

# Generalized Fourier Equations and Thermoconvective Instabilities <sup>\*</sup>

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## Abstract

Thermoconvective instabilities are studied when the Fourier law for heat conduction is replaced by the Maxwell-Cattaneo or the Guyer-Krumhansl constitutive equation. For both the gravity-driven and thermocapillary problem, the convective threshold is determined in terms of the different parameters of the system. Depending on the value of these parameters, the instability can be stationary or oscillatory.

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## I. INTRODUCTION

The Fourier law of heat conduction expresses that the heat flux within a medium is proportional to the local temperature gradient in the system. A well known consequence of this law is that heat perturbations propagate with an infinite velocity. It is possible to get rid of this unphysical result by replacing the Fourier law by the so-called Maxwell-Cattaneo law [1, 2]. This equation contains an extra “inertial” term with respect to the Fourier law and writes as

$$\tau \frac{d\mathbf{q}}{dt} + \mathbf{q} = -\lambda_{liq} \nabla T. \quad (1)$$

In this equation,  $\mathbf{q}$  is the heat flux,  $\tau$  is a relaxation time and  $\lambda_{liq}$  is the heat conductivity of the liquid. When Maxwell- Cattaneo relation is combined with the energy balance equation, a finite phase velocity  $v$  is found for the temperature perturbations which is given by

$$v = \left( \frac{\lambda_{liq}}{\rho c \tau} \right)^{\frac{1}{2}}, \quad (2)$$

where  $\rho$  is the mass density of the fluid and  $c$  the heat capacity. A still more general constitutive equation for the heat flux was also proposed by Guyer and Krumhansl [3–5] for heat transport at low temperatures. Heat propagation at low temperatures has received much attention recently and interesting references on this problem can be found for instance in [6, 7].

The analysis of Guyer and Krumhansl is based on a microscopic approach and results in the following “Guyer-Krumhansl” constitutive equation for the heat flux:

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{1}{3} c c_s^2 \nabla T + \frac{1}{\tau_R} \mathbf{q} = \tau_N \frac{c_s^2}{5} \left( \nabla^2 \mathbf{q} + 2 \nabla \nabla \cdot \mathbf{q} \right). \quad (3)$$

In this equation,  $c_s$  is the mean speed of the phonons,  $\tau_R$  a relaxation time for the momentum nonconserving resistive processes and  $\tau_N$  the relaxation time associated with the momentum preserving processes. Equation (3) can be rewritten under the more convenient form

$$\tau \frac{\partial \mathbf{q}}{\partial t} + \mathbf{q} = -\lambda_{liq} \nabla T + l \left( \nabla^2 \mathbf{q} + 2 \nabla \nabla \cdot \mathbf{q} \right) \quad (4)$$

with  $\tau = \tau_R$ . The heat conductivity  $\lambda_{liq}$  is defined by  $\lambda_{liq} = \frac{1}{3} \tau_R c c_s^2$  and  $l = \frac{1}{5} \tau_N \tau_R c_s^2$ . It is then clear that this constitutive equation reduces to the Maxwell- Cattaneo law when  $l$  tends to zero and to the Fourier law if both  $l$  and  $\tau$  go to zero.

From a physical point of view, it is well known that the Fourier law appears naturally within the framework of the classical theory of thermodynamics [8]. On the other hand, it was shown that the “generalized Fourier equations” (1) and more particularly (4) are appropriately described in the context of EIT (Extended Irreversible Thermodynamics) [9, 10]. EIT received a great attention in the Mexican Scientific community thanks to the impetus and the contributions of L. Garcia-Colin and collaborators (see for instance [11–13]).

When the constitutive equations (1) or (4) are used to describe heat transfer in a moving fluid as in the present work, it is important to recall that objective time derivatives [9] must be introduced instead of the partial time derivatives. In this work, we introduce the so-called Jaumann, or corotational, derivative which is defined by

$$\frac{d_J \mathbf{q}}{dt} = \frac{d\mathbf{q}}{dt} - \Omega \cdot \mathbf{q}, \quad (5)$$

where  $d/dt$  is the material time derivative and  $\Omega$  is the skew-symmetric rotation tensor, whose components are given in terms of the components  $v_k$  of the velocity field by

$$\Omega_{ij} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right). \quad (6)$$

Our objective in this paper is to examine the influence of using generalized Fourier equations instead of Fourier’s law in the analysis of thermoconvective instabilities. Let us recall that these instabilities take place in horizontal thin liquid layers which are heated from below. When the temperature difference between the bottom and the top of the fluid is rather small, heat is transported vertically across the layer by conduction only. On the contrary, when the temperature difference across the system exceeds a critical value, the conductive solution becomes unstable and convection sets in in the fluid. Beyond the critical threshold, one observes the occurrence of nice regular patterns taking generally the form of rolls or hexagonal convective cells. Two different physical mechanisms can be responsible for this instability. In “Rayleigh- Bénard” convection, the motor of the motion is gravity while thermocapillarity is the origin of the instability in the “Bénard-Marangoni” problem. In this latter case the upper surface of the liquid is free, with a temperature dependent surface tension. When both mechanisms cooperate to give rise to instability, the situation is usually referred to as the “Rayleigh-Bénard- Marangoni” problem [14, 15].

Generalized Fourier equations have already been used in the past to study thermoconvective instabilities. For example, in [16], the pure gravity- driven instability is investigated

with the Maxwell-Cattaneo constitutive equation (1) for the heat flux, while the thermo-capillary instability is analysed in [17] with the same constitutive equation. However in this latter work, the boundary condition used for the heat flux at the top surface is questionable. In a more recent paper [18], an interesting analysis of the Rayleigh-Bénard problem is developed by making use of the Guyer-Krumhansl model (4) .

In the present work, the general Rayleigh-Bénard-Marangoni problem is investigated when Fourier's law is replaced by the Maxwell-Cattaneo and the Guyer-Krumhansl models. The mathematical model is presented in Section II and the general equations are formulated. The results of the linear stability analysis are presented in Section III and the influence of the different physical parameters is carefully examined for both the stationary and oscillatory instabilities. Conclusions are drawn in the last section.

## II. MATHEMATICAL MODEL

Let us consider a liquid layer, heated from below, whose horizontal dimensions are much larger than its thickness. Under these conditions, the system can be considered as infinite in the horizontal directions. As long as the temperature difference between the bottom and the top is sufficiently small, heat transport is purely conductive and the velocity of the fluid vanishes. The temperature is then a linear function of the vertical coordinate  $z$  and the heat flux vector is vertical and constant. The general balance equations which govern the evolution of the perturbations with respect to this basic conductive solution are well known (see for instance [14] or [15]) and will be recalled only very briefly. When the Boussinesq hypotheses are taken for granted, the infinitesimal perturbations with respect to the basic state obey the following set of linear partial differential equations:

$$\nabla \cdot \mathbf{u} = 0, \tag{7}$$

$$Pr^{-1} \frac{\partial \mathbf{u}}{\partial t} = -\nabla p + \nabla^2 \mathbf{u} + Ra T \mathbf{e}_z, \tag{8}$$

$$\frac{\partial T}{\partial t} - w = -\nabla \cdot \mathbf{q}. \tag{9}$$

We have chosen the vertical  $z$ -axis in the direction opposite to gravity. The equations are written in a non-dimensional form with distances scaled by  $d$ , the thickness of the liquid layer. The time scale is given by  $d^2/\kappa$ , with  $\kappa$  the heat diffusivity of the liquid. The velocity is scaled by  $\kappa/d$  and the temperature scale is chosen as  $\beta d$ , where  $\beta$  is the vertical

temperature gradient which would exist in a purely conductive state. The heat flux is scaled with  $\kappa\beta$ . Symbols  $\mathbf{u} = (u, v, w)$ ,  $p$ ,  $\mathbf{q}$  and  $T$  represent the non-dimensional perturbations of the velocity, pressure, heat flux and temperature fields respectively. The Rayleigh and Prandtl numbers  $Ra$  and  $Pr$  are defined as

$$Ra = \frac{g\alpha_{liq}\beta d^4}{\nu\kappa}, \quad (10)$$

$$Pr = \frac{\nu}{\kappa}, \quad (11)$$

where  $\nu$  and  $\alpha_{liq}$  are the liquid kinematic viscosity and coefficient of thermal expansion. The non-dimensional linearized constitutive equation for the perturbations  $\mathbf{q}$  is easily deduced from the form of the basic conductive solution and from eqs.(4) and (5):

$$2Ca \left( \frac{\partial \mathbf{q}}{\partial t} - \frac{1}{2} \left( \frac{\partial \mathbf{v}}{\partial z} - \nabla w \right) \right) + \mathbf{q} = -\nabla T + L \left( \nabla^2 \mathbf{q} + 2\nabla \nabla \cdot \mathbf{q} \right). \quad (12)$$

In this equation  $Ca$  and  $L$  are the so-called Cattaneo and Guyer-Krumhansl non-dimensional numbers, defined by

$$Ca = \frac{\kappa\tau}{2d^2}, \quad (13)$$

$$L = \frac{l}{d^2}. \quad (14)$$

Equations (7,8,9,12) are then modified in a standard way to eliminate the pressure and to obtain 3 scalar equations for the vertical component of the velocity field  $w$ , the temperature perturbations  $\theta$  and the vertical component  $q_z$  of the heat flux perturbations. These equations are given by

$$Pr^{-1} \frac{\partial \nabla^2 w}{\partial t} = Ra \nabla_h^2 \theta + \nabla^4 w, \quad (15)$$

$$\begin{aligned} & \nabla^2 \theta + 3L \frac{\partial \nabla^2 \theta}{\partial t} - \frac{\partial \theta}{\partial t} - 2Ca \frac{\partial^2 \theta}{\partial t^2} \\ & = -Ca \nabla^2 w - w + 3L \nabla^2 w - 2Ca \frac{\partial w}{\partial t}, \end{aligned} \quad (16)$$

$$\begin{aligned} & 2Ca \frac{\partial q_z}{\partial t} = -q_z - \frac{\partial \theta}{\partial z} \\ & + L \left( \nabla^2 q_z + 2 \frac{\partial w}{\partial z} - 2 \frac{\partial^2 \theta}{\partial z \partial t} \right), \end{aligned} \quad (17)$$

where  $\nabla_h = \mathbf{e}_x(\partial/\partial x) + \mathbf{e}_y(\partial/\partial y)$  is the horizontal nabla operator. Following the normal mode technique, one seeks solutions of the form

$$w = W(Z) \exp i(k_x x + k_y y) \exp \sigma t, \quad (18)$$

$$\theta = \Theta(Z) \exp i(k_x x + k_y y) \exp \sigma t, \quad (19)$$

$$q_z = Q(Z) \exp i(k_x x + k_y y) \exp \sigma t, \quad (20)$$

where  $\sigma$  is the growth rate of the perturbations while  $(k_x, k_y)$  is their horizontal wave vector. When these relations are introduced in equations (15-17), the following differential equations for the amplitudes  $W$ ,  $\Theta$  and  $Q$  are obtained:

$$\sigma (D^2 - k^2) W = -Ra Pr k^2 \Theta + Pr (D^2 - k^2)^2 W, \quad (21)$$

$$\begin{aligned} (1 + 2Ca\sigma) \sigma \Theta &= (1 + 2Ca\sigma) W \\ &+ (1 + 3L\sigma) (D^2 - k^2) \Theta \\ &- (3L - Ca) (D^2 - k^2) W, \end{aligned} \quad (22)$$

$$\begin{aligned} -Q - D\Theta + L (D^2 - k^2) Q + 2L DW &= \\ &2\sigma (Ca Q + L D\Theta), \end{aligned} \quad (23)$$

where  $k$  is the modulus of the horizontal wave vector and  $D = d/dz$  denotes the derivative with respect to the vertical coordinate  $z$ .

Let us now describe the boundary conditions needed to solve these equations. The bottom of the fluid layer is assumed in contact with a rigid plate so that the velocity vanishes at  $z = 0$ . Moreover, the temperature is fixed. In terms of  $W$  and  $\Theta$ , the corresponding conditions write

$$W = DW = \Theta = 0 \text{ at } z = 0. \quad (24)$$

The upper free surface of the liquid is assumed to remain undeformed and a balance is assumed between the viscous stress and the Marangoni stress due to the variations of surface tension with temperature. The mathematical expression for these conditions is

$$W = D^2 W + Ma k^2 \Theta = 0 \text{ at } z = 1, \quad (25)$$

where the Marangoni number  $Ma$  is defined as

$$Ma = \frac{\gamma \beta d^2}{\mu \kappa}. \quad (26)$$

In this definition,  $\mu$  is the dynamic viscosity of the liquid and  $\gamma$  is the sign-changed derivative of the surface tension with respect to the temperature.

The thermal boundary condition at the free surface is a bit more delicate to derive when generalized Fourier equations are used. It is assumed first that the temperature and vertical component of the heat flux are continuous across the surface. One also takes for granted Newton's law of cooling, which expresses that the vertical flux in the gas is proportional to the temperature difference between the gas at the interface and the gas far from this interface. In dimensional variables, this condition writes as:

$$q_z^{gas}(z=0) = h(T^{gas}(z=0) - T_\infty), \quad (27)$$

where  $h$  is the heat transfer coefficient. The upper index "gas" means that the corresponding quantities are considered in the gas and  $T_\infty$  is the temperature of the gas far from the interface with the liquid. When the continuity of  $q_z$  and  $T$  across the interface is used with the normal mode decomposition, eq. (27) reads, in non-dimensional form:

$$Q = Bi \Theta, \quad (28)$$

with  $Bi = h d / \kappa$  the Biot number.

To proceed further with this boundary condition, it is necessary to consider separately the Cattaneo equation (1) and the Guyer-Krumhansl model (4). In terms of  $Q$  and  $\Theta$ , the Cattaneo constitutive equation is deduced from (23) by taking  $L = 0$ . This results in:

$$-Q - D\Theta = 2\sigma Ca Q. \quad (29)$$

When this relation is combined with (28),  $Q$  disappears from the boundary condition which eventually takes the form:

$$D\Theta + Bi \Theta = -2\sigma Ca Bi \Theta \text{ at } z = 1. \quad (30)$$

It is important to notice that this relation is not exactly the same as the boundary condition used by Lebon and Cloot in [17]. The term proportional to  $\sigma$  is indeed missing in the latter

work and it is thus important to revisit here the analysis by using the more correct relation (30).

Since  $Q$  is absent not only in the boundary conditions, but also in the field equation for  $W$  and  $\Theta$ , the stability problem for the Cattaneo model reduces to the eigenvalue problem defined by equations (21)- (22) (with  $L = 0$ ) and the boundary conditions (24)- (25),(30).

When the more complicated Guyer-Krumhansl model is used, it is not easy to deduce exact boundary conditions as in the case of the Cattaneo relation. Eq. (23) is indeed of second order with respect to  $Q$  and two additional boundary conditions would be needed for this unknown. Since no experimental or theoretical informations about these additional boundary conditions are available, we assume in a heuristic way that the traditional Fourier law remains valid at the fluid boundary  $z = 1$ , even if the Guyer-Krumhansl model is used in the bulk of the system. As a consequence of this hypothesis, the heat flux amplitude  $Q$  in eq. (28) can be directly replaced by  $-D\Theta$ . The boundary condition (28) can thus be rewritten as

$$D\Theta + Bi\Theta = 0 \text{ at } z = 1. \quad (31)$$

Within the simplifying assumption of a Fourier law valid at the upper surface, the unknown  $Q$  disappears from the boundary conditions as in the Cattaneo model. The complete stability eigenvalue problem corresponding to the Guyer- Krumhansl constitutive equation is then formed by equations (21)- (22), coupled with the boundary conditions (24)- (25),(31).

### III. RESULTS

For both the Cattaneo and the Guyer-Krumhansl constitutive equations, we use a Chebyshev spectral-tau method to solve the stability problem. To implement this technique [19, 20], the unknown amplitudes  $W$  and  $\Theta$  are expanded in series of Chebyshev polynomials and the resulting expressions are introduced in the bulk equations and boundary conditions. These relations are then projected on the same Chebyshev polynomials in order to transform the differential equations in an algebraic eigenvalue problem, which is solved by using standard software.

The critical temperature difference above which convection starts depends on several parameters. First of all, it is important to note that both the Rayleigh and Marangoni numbers are proportional to the temperature difference  $\Delta T = \beta d$ . In fact, it is often



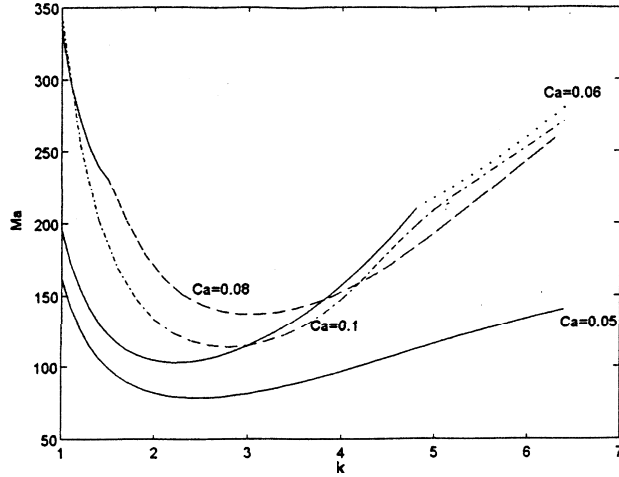


FIG. 1: Marginal stability curves for different values of the Cattaneo number  $Ca$  and for  $L = 0$ ,  $Ra = 0$ ,  $Bi = 1$  and  $Pr = 1$

interesting to replace these two non-dimensional groups by two other non-dimensional numbers chosen in such a way that only one of them depends on  $\Delta T$ . In this work we introduce two dimensionless quantities  $\alpha$  and  $\lambda$  defined by:

$$(1 - \alpha) \frac{Ra}{Ra_0} = \alpha \frac{Ma}{Ma_0}, \quad \lambda = \frac{Ra}{Ra_0} + \frac{Ma}{Ma_0}. \quad (32)$$

The classical  $Ra$  and  $Ma$  numbers are then related to these numbers through

$$Ra = Ra_0 \alpha \lambda, \quad Ma = Ma_0 (1 - \alpha) \lambda. \quad (33)$$

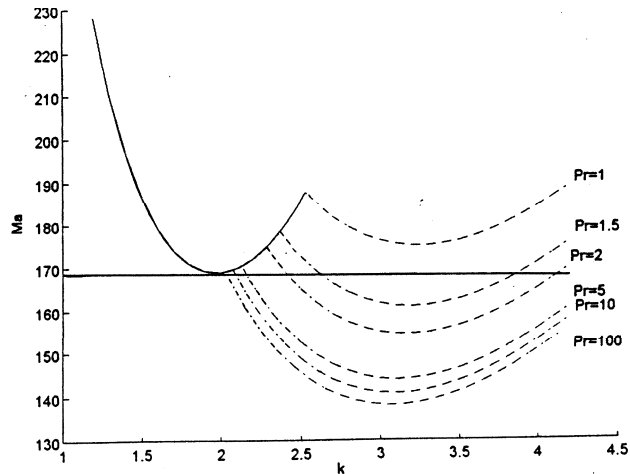


FIG. 2: Marginal stability curves for different values of the Prandtl number  $Pr$  and for  $Ca = 0.07$ ,  $L = 0.02$ ,  $Ra = 0$ , and  $Bi = 1$

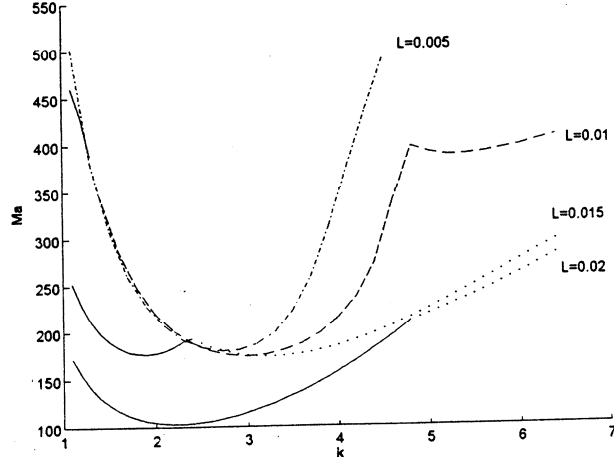


FIG. 3: Marginal stability curves for different values of the Guyer-Krumhansl number  $L$  and for  $Ca = 0.06$ ,  $\alpha = 0.1$ ,  $Bi = 1$  and  $Pr = 1$

In these relations  $Ra_0$  and  $Ma_0$  are two arbitrary constants which are fixed in the following to 669 and 79.6 respectively. It is then easily seen that only  $\lambda$  depends on the temperature difference across the liquid layer while  $\alpha$ , which can be expressed as  $(1 + Ra_0 Ma_0^{-1} \gamma (\alpha_{liq} g d^2 \rho)^{-1})^{-1}$ , is the only parameter depending on  $d$ . In the analysis of the results, we shall determine the convective threshold for a fixed value of this parameter  $\alpha$ . For a given liquid, this procedure amounts to determine the stability threshold as a function of the thickness of the layer. Let us also mention that for  $\alpha = 0$ , the instability is purely

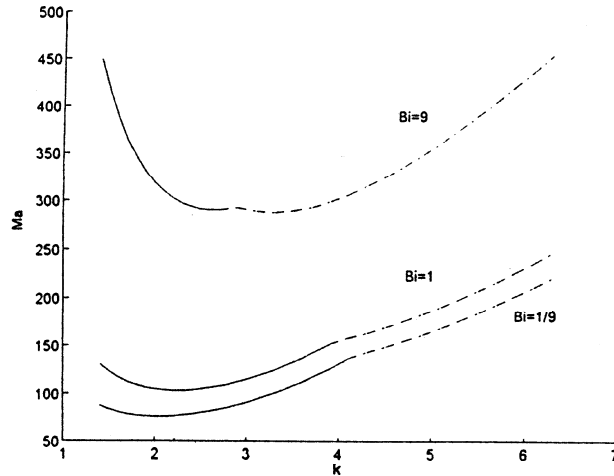


FIG. 4: Marginal stability curves for different values of the Biot number  $Bi$  and for  $Ca = 0.06$ ,  $L = 0.02$ ,  $\alpha = 0.1$ , and  $Pr = 5$

thermocapillary ( $Ra = 0$ ) while it is purely gravitational ( $Ma = 0$ ) when  $\alpha = 1$ .

Besides this “thickness parameter”  $\alpha$ , the problem depends also on the Biot, Prandtl, Cattaneo and Guyer-Krumhansl numbers. The parameter space is thus rather large and only the most significant results will be reported. In Fig. 1, the Bénard-Marangoni instability ( $\alpha = 0$ ) is considered for the Cattaneo constitutive equation ( $L = 0$ ). The Biot and Prandtl numbers are fixed and the marginal stability curves are represented for different values of the Cattaneo number  $Ca$ . In all the figures, the solid curves correspond to stationary instabilities ( $\sigma = 0$ ) while oscillatory thresholds ( $\text{Re}(\sigma) = 0$ ,  $\text{Im}(\sigma) \neq 0$ ) are represented with dashed lines. It is clear from Fig. 1 that oscillatory instabilities are favoured when the Cattaneo number is progressively increased. For large values of  $Ca$ , the stationary curve even disappears completely.

The influence of the variations of the Prandtl number are examined in Fig. 2. In this figure, the Bénard-Marangoni instability ( $\alpha = 0$ ) is considered again and the Guyer-Krumhansl model is used ( $Ca = 0.07$ ,  $L = 0.02$ ). As expected the stationary instability does not depend on the value of  $Pr$  but the dashed lines move downwards as  $Pr$  is progressively increased. For small Prandtl numbers, the instability is thus stationary while oscillations are expected for large  $Pr$  only. For some “critical” value between 1 and 1.5 of the Prandtl number, both the stationary and oscillatory modes are simultaneously unstable. In this case, a non-linear analysis of the interactions between the two modes would be interesting in order to study the behaviour of the system.

The influence of the Guyer-Krumhansl number is reported on Fig. 3, in which the coupled Rayleigh-Bénard-Marangoni problem is considered ( $\alpha = 0.1$ ) for different values of parameter  $L$ . An interesting point is that the oscillatory instabilities tend to disappear when  $L$  is increased. We can also note that the stable curves are shifted downwards when  $L$  becomes larger, which means that the system becomes more unstable.

Eventually, we have analysed the effect of changing the Biot number. The general behaviour is depicted in Fig. 4. As expected, the decrease of the Biot number makes the system more unstable since the temperature perturbations arriving at the upper surface are more easily “reflected” in the bulk. It is also very important to stress that oscillatory instabilities are favoured by large values of the Biot number.

## IV. CONCLUSION

In this paper, we have analysed how thermoconvective instabilities in fluid layers heated from below are modified when heat transfer is modelled with generalized Fourier laws, namely Cattaneo and Guyer-Krumhansl equations. Both gravity-driven and thermocapillary instabilities are considered.

An important conclusion of our analysis is that oscillatory instabilities are predicted when the Cattaneo number (i.e. the relaxation time) becomes large enough. The origin of these oscillations is the inertial term in the constitutive equations which delays the response of the heat flux to the variations of the temperature field. We have also shown that the oscillatory solutions are more and more favoured when the Prandtl or the Biot number at the top surface are increased. On the contrary, when the Guyer-Krumhansl number  $L$  is increased, the instability becomes stationary again.

It should finally be stressed that in the case of the Guyer-Krumhansl model, only an approximate thermal boundary condition at the top surface was used because no other precise condition exists to our knowledge. Of course, it would be interesting to have a definite expression for this condition and to introduce it in the stability analysis.

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