A NON-LOCAL MLT TREATMENT FITTING 3D SIMULATIONS

Marc-Antoine Dupret*[1], M.-J. Goupil[1], R. Samadi[1], A. Grigahcène[2], and M. Gabriel[3]

[1]*Observatoire de Paris, LESIA, CNRS UMR 8109, 92195 Meudon, France*
[2]*CRAAG - Algiers Observatory BP 63 Bouzareah 16340, Algiers, Algeria*
[3]*Institut d'Astrophysique et de Géophysique, Université de Liège, Belgium*

ABSTRACT

It is known that the mixing-length approach gives a very crude description of convection. First it is local and second it neglects the spectral nature of turbulence. However, it is still widely used in stellar evolutionary codes. Moreover, perturbative theories of the MLT can be derived, allowing a non-adiabatic modelling of the convection-oscillations interaction. We propose here a generalization of the mixing-length theory to the non-local case, introducing 2 non-local parameters and 2 free functions associated with the closure of the problem. The description of the convective envelope (including the overshooting region) as predicted by 3D hydrodynamic simulations (horizontal and time averages) can be reproduced with our treatment by adjusting these free parameters and functions. A perturbative theory can be derived with our new treatment, as in the MLT, allowing the theoretical determination of the modes damping rates for structure models with 3D description of the convective zone top.

Key words: Convection; Asteroseismology.

1. INTRODUCTION

This paper deals with the derivation of convection models that can be implemented both in stellar evolution codes for the stationary case, and in non-adiabatic pulsation codes for the time-dependent case. At present, only analytical approaches can be followed in this context, most of the time following the mixing-length phenomenology, or more sophisticated semi-analytical approaches (e.g. Canuto et al. 1996). But it is known that the Mixing-Length Theory (MLT) gives a very crude description of the non-adiabatic part of convective envelopes. To describe correctly these layers, 3D hydrodynamic simulations are required (e.g. Stein & Nordlund). Taking appropriate horizontal and time averages of these 3D results, we get a 1D description of the convective envelope. Our goal is to reproduce this 1D stratification with simple semi-analytical models. We wish also that our models could be "easily" perturbed, allowing a linear study of the interaction between convection and oscillations. Such formalism would have two big advantages: first it could be easy to implement in stellar evolution codes; and second its perturbation could allow the determination of the damping rate of oscillation modes for more realistic models.

We propose here a formalism reaching these two goals. In the first part of this paper, we recall the main lines of the old MLT model: Sect. 2 summarizes the phenomenological description of convection proposed by Unno (1967), and Sect. 3 summarizes some important aspects of a perturbative theory following the Unno approach, as derived by Gabriel (1996) and Grigahcène et al. (2005). In the second part, we present our new models. In Sect. 4 we present our generalization of the Unno approach allowing to reproduce the stratification of 3D simulations. The perturbation of this formalism is considered in Sect. 5. It can be implemented in non-adiabatic pulsation codes, allowing the determination of the damping rate of the oscillation modes for models including the more realistic stratification coming from 3D hydrodynamic simulations. The results obtained (damping rates, ...) are presented in Dupret et al. (these proceedings, paper III).

2. A MIXING-LENGTH THEORY

Unno (1967) proposed a new phenomenological description of convection, compared to the classical MLT of Böm-Vitense (1958), but leading to the same results in the stationary case. As usual in turbulence, equations and variables are splitted in average values (horizontal average on a scale smaller than the horizontal wavelength of the considered acoustic mode) and convective fluctuations: the corresponding notations are $y = \overline{y} + \Delta y$ for scalars and $\vec{v} = \vec{u} + \vec{V}$ for the velocity. In the Unno formalism, the continuity, movement and energy equations describing the convective fluctuations are respectively:

$$\nabla \cdot \vec{V} = 0, \tag{1}$$

$$\overline{\rho}\,\frac{\mathrm{d}\vec{V}}{\mathrm{d}t} = \frac{\Delta\rho}{\overline{\rho}}\nabla\overline{p} - \nabla\Delta p - \rho\vec{V}\cdot\nabla\vec{u} - \Lambda\frac{\overline{\rho}\vec{V}}{\tau_{\rm c}}, \tag{2}$$

*Chargé de Recherche de 1ère classe at the CNRS

$$\frac{\Delta\left(\rho T\right)}{\overline{\rho T}}\frac{\mathrm{d}\overline{s}}{\mathrm{d}t}+\frac{\mathrm{d}\Delta s}{\mathrm{d}t}+\vec{V}\cdot\nabla\overline{s}=-\frac{\omega_{\rm R}\tau_{\rm c}+1}{\tau_{\rm c}}\Delta s, \quad (3)$$

where $\omega_{\rm R}$ is the characteristic frequency of radiative energy lost by turbulent eddies, $\tau_{\rm c}$ is the characteristic life time of the convective elements, $\overline{\rho\vec{V}\vec{V}}=\overline{p}_{\rm t}\,\mathbb{I}-\overline{\beta}_{\rm t}$ is the Reynolds stress tensor, $\rho\epsilon_2=(\beta_{\rm g}+\beta_{\rm R})\otimes\nabla\vec{V}$ is the rate of dissipation of turbulent kinetic energy into heat per unit volume, $\otimes$ stands for the tensorial product, $\beta_{\rm g}$ and $\beta_{\rm R}$ are the non-diagonal components of the gas and radiation stress tensors.

To obtain these equations, crude approximations have been made for the closure of the movement and energy fluctuation equations:

$$\begin{aligned}\Lambda\frac{\overline{\rho}\vec{V}}{\tau_{\rm c}} &= \frac{\Delta\rho}{\overline{\rho}}\nabla\cdot(\overline{\beta}_{\rm g}+\overline{\beta}_{\rm R}+\overline{\beta}_{\rm t})\\ &\quad -\nabla\cdot(\Delta\beta_{\rm g}+\Delta\beta_{\rm R}+\Delta\beta_{\rm t}), \quad (4)\end{aligned}$$

$$\begin{aligned}\overline{\rho T}\frac{\Delta s}{\tau_{\rm c}} &= -\overline{\rho T}\vec{V}\cdot\nabla\overline{s}-\rho\epsilon_2+\overline{\rho\epsilon_2}\\ &\quad +(\rho T\nabla s)\cdot\vec{V}-\overline{(\rho T\nabla s)\cdot\vec{V}}, \quad (5)\end{aligned}$$

$$\nabla\cdot\Delta\vec{F}_{\rm R} = -\omega_{\rm R}\Delta s\overline{\rho T}, \quad (6)$$

$$l = \alpha H_{\rm p}=\alpha|{\rm d}r/{\rm d}\ln P|=|\vec{V}|\tau_{\rm c}. \quad (7)$$

Eq. (7) is the usual closure equation of the MLT. Stationary solutions of the above equations, assuming constant coefficients and $\Lambda=8/3$, lead to the classical solutions of the MLT:

$$\Gamma(\Gamma+1) = \mathcal{A}\left(\nabla-\nabla_{\rm ad}\right), \quad (8)$$

$$\frac{9}{4}\,\Gamma^3+\Gamma^2+\Gamma = \mathcal{A}\left(\nabla_{\rm rad}-\nabla_{\rm ad}\right), \quad (9)$$

$$F_C=\frac{\alpha^2 c_p\rho T}{4}\sqrt{\frac{P_T P}{2P_\rho\rho}}\left[\frac{\Gamma(\nabla-\nabla_{\rm ad})}{\Gamma+1}\right]^{3/2}, \quad (10)$$

$$P_{\rm turb}=\frac{\alpha^2}{8}\,\frac{P_T P}{2P_\rho}\,\frac{\Gamma}{\Gamma+1}(\nabla-\nabla_{\rm ad}), \quad (11)$$

where $\mathcal{A}=P_T P/(2P_\rho\rho)[\kappa c_p\rho^3 g l^2/(12acT^3P)]^2$ and $\Gamma=(\omega_{\rm R}\tau_{\rm c})^{-1}$.

3. A MIXING-LENGTH PERTURBATIVE THEORY

A perturbation of the above formalism was proposed by Gabriel (1996) and improved by Grigahcène et al. (2005), allowing the study of the coherent interaction between convection and oscillations. The goal here is to obtain a set of differential equations relating the perturbation of correlated quantities such as the convective flux and the turbulent pressure to the usual eigenfunctions. These equations could then be implemented in a non-adiabatic pulsation code.

In a very short summary, the procedure to obtain this perturbative theory is the following. We begin by perturbing Eqs. (1)-(3). The closure of the problem is crude in the MLT approach and many complex physical processes, including the whole cascade of energy are extremely simplified compared to reality. Therefore, it is clear that much uncertainty is associated to the perturbation of the closure terms of these equations (Eqs. (4)-(7)). Because of these uncertainties and problems of short wavelength oscillations of the eigenfunctions, Grigahcène et al. (2005) proposed to introduce a free complex parameter β in the perturbation of the thermal closure equations:

$$\delta\left(\frac{\Delta s}{\tau_{\rm c}}\right)=\frac{\Delta s}{\tau_{\rm c}}\left((1+\beta\sigma\tau_{\rm c})\frac{\delta\Delta s}{\Delta s}-\frac{\delta\tau_{\rm c}}{\tau_{\rm c}}\right). \quad (12)$$

With this free parameter β, phase lags are allowed to occur between the oscillations and the way the turbulence cascade adapts to them.

We search then for solutions of the perturbed convective fluctuation equations of the form $\delta\left(\Delta X\right)=\delta\left(\Delta X\right)_{\vec{k}}\,e^{i\,\vec{k}\cdot\vec{r}}\,e^{i\sigma\,t}$, assuming constant coefficients. Then we integrate these particular solutions over all values of k_θ and k_ϕ such that $k_\theta^2+k_\phi^2=A\,k_r^2$, assuming A constant and that every direction of the horizontal component of $\vec{k}$ has the same probability. A defines the anisotropy of turbulence: $A=0.5\overline{\rho V_r^2}/\overline{\rho V_\theta^2}$ ($A=1/2$ in the isotropic case). We have to introduce this distribution of $\vec{k}$ values to obtain an expression for the perturbation of the Reynolds tensor which allows the proper separation of the variables in term of spherical harmonics in the equation of motion (Gabriel 1987). Finally, taking appropriate correlations of the solutions gives differential equations for the different perturbed convective quantities (Grigahcène et al. 2005).

The theory presented here is local, but it can easily be generalized to the non-local case as detailed in the next sections and in Dupret et al. (2006).

4. FITTING 3D HYDRODYNAMIC RESULTS

3D hydrodynamic simulations (e.g. Stein & Nordlund 1998) give a much better description of the non-adiabatic part of the convective envelope than the MLT. We propose here an improvement of the MLT approach fitting the mean stratification predicted by 3D hydrodynamic simulations.

The idea is simple and consists in introducing appropriate free functions and parameters in the theory. As most

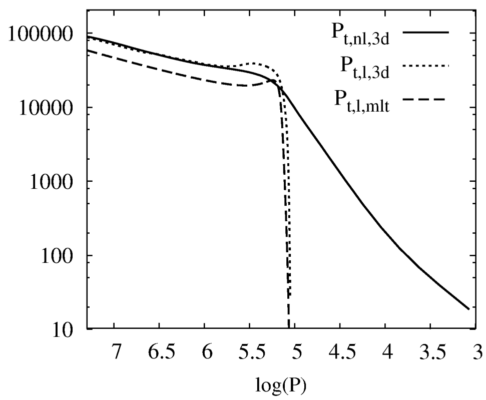

Figure 1. Turbulent pressure obtained as averages of 3D hydrodynamic models by Stein & Nordlund (1998) ($P_{\rm t,nl,3d}$, solid line), local counterpart of these 3D values ($P_{\rm t,l,3d}$, dotted line) and local MLT result ($P_{\rm t,l,mlt}$, dashed line) for the Sun.

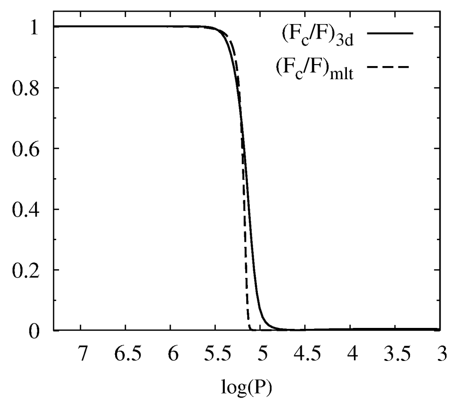

Figure 2. Convective to total flux ratio obtained as averages of 3D hydrodynamic models (solid line) and local MLT result (dashed line) for the Sun.

of the uncertainties are in the closure terms (Eqs. (4)-(7)), we introduce at this level two free functions varying with depths: Ω and α. Ω has the same meaning as in the formalism of Canuto & Mazzitelli (1991); and we can assume as proposed by these authors that it is a function of the convective efficiency Γ. α is the usual ML parameter varying now with depth. More precisely, we multiply the left hand side of Eq. (4) by $\Omega(\Gamma)$ and the left hand side of Eqs. (5) and (6) by $1/\Omega(\Gamma)$. Hence, Eqs. (2) and (3) are replaced by the following equations:

$$\overline{\rho}\,\frac{{\rm d}\vec{V}}{{\rm d}t} = \frac{\Delta\rho}{\overline{\rho}}\nabla\overline{p} - \nabla\Delta p - \rho\vec{V}\cdot\nabla\vec{u} - \Omega(\Gamma)\Lambda\frac{\overline{\rho}\vec{V}}{\tau_{\rm c}}, \quad (13)$$

$$\frac{\Delta\,(\rho T)}{\overline{\rho T}}\frac{{\rm d}\overline{s}}{{\rm d}t} + \frac{{\rm d}\Delta s}{{\rm d}t} + \vec{V}\cdot\nabla\overline{s} = -\frac{\omega_{\rm R}\tau_{\rm c}+1}{\Omega(\Gamma)\tau_{\rm c}}\,\Delta s. \quad (14)$$

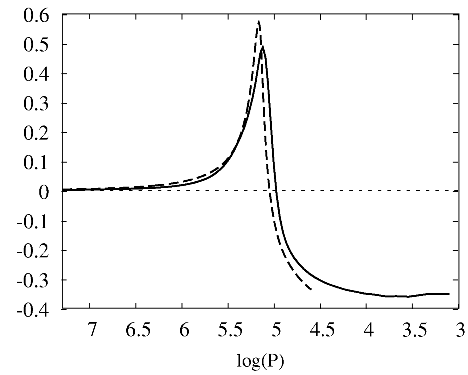

Figure 3. Superadiabatic gradient $\nabla - \nabla_{\rm ad}$ obtained as averages of 3D hydrodynamic models by (solid line) and local MLT result (dashed line) for the Sun.

In the stationary case and assuming constant coefficients, the plane wave solutions of these new equations have a form similar to the old ones. Eq. (8) remains unchanged, giving the same meaning to Γ as in the previous case. Eq. (11) is still verified (with varying α), but Eqs. (9) and (10) are slightly modified and are now:

$$\frac{9}{4}\,\Omega(\Gamma)\,\Gamma^3 + \Gamma^2 + \Gamma = \mathcal{A}\left(\nabla_{\rm rad} - \nabla_{\rm ad}\right), \quad (15)$$

$$F_C = \frac{\Omega(\Gamma)\alpha^2 c_p\rho T}{4}\sqrt{\frac{P_T P}{2P_\rho\rho}}\left[\frac{\Gamma(\nabla-\nabla_{\rm ad})}{\Gamma+1}\right]^{3/2}. \quad (16)$$

By adjusting the free functions Ω and α, we can reproduce exactly the 3D results.

$F_{\rm C}$, $P_{\rm turb}$, $(\nabla - \nabla_{\rm ad})$ and the other thermodynamic quantities are directly deduced from the 3D simulations by taking appropriate horizontal and time averages. Their values are very different from those of MLT models, as can be seen in Figs. 1, 2 and 3 where we compare MLT and 3D results for a Solar model. In the present application, we obtained the 3D results using the hydrodynamic code of Stein & Nordlund (1998) with a resolution of $125 \times 125 \times 82$ over the span of 1 hr.

In the local approach, these quantities are directly injected in Eqs. (8), (11) and (16), which are solved for Ω, α and Γ.

But a non-local approach can also be followed, it better fits the 3D results and is more appropriate for the perturbative theory proposed in next section. The basic ideas of this non-local treatment are due to Spiegel (1963) and can be understood doing an analogy with radiative transfer. The local solutions obtained for example using the MLT are considered as a source term (as the Planck function for radiative transfer with Local Thermodynamic Equilibrium), and then the non-local solutions are obtained by letting diffuse these local solutions. More precisely, the

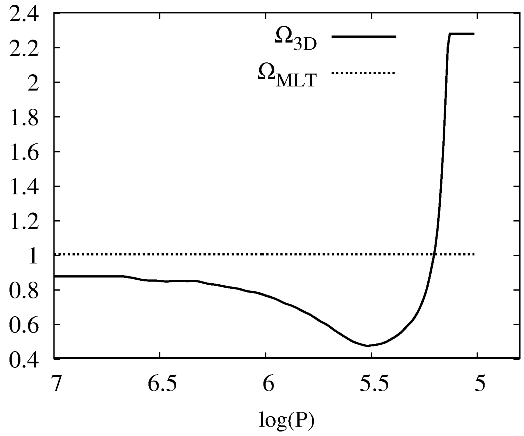

Figure 4. Values of the free function Ω obtained to fit the mean stratification of 3D hydrodynamic models of the Sun (solid line) and $\Omega = 1$ MLT value.

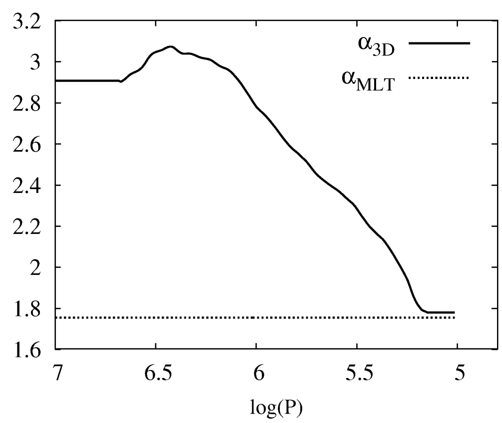

Figure 5. Non-constant values of the Mixing-Length parameter α obtained to fit the mean stratification of 3D hydrodynamic models (solid line) and constant solar calibrated value (dotted line).

non-local solutions are an average of the local ones, according to the following equations:

$$P_{\mathrm{t,nl}}(\zeta_0) = \int_{-\infty}^{+\infty} P_{\mathrm{t,l}} e^{-b|\zeta-\zeta_0|} \mathrm{d}\zeta \,, \quad (17)$$

$$F_{\mathrm{c,nl}}(\zeta_0) = \int_{-\infty}^{+\infty} F_{\mathrm{c,l}} e^{-a|\zeta-\zeta_0|} \mathrm{d}\zeta \,. \quad (18)$$

The free non-local parameters a and b appearing in these equations were introduced by Balmforth (1992) and $\mathrm{d}\zeta = \mathrm{d}r/l$. Taking the second order derivative gives the two very simple differential equations:

$$\mathrm{d}^2 P_{\mathrm{t,nl}}/\mathrm{d}\zeta^2 = b^2 (P_{\mathrm{t,nl}} - P_{\mathrm{t,l}}) \,, \quad (19)$$

$$\mathrm{d}^2 F_{\mathrm{c,nl}}/\mathrm{d}\zeta^2 = a^2 (F_{\mathrm{c,nl}} - F_{\mathrm{c,l}}) \,. \quad (20)$$

In the case of our new theory, F_{C} and P_{turb} are deduced from the 3D stratification and are interpreted as non-local

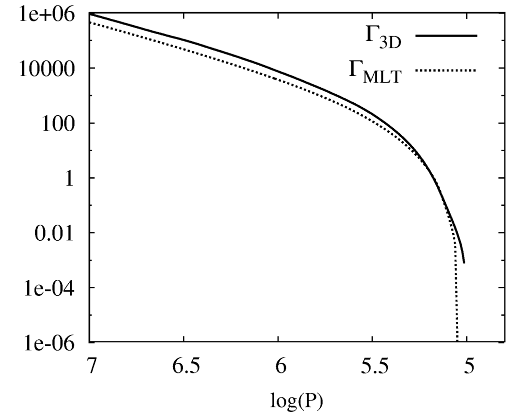

Figure 6. Convective efficiency Γ obtained to fit the mean stratification of 3D hydrodynamic models (solid line) and values obtained with local MLT models (dotted line).

quantities. The non-local parameters a and b are obtained by fitting F_{C} and P_{turb} in the overshooting region (Dupret et al. 2006), which gives: $a = 10$, $b = 3$. The local counterparts of these non-local quantities are deduced from Eqs. (19) and (20). Then these local solutions are injected in Eqs. (8), (11) and (16), which are solved for Ω, α and Γ.

As an example, we illustrate the case of the turbulent pressure in Fig. 1. The solid line is the turbulent pressure as deduced from the 3D hydrodynamic simulations ($P_{\mathrm{t,nl,3D}}$); we can note the exponential decrease in the overshooting region, which justifies our non-local formalism. The dotted line gives its local counterpart ($P_{\mathrm{t,l,3D}}$) according to Eq. (19), and the dashed line gives the local MLT values ($P_{\mathrm{t,MLT}}$).

We illustrate in Figs. 4, 5 and 6 the functions Ω, α and Γ obtained for our Solar model and compare them with the MLT case. We see in Fig. 4 that the values obtained for α are higher than the MLT constant value and decreases towards the surface. Ω is below 1 in most of the convective envelope, and then it increases quickly towards the top. The 3D stratification corresponds to a more efficient convection (higher Γ) than MLT.

In this paper we considered only one model. By considering a grid of 3D hydrodynamic simulations, it would be possible to determine the functions $\Omega(\Gamma)$ and $\alpha(\Gamma)$ and the non-local parameters a and b for different T_{eff} and $\log g$. Then, in a way similar to the semi-analytical treatment of Canuto et al. (1996), these functions could be fitted by polynomials. By this way, our new treatment could be easily implemented in a stellar evolution code.

5. A PERTURBATIVE THEORY FOR MODELS FITTING THE 3D STRATIFICATIONS

It is easy to generalize the perturbative theory presented in Sect. 3 to the case where the structure models fit the stratification of 3D hydrodynamic results (Sect. 4). We follow exactly the same procedure as in Sect. 3, but perturbing Eqs. (13) and (14) instead of Eqs. (2) and (3). Assuming again constant coefficient and searching for solutions in the form of plane waves, we obtain then new expressions for the perturbed convective flux, turbulent pressure, ...

The main uncertainties in this approach appear in the way to perturb Ω and α. The free parameter β introduced in Eq. (12) is somehow related to these uncertainties. At present, we have no theoretical prescriptions for $\delta\Omega$ and $\delta\alpha$ and we neglect these perturbations. But we have no reasons to expect them to be small, and for this reason we have not to be too optimistic when using this new perturbative treatment in a predictive way.

We do not give here the details of the derivations which are very similar to those of Grigahcène et al. (2005). The final results for the variations of the radial components of the local convective velocities and convective flux obtained with our new treatment ($\delta\Omega$ and $\delta\alpha$ neglected) are given in Eqs. (21) and (22). They are similar to the old expressions (Grigahcène et al. 2005, Eqs. (12), (18), (21)).

$$\begin{aligned}\frac{\overline{V_r \delta V_r}}{\overline{V_r^2}} &= \frac{1}{B + ((i\Omega+\beta)\sigma\tau_c + 1)D} \\ &\cdot\left\{ -\frac{\delta c_p}{c_p} - \frac{\delta Q}{Q} - \frac{\delta\rho}{\rho} + \frac{d\delta p}{dp} - \frac{d\xi_r}{dr} \right. \\ &-i\Omega\sigma\tau_c D\,\frac{(Q+1)}{Q}\frac{\delta s}{c_p} + C\left[\frac{d\delta s}{ds} - \frac{d\xi_r}{dr}\right] \\ &-\frac{A}{A+1}\frac{i\sigma\tau_c}{\Omega\Lambda}\left(\frac{d\xi_r}{dr} + \frac{1}{A}\frac{\xi_r}{r} - \frac{\ell(\ell+1)}{2A}\frac{\xi_h}{r}\right) \\ &-\omega_R\tau_c D\left(3\frac{\delta T}{T} - \frac{\delta c_p}{c_p} - \frac{\delta\kappa}{\kappa} - 2\frac{\delta\rho}{\rho}\right) \\ &\left. + ((i\Omega+\beta)\sigma\tau_c + 3\omega_R\tau_c + 2)D\,\frac{\delta l}{l}\right\}, \qquad (21)\end{aligned}$$

$$\begin{aligned}\frac{\delta F_{c,r}}{F_{c,r}} &= \frac{\delta\rho}{\rho} + \frac{\delta T}{T} - i\Omega\sigma\tau_c D\frac{(Q+1)}{Q}\frac{\delta s}{c_p} \\ &+C\left[\frac{d\delta s}{ds} - \frac{d\xi_r}{dr}\right] \\ &-\omega_R\tau_c D\left(3\frac{\delta T}{T} - \frac{\delta c_p}{c_p} - \frac{\delta\kappa}{\kappa} - 2\frac{\delta\rho}{\rho}\right) \\ &+((i\Omega+\beta)\sigma\tau_c + 2\omega_R\tau_c + 1)D\frac{\overline{\delta V_r}}{V_r} \\ &+(2\omega_R\tau_c + 1)D\,\frac{\delta l}{l}, \qquad (22)\end{aligned}$$

where

$$\begin{aligned}B &= \frac{i\sigma\tau_c + \Omega\Lambda}{\Omega\Lambda}, \\ C &= \frac{\omega_R\tau_c + 1}{(i\Omega+\beta)\sigma\tau_c + \omega_R\tau_c + 1}, \\ D &= \frac{1}{(i\Omega+\beta)\sigma\tau_c + \omega_R\tau_c + 1}.\end{aligned}$$

and $Q = \left.\frac{\partial \ln T}{\partial \ln \rho}\right|_p$.

We recall that the turbulent pressure perturbation is:

$$\frac{\delta p_t}{p_t} = \frac{\delta\rho}{\rho} + 2\frac{\overline{V_r \delta V_r}}{\overline{V_r^2}}. \qquad (23)$$

We have implemented these new expressions for the perturbed convective flux and turbulent pressure in our non-radial non-adiabatic pulsation code. The results obtained for the solar case are presented in paper III.

6. CONCLUSIONS

We have presented a new non-local treatment of convection, which is a generalization of the MLT. Thanks to the introduction of free fitting parameters and functions, this treatment can reproduce more realistic descriptions of stellar convective envelopes. For example, it is able to reproduce the mean stratification coming from 3D hydrodynamic models (Stein & Nordlund 1998). Our formalism has two big qualities. First it could easily be implemented in stellar evolution codes. And second it can be perturbed, allowing the study of the coherent interaction between convection and oscillations. By implementing this perturbed treatment in a non-adiabatic pulsation code, we can compute the modes damping rates for models with a more realistic description of the convective envelope; the results obtained are presented in paper III. However, we must admit that the the perturbation of the closure terms appearing in our treatment are still subject to large uncertainties, which limits its predictive capacity.

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