic homogeneity definitions are recalled in Appendix.

Theorem 1 For a given dilation $\delta_\epsilon(x)$, let $f_h(x)$, $g_h^i(x)$ and $V_h(x)$ be the homogeneous approximations of the vector fields $f(x)$ and $g^i(x)$ and of the CLF $V(x)$, respectively. Denote by d_f, d_{g^i} and d_V the degrees of these approximations.

If V_h is a CLF for the homogeneous system

$$\dot{x} = f_h(x) + g_h(x)u = f_h(x) + \sum_{i=1}^{m} g_h^i(x)u_i, \tag{3.3}$$

then $R_S^{-1}(x)$ is bounded on compact sets provided that

$$d_f \geq 2d_{g^i} + d_v, \quad i \in \{1, \ldots, m\} \tag{3.4}$$

Similarly, if $V_h(x)$ is an α-CLF for (3.3) with respect to $\alpha_h(x)$, the homogeneous approximation of $\alpha(x)$, then $R_F^{-1}(x)$ is bounded on compact sets.

Proof: We only prove the theorem for Sontag's formula (the proof is analog for Freeman's formula).

If $L_fV(x) \leq 0$, we have $0 \leq L_fV + \sqrt{(L_fV)^2 + (L_gV(L_gV)^T)^2} \leq L_gV(L_gV)^T$ which implies

$$c_0 \leq c_0 + \frac{L_fV + \sqrt{(L_fV)^2 + (L_gV(L_gV)^T)^2}}{L_gV(L_gV)^T} \leq 1 + c_0 \tag{3.5}$$

Hence, $R_S^{-1}(x)$ is uniformly bounded in the region of the state space where $L_fV(x) \leq 0$ and we only need to consider regions where $L_fV(x) > 0$.

On any compact set which does not contain the origin, $L_gV(x)$ must be bounded away from 0 for those x for which $L_fV > 0$. Otherwise, because L_gV is continuous, there must exists $x_0 \neq 0$ such that $L_gV(x_0) = 0$ and $L_fV(x_0) \geq 0$ which violates the assumption that V is a CLF. This means that we only need to prove that $R_S^{-1}(x)$ is bounded on an arbitrarily small open ball $U_\delta := \{x \in I\!R^n : ||x|| < \delta\}$, $\delta > 0$, around of the origin.

Because V_h is a homogeneous CLF for the homogeneous system (3.3), a standard argument (see for instance [3, 13]) shows that $\dot{V}_h$ is rendered strictly negative by the homogeneous control law

$$u_i(x) = -c||x||_h^{d_f - 2d_{g^i} - d_v}(L_{g_h^i}V_h(x))$$

where $c > 0$ is a sufficiently large constant and $||.||_h$ is any homogeneous norm (the homogeneous scaling factor is used to adjust the homogeneity degrees of f_h and $g_h^i u_i$). With this control law, the closed-loop vector field is homogeneous of degree d_f and $\dot{V}_h$ satisfies

$$\dot{V}_h = L_{f_h}V_h - c\sum_{i=1}^{m}(L_{g_h^i}V_h)^2||x||_h^{d_f - 2d_g - d_v} < -\mu||x||_h^{d_f + d_v}$$

for some constant $\mu > 0$. If (3.4) holds, then for $||x||_h$ small, we have a fortiori

$$L_{f_h}V_h - c\sum_{i=1}^{m}(L_{g_h^i}V_h)^2 \leq \dot{V}_h < -\mu||x||_h^{d_f + d_v}$$

Returning to the original system, and applying the control law $u = -c(L_gV)^T$, we write $\dot{V}$ in the form

$$\dot{V}\mid_{u=-c(L_gV)^T} = \dot{V}_h\mid_{u=-c(L_{g_h}V_h)^T} + 0(L_{f_h}V_h) < -\mu||x||_h^{d_f+d_v} + 0(||x||_h^{d_f+d_v})$$

By definition, this means that $\dot{V}_{u=-c(L_gV)^T}$ is negative definite in a neighborhood U_δ of the origin, that is

$$L_fV - c||L_gV||^2 < 0, \ x \neq 0$$

Hence, we obtain

$$0 < \frac{1}{c} < \frac{||L_gV(x)||^2}{L_fV(x)}$$

for all $x \in U_\delta$ for which $L_fV(x) > 0$. This implies that in a neighborhood of the origin

$$R_S(x) = \frac{||L_gV(x)||^2}{c_0||L_gV(x)||^2 + L_fV + \sqrt{(L_fV)^2 + ||L_gV(x)||^4}} I > \epsilon I > 0$$

Therefore, $R_S^{-1}(x)$ is bounded on compact sets and the domination redesign applies.

□

If the standard dilation $\delta_\epsilon(x) = \epsilon x$ is considered, then the homogeneous approximation of the system typically reduces to its Jacobian linearization. If the CLF has a nontrivial quadratic approximation, the homogeneity degrees defined in Theorem 1 are $d_f = 0$, $d_{g^i} = -1$, and $d_v = 2$, so that the condition (3.4) is satisfied. This particular case of Theorem 1 yields the following corollary.

Corollary 1 If the quadratic approximation of V is a CLF for the Jacobian linearization of $\dot{x} = f(x) + g(x)u$, then $R_S^{-1}(x)$ is bounded on compact sets. Similarly, $R_F^{-1}(x)$ is bounded on compact sets if the quadratic approximation of $V(x)$ is an α-CLF for the Jacobian linearization of $\dot{x} = f(x) + g(x)u$ with respect to the quadratic approximation of $\alpha(x)$. □

The degree condition (3.4) of Theorem 1 excludes situations where the degree of the drift vector field is lower than the degree of the control vector field, i.e. $d_f \leq d_g$. Example 1 has indeed illustrated that this is a typical situation where the domination redesign might fail, regardless of the choice of the CLF.

In other situations, the conditions of Theorem 1 may be violated or satisfied depending on the choice of the dilation and of the CLF. The next example illustrates that the conditions of Theorem 1 may then guide the choice of a CLF which will be suitable for a domination redesign.

Example 2 For the system

$$\begin{aligned} \dot{x}_1 &= -x_1^5 + x_1x_2 \\ \dot{x}_2 &= x_2^2 + u \end{aligned} \tag{3.6}$$

the CLF $W = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$ is not suitable for domination redesign. Indeed, substituting $L_fW = -x_1^6 + x_1^2x_2 + x_2^4$ and $L_gW = x_2$ into Sontag's formula we obtain

$$R_S^{-1}(x) = \frac{-x_1^6 + x_1^2x_2 + x_2^4 + \sqrt{(-x^6 + x_1^2x_2 + x_2^4)^2 + x_2^4}}{x_2^2}$$

As x converges to the origin along the curve defined by $x_2 = x_1^3$, R_S^{-1} grows to infinity because

$$R_S^{-1}(x) \geq \frac{-x_1^6 + x_1^2x_2 + x_2^4}{x_2^2} = \frac{-x_1^6 + x_1^5 + x_1^{12}}{x_1^6} = \frac{(-x_1 + 1 + x_1^7)x_1^5}{x_1^6} \geq \frac{1}{2x_1} \tag{3.7}$$

Hence, $R_S^{-1}(x)$ is not bounded in a neighborhood of the origin and the domination redesign does not apply.

A reason why the quadratic CLF is not suitable for domination redesign is suggested by the fact that this CLF violates the assumptions of Corollary 1: the function $x_1^2 + x_2^2$ is *not* a CLF for the Jacobian linearization of (2) because the stabilizing term $-x_1^5$ is not retained in the linear approximation.

To retain the term $-x_1^5$ in the approximation, we choose the dilation $\delta_\epsilon(x) = (\epsilon^{1/4} x_1, \epsilon x_2)$. The homogeneous approximation of (2) now becomes

$$\begin{aligned} \dot{x}_1 &= -x_1^5 + x_1 x_2 \\ \dot{x}_2 &= u \end{aligned} \tag{3.8}$$

The degrees are $d_f = 1$ and $d_g = -1$ so that the degree condition (3.4) will be satisfied with a CLF of degree $d_v = 2$. This suggests the homogeneous CLF $V_h = x_1^8 + x_2^2$. Because V_h is also a CLF for the original system, Theorem 1 shows that a domination redesign applies to the CLF $V = V_h = x_1^8 + x_2^2$.

□

4 Conclusion

Because of the guaranteed robustness to input uncertainties, we are interested in designing control laws which are optimal with respect to cost functionals of the form $\int_0^\infty l(x) + u^T u \, dt$. The domination redesign method achieves this goal provided that $R^{-1}(x)$ obtained from Sontag's or Freeman's formula is bounded in a neighborhood of the origin. However, this may not be the case, and for some nonlinear systems our inverse optimal control problem cannot be solved regardless of the choice of a CLF.

We have shown that the domination redesign applies when the CLF property is retained for some local homogeneous approximation of the system and of the CLF. This local condition may be used to select or construct a CLF which is suitable for the domination redesign.

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5 Appendix: homogeneous approximations

A standard reference for the use of homogeneous approximations as local approximations of control systems is [6]. Here we recall the basic definitions.

A homogeneous approximation is defined with respect to a given *dilation*

$$\delta_\epsilon(x) = (\epsilon^{r_1} x_1, \ldots, \epsilon^{r_n} x_n), \quad \epsilon > 0$$

where the weights $(r_1, \ldots, r_n)$ are fixed real numbers, possibly different for each variable. The standard dilation is defined by $r_1 = \ldots = r_n = 1$.

For a given dilation, a continuous function $l : \mathbb{R}^n \to \mathbb{R}$ is homogeneous of degree $d \geq 0$ if $l(\delta_\epsilon(x)) = \epsilon^d l(x)$ for every $\epsilon > 0$. A continuous vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is homogeneous of degree d if $L_f l$ is homogeneous of degree $d + \tau$ whenever l is homogeneous of degree τ. In coordinates, this means that $f_i(x)$, the i-th component of $f(x)$, is homogeneous of degree $r_i + d$. A homogeneous norm $||.||_h$ is a norm which is homogeneous of degree one, for instance

$$||x||_h = (\sum_{i=1}^{n} |x_i|^{\frac{p}{r_i}})^{\frac{1}{p}}$$

with p large enough for the norm to be differentiable at the origin.

Local homogeneous approximations are defined by extension of the classical notion of approximation (which is defined with respect to the standard dilation). For a given dilation, the local approximation $l_h(x)$ of a smooth function l which satisfies $l(0) = 0$ is the first non trivial term in the series expansion of $l(x)$ in homogeneous terms: $l(x) = l_h(x) + 0(l_h(x))$ where

$$\lim_{x \to 0} \frac{0(l_h(x))}{||x||_h^d} = 0$$

if d is the degree of $l_h(x)$. The local approximation of a vector field is defined in a similar way.

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