Summary

A general expression for the velocity distribution in a region of the stellar system in terms of the integrals of the "local" orbits is given. The special cases of ellipsoidal and stationary distributions are considered. The normal velocity ellipsoid demanded by the rotation theory of the Galaxy is discussed, and the deviations from this ellipsoid are classed as "secular" and "accidental." The secular variations depend on differences between widely different phases in the "relative" orbits, the accidental on variations in the region between motions of nearly the same phase. Neglecting the accidental variations, which are assumed to cancel out when computing the moments of the velocities, the secular variations will appear as disturbing streams through our region. A development of the stream theory starting from general features of the stellar system is given in adherence to earlier work.

The high-velocity stars, the distribution of the groups of B stars in our neighbourhood and the local disturbances in the differential rotation are briefly discussed and shown to be phenomena in general accordance with the theory.

Stockholm Observatory,
Saltsjöbaden :
1936 *September.*

ON THE DISTRIBUTION OF THE ABSORBING ATOMS IN THE REVERSING LAYERS OF STARS AND THE FORMATION OF BLENDED ABSORPTION LINES.

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Introduction.—Numerous investigations which have been made regarding the effect of spectral type and surface gravity on the ionization of atoms and on the dissociation of molecules have dealt with the total numbers of atoms or molecules down to a certain optical thickness τ_1. In these calculations the temperature is usually considered as a constant; in most of the cases the pressure is also taken as a constant and equal to half of the pressure at the level corresponding to τ_1.

It is also of some interest to compare carefully the distributions of the absorbing atoms of given elements in the reversing layer of a star, as functions of the depth h or of the optical thickness τ. The problem is the following. Let us consider a star characterized by an effective temperature T_e and surface gravity g, and having an electron pressure p_ϵ at the level corresponding to $\tau = 2/3$; let us suppose that all the atoms are perfectly mixed, with determined abundance ratios. What will be the distributions of specified atoms or ions, in given electronic states, as functions of h or τ? How will these distributions vary in stars of different T_e and g?

This problem may be important from several points of view.* As we shall see later, the distributions depend rather sensitively on the actual values of g and T_e; they might thus give rise to intensities of spectral lines which may appear unusual at first sight; it seems to us that an examination of the distributions of the absorbing atoms may be useful for later discussions on spectral classifications or abundance problems. A crude application of the ionization equation, assuming a uniform temperature and a constant pressure (half the pressure at the bottom), may be insufficient in certain special cases. It may thus happen that, owing to the distributions of T and p_ϵ, the absorbing atoms of an element concentrate preferably in a determined region of the reversing layer and give rise to intensities corresponding to the temperature and pressure of that region.†

On the other hand, Swings and Struve in a recent note ‡ have called attention to the abnormal intensities of the OII lines appearing in the wings of H_γ, as well as of the Ca^+ line superposed on a wing of H_ϵ. Their general result shows a marked departure from the application of Eddington's formula, and it seems tempting to attribute the observed effect to some kind of stratification. Owing to the great microphotometric difficulties involved in this problem, it would be important to rediscuss the observations of Swings and Struve; this would bring complementary observational information which would be valuable in connection with the distribution of absorbing atoms. In their paper, Swings and Struve suggest some kind of chemical stratification of the elements. But actually a similar result may be obtained if all the atoms are perfectly mixed and if, owing to the effects of T and p_ϵ, the distributions of the absorbing elements are different.

It is obvious that this question is also of some importance for the molecular problem in late-type stars. This will not be considered in this note, which deals only with the case of atoms. Actually, the examples which will be considered here are those which correspond to the observations, namely:

(*a*) In the early A-type stars: distributions of the H and Ca^+ atoms giving rise respectively to the Balmer and to the H and K lines.

(*b*) In the early B-type stars: distributions of the H and O^+ atoms absorbing respectively the Balmer and the visible OII lines.

It would be interesting to investigate theoretically (following a method similar to that of this paper) the distribution of the excitation and ionization of the different abundant elements in the atmosphere of ζ Aurigæ, for which direct observations are now available. §

* In a recent paper (*Ap. J.*, 83, 202, 1936) R. Wildt has reinvestigated the equilibrium of stellar atmospheres under a temperature gradient; if the temperature gradient maintained by the radiation flux is smaller than the adiabatic gradient of the stellar matter, the different constituents of the atmosphere will be separated by diffusion; this gives thus a chemical stratification, which is quite different from the physical distribution examined here.

† In connection with this, see the recent discussions on the spectral classification of early-type stars: Struve, *Ap. J.*, 78, 73, 1933; Russell, Payne-Gaposchkin and Menzel, *ibid.*, 81, 107, 1935; E. G. Williams, *ibid.*, 83, 305, 1936.

‡ *Ap. J.*, 83, 238, 1936.

§ W. H. Christie and O. C. Wilson, *Ap. J.*, 81, 426, 1935.

1. *General Formulæ.*—Following the notation of Fowler and Milne, let

x_0 be the fraction of any atomic element in the neutral state and in the lowest electronic level ;

x_1 the fraction of singly ionized atoms ;

x_2 the fraction of doubly ionized atoms ;

$(n_r)_s$ the fraction of the atoms of one type which are r times ionized and in their sth state ;

χ_r the ionization potential of the r times ionized atom in its lowest level ;

$(\chi_r)_s$ the ionization potential in the sth level ;

$(\varpi_r)_s$ the statistical weight of that sth level ;

$u_r(T)$ the partition factors.

We have the following relations :—

$$\frac{x_{r+1}}{x_r}=\frac{0\cdot664}{p_\epsilon}\cdot\frac{u_{r+1}(T)}{u_r(T)}\cdot T^{5/2}\cdot e^{-\chi_r/kT}; \tag{1}$$

$$\Sigma x_r=1; \tag{2}$$

$$(n_r)_s=\frac{x_r\,.\,(\varpi_r)_s\,.\,e^{-\frac{\chi_r-(\chi_r)_s}{kT}}}{u_r(T)}. \tag{3}$$

(a) *Hydrogen Atoms.*—For neutral atoms in the sth state we have

$$(n_0)_s=\frac{(\varpi_0)_s\,.\,e^{(\chi_0)_s/kT}}{u_0(T)\,.\,e^{\chi_0/kT}+0\cdot664\,.\,u_1(T)\,.\,\frac{T^{5/2}}{p_\epsilon}}. \tag{4}$$

In this formula we must introduce the values corresponding to the hydrogen atom : *

$\chi_0=13\cdot54$; $(\chi_0)_s=3\cdot385$; $u_0(T)=2$; $u_1(T)=1$; $(\varpi_0)_1=8.$

(b) *Singly Ionized Oxygen Atoms.*—At the temperatures of the early B-type stars there is almost no neutral oxygen left, as it appears from an immediate application of the ionization formula (1).† Consequently, the formula for OII is exactly similar to that for hydrogen :

$$(n_1)_s=\frac{(\varpi_1)_s\,.\,e^{(\chi_1)_s/kT}}{u_1(T)\,.\,e^{\chi_1/kT}+0\cdot664u_2(T)\,.\,\frac{T^{5/2}}{p_\epsilon}}. \tag{5}$$

The specific factors are : ‡

$$\chi_1=35\,;\quad (\chi_1)_s=11\cdot53\,;$$
$$u_1(T)=4+10e^{-3\cdot37/kT}+6e^{-5\cdot04/kT}\,;$$
$$u_2(T)=9\,;$$
$$(\varpi_1)_s=6.$$

* The ionization potentials will be expressed in electron-volts.

† For the range of electron pressures which seem admissible and for $T=20000°$, x_1/x_0 varies from 10^4 to 10^8.

‡ We have taken the same values as were adopted by R. H. Fowler and which concern λλ 4417 and 4415 ; for any other visible line the calculations are very similar.

(c) *Singly Ionized Calcium Atoms.*—For the normal lines of these ionized atoms we must retain three stages; in other words, we must apply formula (1) for $r=0$ and $r=1$, and consider formula (3) for $r=1$ and the ground level. We get in that way:

$$n_1 = \frac{\varpi_1}{u_1(T)+u_0(T)\,.\,\dfrac{p_\epsilon}{0\cdot664}\,.\,T^{-5/2}\,.\,e^{\chi_0/kT}+u_2(T)\,.\,\dfrac{0\cdot664}{p_\epsilon}\,.\,T^{5/2}\,.\,e^{-\chi_1/kT}}\,; \tag{6}$$

with the following values of the factors:

$$\chi_0=6\cdot08\,; \quad \chi_1=11\cdot82\,; \quad u_0(T)=1\,; \quad u_1(T)=2+10e^{-\frac{1\cdot69}{kT}}\,; \quad u_2(T)=1.$$

2. *Variation of T and p_ϵ in the Stellar Atmosphere.*—In the early-type stars which are considered here we may assume that, inside of a reversing layer, the electron pressure p_ϵ is a constant fraction of the gas pressure. We know that the temperature is related to the optical thickness τ by the Milne formula

$$T^4 = \frac{T_e^4}{2}(1+\tfrac{3}{2}\tau). \tag{7}$$

For the distribution of pressures we shall use the following formula:—*

$$p_\epsilon = aT^{19/4}\,.\,\sqrt{1-\left(\frac{T_0}{T}\right)^{19/2}}, \tag{8}$$

a being a constant, and T_0 the temperature at the boundary ($T_0=0\cdot871\,T_e$). All the calculations will be made for three values of a, namely:

a_0, which will correspond to an intermediary star;
$100a_0$, corresponding to a dwarf;
$a_0/100$, corresponding to a giant.

3. *Hydrogen and Ca^+ Atoms in the A0 Stars.*—We shall assume here:

$$T_e = 10000°,$$

and

$$a_0 = 3\cdot10^{-17}.$$

This value of a_0 gives an electron pressure 270 dynes/cm.$^{-2}$ for $\tau=2/3$, which seems quite normal.

Table I gives the results of the calculations for the three values of the constant a. The maximum value of τ which has been considered is $\tau=4$, which, following Milne, defines the greatest depth to which we can see in the photosphere at the centre of the solar disk. Evidently, the most interesting part of Table I concerns the values of τ extending from $\tau=\cdot01$ to $\cdot666$.

Table I brings the following results:—

(a) *Case* $a=a_0$.—The fraction of absorbing H atoms increases steadily with increasing τ, varying from $3\cdot24\,.\,10^{-6}$ at $\tau=\cdot01$ to $25\cdot7\,.\,10^{-6}$ at $\tau=2/3$. On the other hand, the fraction of Ca^+ absorbing atoms increases slowly, reaches a maximum at about $\tau=\cdot05$ or $\cdot10$, and then decreases slowly. This

* S. Chandrasekhar, *M.N.*, **92**, 186, 1932.

TABLE I

Distribution of the Absorbing H^- and Ca^+ Atoms in A0-type Stars

			With $a=a_0$			With $a=100a_0$			With $a=a_0/100$		
τ	T	$\frac{a}{a_0}p_\epsilon$	nBal. in Units 10^{-6}	$n(Ca^+)$ in Units 10^{-2}	$\frac{n\text{Bal.}}{n(Ca^+)}$ in Units 10^{-5}	nBal. in Units 10^{-6}	$n(Ca^+)$ in Units 10^{-1}	$\frac{n\text{Bal.}}{n(Ca^+)}$ in Units 10^{-5}	nBal. in Units 10^{-7}	$n(Ca^+)$ in Units 10^{-3}	$\frac{n\text{Bal.}}{n(Ca^+)}$ in Units 10^{-5}
0·01	8440	25·71	3·24	44·8	0·719	3·46	6·62	0·523	4·34	13·24	3·277
·02	8470	34·77	3·45	47·9	·720	3·65	6·60	·553	5·54	16·52	3·354
·05	8560	57·21	4·05	51·5	·786	4·22	6·56	·643	8·07	22·22	3·632
·1	8710	82·32	5·11	51·5	·992	5·34	6·48	·824	10·30	23·26	4·43
·2	8980	122·97	7·65	48·9	1·56	8·02	6·28	1·27	12·92	20·10	6·43
·5	9670	219·39	18·3	38·1	4·80	20·54	5·92	3·47	15·42	10·26	15·03
·666	10000	269·4	25·7	32·5	7·91	30·53	5·74	5·32	15·62	7·22	21·6
1·0	10575	368·52	42·8	23·8	17·98	58·53	5·46	10·72	15·39	4·12	37·3
1·5	11290	522·48	66·1	7·7	85·8	116·7	5·14	22·7	14·89	2·16	68·9
2	11890	669·84	83·1	5·3	157	196·04	4·88	40·2	14·13	1·34	105
3	12870	985·86	102	2·96	345	411·3	4·44	92·7	13·31	0·68	196
4	13680	1322·64	110	1·88	585	688·4	4·08	169	12·83	0·40	321

means that, on the average and in comparison with the H and K lines, the hydrogen lines will be formed in deeper layers.

(b) *Case* $a=100a_0$ (*Dwarf*).—A similar result is reached. The increase of the hydrogen-absorbing fraction is even faster than in the first case, while the Ca^+ fraction is decreasing continuously and slowly.

(c) *Case* $a=a_0/100$ (*Giant*).—The increase of the hydrogen fraction is much slower than for $a=a_0$. The Ca^+ fraction reaches a maximum at $\tau=0{\cdot}10$; from there it decreases fast, and is at $\tau=2/3$ only a third of its maximum value. In all cases the ratio $\frac{n(\text{Balmer})}{n(Ca^+)}$ is increasing steadily with τ, reaching higher and higher values in the deeper layers of the atmosphere.*

4. *Hydrogen and Ionized Oxygen in Early B-type Stars.*—We shall assume here

$$T_e=20{,}000^\circ \quad \text{and} \quad a_0=2\cdot10^{-9}.$$

This value of a_0 gives $p_\epsilon=50$ dynes/cm.$^{-2}$ at $\tau=2/3$. Table II gives the results of the calculations. It shows that the behaviour of the hydrogen-absorbing atoms is the same for giant, dwarf and intermediary stars, but that it is extremely different for the O^+ atoms. In the giant ($a=a_0/100$),

* As we assume that p_ϵ is proportional to the gas density, it would be an easy matter to convert these fractions n into actual numbers of atoms; this may be useful for a discussion of certain spectra, but not for the comparison between two atoms.

TABLE II

Distribution of the Absorbing H and O^+ Atoms in Early B-type Stars

τ	T	$\frac{a}{a_0}p_e$	With $a=a_0$			With $a=100a_0$			With $a=a_0/100$		
			nBal. in Units 10^{-8}	$n(O^+)$ in Units 10^{-8}	$\frac{n\text{Bal.}}{n(O^+)}$ in Units 10^{-1}	nBal. in Units 10^{-6}	$n(O^+)$ in Units 10^{-7}	$\frac{n\text{Bal.}}{n(O^+)}$ in Units 10	nBal. in Units 10^{-10}	$n(O^+)$ in Units 10^{-9}	$\frac{n\text{Bal.}}{n(O^+)}$ in Units 10^{-1}
0·01	16880	4·617	1·536	8·397	1·829		1·115	1·38		3·283	0·469
·02	16940	6·229	2·041	9·258	2·204	*Idem*	1·17	1·74	*Idem*	4·225	·484
·05	17120	10·263	3·197	11·505	2·78		1·395	2·28		6·216	·514
·1	17420	14·771	4·234	14·72	2·88	$a=a_0$	1·806	2·34	$a=a_0$	7·534	·56
·2	17960	22·044	5·479	21·39	2·56		2·832	1·93		8·388	·65
·5	19340	39·328	6·941	38·65	1·79		7·926	0·87		7·395	·94
·666	20000	48·31	7·326	43·84	1·67		12·27	·59		6·719	1·09
1	21150	66·087	7·818	45·92	1·70		24·4	·32		5·593	1·22
1·5	22580	93·7	8·384	41·82	2·00		49·02	·17		4·524	1·88
2	23780	120·127	8·657	36·34	2·38		77·68	·11		3·776	2·29
3	25740	176·809	9·192	29·85	3·08		126·5	·073		2·961	3·11
4	27360	237·191	9·683	25·15	3·85		154·5	·062		2·532	3·82

$n(O^+)$ remains almost constant from $\tau=·01$ until $\tau=4$; while in the dwarf ($a=100a_0$) it increases from $1·115 . 10^{-7}$ to $154·5 . 10^{-7}$. In the dwarf it appears from the behaviour of $\frac{n(\text{Balmer})}{n(O^+)}$ that, when compared with the Balmer lines, the O^+ lines have a tendency to originate rather from deeper layers * (see the variation of $n(\text{Bal})/n(O^+)$ between $\tau=0·1$ and $\tau=0·666$).

5. *The Formation of Blended Absorption Lines in a Stratified Atmosphere.*—Before proceeding to a detailed theoretical discussion of the effect observed by Swings and Struve, it would be necessary to get more observations and more accurate measures. Nevertheless, owing to the general interest of the problem and its various possible applications, it seems worth while to examine what the effect of the different distributions of the absorbing atoms will be on the intensities of blended lines. It is quite certain that the general result of Swings and Struve indicating a marked departure from a straightforward application of Eddington's formula is essentially correct, and the bearing of a plausible "stratified atmosphere" on the formation of superposed absorption lines is well worth examining.

We consider the simplified model of a plane atmosphere containing the absorbing element 1 in the outer region from $\tau=0$ to $\tau=\tau_0$ (say), and the absorbing element 2 at optical depths greater than τ_0. (τ is the optical depth

* It must be remembered that photometric observations of the type carried on by Swings and Struve are only possible when the Balmer lines have rather broad wings, *i.e.* in stars having a dwarf character.

measured in the continuous spectrum in the immediate background of the line.)

Let

κ_ν be the coefficient of continuous absorption;
l_1 and l_2 the coefficients of scattering or of line absorption due to the elements 1 and 2 respectively; *
η_1 and η_2 the customary ratios l_1/κ_ν and l_2/κ_ν respectively.

Further, let

$$\mathfrak{J}_\nu = \tfrac{1}{2}\int_0^\pi I_\nu \sin\theta d\theta; \qquad F_\nu = 2\int_0^\pi I_\nu \sin\theta\cos\theta d\theta; \qquad K_\nu = \tfrac{1}{2}\int_0^\pi I_\nu \sin\theta\cos^2\theta d\theta.$$

Let B_ν be the Planck function.

The equation of transfer is as usual:

$$\cos\theta\frac{dI_\nu}{\rho dx} = -(\kappa_\nu + l_1 + l_2)I_\nu + (l_1 + l_2)\mathfrak{J}_\nu + \kappa_\nu B_\nu.$$

Multiplying the equation of transfer successively by $\frac{1}{2}\sin\theta d\theta$ and $\frac{1}{2}\sin\theta\cos\theta d\theta$ and integrating from O to π, we get

$$\frac{1}{4}\frac{dF_\nu}{d\tau} = \mathfrak{J}_\nu - B_\nu, \tag{9}$$

and

$$\frac{dK_\nu}{d\tau} = \tfrac{1}{4}(1+\eta_i)F_\nu, \qquad \begin{Bmatrix} i=1 \text{ for } \tau \leqslant \tau_0 \\ i=2 \text{ for } \tau > \tau_0 \end{Bmatrix} \tag{10}$$

where we have introduced the optical depth τ by the equation

$$d\tau = -\kappa_\nu \rho dx.$$

We consider the case where η_1 and η_2 are constants. Equation (10) then leads to

$$\frac{d^2K_\nu}{d\tau^2} = (1+\eta_i)(\mathfrak{J}_\nu - B_\nu);$$

or with the usual approximation $\mathfrak{J}_\nu = 3K_\nu$,

$$\frac{d^2\mathfrak{J}_\nu}{d\tau^2} = 3(1+\eta_i)(\mathfrak{J}_\nu - B_\nu). \tag{11}$$

We shall now consider what Eddington calls the "standard case," where we have with sufficient accuracy

$$B_\nu = B_0(1+\tfrac{3}{2}\tau) = \tfrac{1}{2}F(0)(1+\tfrac{3}{2}\tau). \tag{12}$$

In (12), $F(0)$ is the emergent flux in the continuous spectrum in the immediate background of the lines. Equation of the Milne type (12) is strictly true for the integrated radiation for grey material in radiative equilibrium,

* l_1 and l_2 are, of course, functions of the frequency ν, but we have suppressed the suffixes.

but in certain special cases it will also be true in the separate frequencies.* We shall accordingly consider this case first. A better approximation is considered in § 9.

Combining equations (11) and (12), we have

$$\frac{d^2(J_\nu - B_\nu)}{d\tau^2} = 3(1+\eta_i)(J_\nu - B_\nu). \qquad (13)$$

The required solution of (13) is

$$J_\nu = B_0(1+\tfrac{3}{2}\tau) + \alpha e^{-q_1\tau} + \beta e^{+q_1\tau}, \qquad (\tau < \tau_0), \qquad (14')$$

$$J_\nu = B_0(1+\tfrac{3}{2}\tau) + \gamma e^{-q_2\tau}, \qquad (\tau \geqslant \tau_0), \qquad (14'')$$

where

$$q_1^2 = 3(1+\eta_1)\,; \qquad q_2^2 = 3(1+\eta_2).$$

In (14), α, β and γ are constants of integration.

By differentiation of (14) we find

$$\tfrac{3}{4}(1+\eta_1)F_\nu = \tfrac{3}{2}B_0 + q_1(\beta e^{q_1\tau} - \alpha e^{-q_1\tau}), \qquad (\tau < \tau_0) \qquad (15')$$

$$\tfrac{3}{4}(1+\eta_2)F_\nu = \tfrac{3}{2}B_0 - q_2\gamma e^{-q_2\tau}. \qquad (\tau \geqslant \tau_0) \qquad (15'')$$

The boundary conditions are, that at $\tau = \tau_0$ the J's given by (14′) and (14″) should be the same, and similarly the F's given by (15′) and (15″) should also be the same. Finally, at $\tau = 0$ we use the boundary condition that $F = 2J$. These conditions yield the following equations for the determination of the constants α, β and γ:—

$$\left.\begin{aligned}
&\alpha(1+\eta_1+\tfrac{2}{3}q_1) + \beta(1+\eta_1-\tfrac{2}{3}q_1) + \eta_1 B_0 = 0,\\
&\alpha e^{-q_1\tau_0} + \beta e^{+q_1\tau_0} - \gamma e^{-q_2\tau_0} = 0,\\
&\alpha e^{-q_1\tau_0}(1+\eta_2)q_1 - \beta e^{+q_1\tau_0}(1+\eta_2)q_1 - \gamma e^{-q_2\tau_0}(1+\eta_1)q_2 + \tfrac{3}{2}B_0(\eta_1-\eta_2) = 0.
\end{aligned}\right\} \qquad (16)$$

6. *Formulæ for the Residual Intensity.*—For this problem we need only the two constants α and β to determine the emergent flux at $\tau = 0$. Using the abbreviations

$$\xi_1 = (1+\eta_1) \quad \text{and} \quad \xi_2 = (1+\eta_2),$$

we obtain finally that

$$\tfrac{1}{2}\xi_1 F_\nu(0) = B_0 + q_1 B_0 \frac{2\xi_1(\xi_1-\xi_2) + \tfrac{2}{3}(\xi_1 - 1)[(\xi_2 q_1 + \xi_1 q_2)e^{q_1\tau_0} - (\xi_2 q_1 - \xi_1 q_2)e^{-q_1\tau_0}]}{(\xi_2 q_1 - \xi_1 q_2)(\xi_1 - \tfrac{2}{3}q_1)e^{-q_1\tau_0} + (\xi_2 q_1 + \xi_1 q_2)(\xi_1 + \tfrac{2}{3}q_1)e^{q_1\tau_0}}.$$

The residual intensity is obtained by dividing $F_\nu(0)$ by $2B_0$ (*cf.* equation (12)), and we have

$$r_\nu(\xi_1\,;\ \xi_2) = \frac{1}{\xi_1}\left\{1 + q_1 \frac{2\xi_1(\xi_1-\xi_2) + \tfrac{2}{3}(\xi_1 - 1)[(\xi_2 q_1 + \xi_1 q_2)e^{q_1\tau_0} - (\xi_2 q_1 - \xi_1 q_2)e^{-q_1\tau_0}]}{(\xi_2 q_1 - \xi_1 q_2)(\xi_1 - \tfrac{2}{3}q_1)e^{-q_1\tau_0} + (\xi_2 q_1 + \xi_1 q_2)(\xi_1 + \tfrac{2}{3}q_1)e^{q_1\tau_0}}\right\}. \qquad (17)$$

One easily verifies from (17) that for either of the cases (i) $\tau_0 = 0$, (ii) $\tau_0 \to \infty$ or (iii) $\xi_1 = \xi_2$ it reduces to the "classical formula"

$$r_\nu = \frac{1+\tfrac{2}{3}q}{1+\eta+\tfrac{2}{3}q}, \qquad \begin{pmatrix}\tau_0 = 0\\ \tau_0 \to \infty\\ \xi_1 = \xi_2\end{pmatrix}. \qquad (17')$$

* See A. S. Eddington, *M.N.*, **89**, 620, 1929, § 5 of this paper.

For an absorption line due to element 1 only we have to take $\xi_1 > 1$ and $\xi_2 = 1$, and similarly for an absorption line due to element 2, $\xi_1 = 1$ and $\xi_2 > 1$. But if the *absorption line of element* 1 *appears in the wing of the absorption line due to element* 2, then the residual intensity R_ν in the line due to element 1 will be given by

$$R_\nu(1 \text{ in } 2) = \frac{r_\nu(\xi_1(\nu)\,;\ \xi_2(\nu))}{r_\nu(\xi_1(\nu)\,;\ 1)}, \tag{18}$$

where r_ν is given by (17). Similarly,

$$R_\nu(2 \text{ in } 1) = \frac{r_\nu(\xi_1(\nu)\,;\ \xi_2(\nu))}{r_\nu(1\,;\ \xi_2(\nu))}. \tag{18'}$$

7. *The Coefficients ξ_1 and ξ_2.*

(a) *Determination of ξ_1.*—We consider an absorption line due to element 1 at frequencies at which there is no blending with an absorption line due to element 2. Then $\xi_2 = 1$ (or $\eta_2 = 0$). Further, $q_2 = \sqrt{3}$. By (17), then,

$$r_\nu(\xi_1\,;\ 1) = \frac{1}{\xi_1}\left\{1 + 2q_1(\xi_1 - 1)\frac{\xi_1 + \frac{1}{3}[(\sqrt{3}\xi_1 + q_1)e^{q_1\tau_0} + (\sqrt{3}\xi_1 - q_1)e^{-q_0\tau_1}]}{(\sqrt{3}\xi_1 + q_1)(\frac{2}{3}q_1 + \xi_1)e^{q_1\tau_0} - (\sqrt{3}\xi_1 - q_1)(\xi_1 - \frac{2}{3}q_1)e^{-q_1\tau_0}}\right\}. \tag{19}$$

In order to facilitate subsequent work with the above formula, we have tabulated in Table III a number of values of ξ_1 and the corresponding calculated central absorptions * for three values of τ_0.

TABLE III

Relation between ξ_1 and $r_\nu(\xi_1\,;\ 1)$

ξ_1	Percentage Central Absorption for $\tau_0 = \frac{1}{6}$	Percentage Central Absorption for $\tau_0 = \frac{1}{3}$	Percentage Central Absorption for $\tau_0 = \frac{1}{2}$
1·1	1·2	2	2·6
1·2	2·3	3·9	5·2
1·3	3·4	5·7	7·4
1·8	8·5	13·8	17·2
2	10·4	16·6	20·5
2·5	14·8	22·8	27·4
3	18·7	28	33
4	25·2	36·3	41·5
5	31·2	42·8	47·8
10	49·6	60·4	63·8
15	59·8	68·5	70·9

* In Table III and also in Table IV we have used the percentages of central absorption instead of the residual intensities.

(b) *Determination of* ξ_2.—We next consider an absorption line due to element 2 at frequencies at which there is no blending with an absorption line due to element 1. Then $\xi_1 = 1$ and $q_1 = \sqrt{3}$. Then

$$(1\,;\,\xi_2) = 1 - \frac{\xi_2 - 1}{\xi_2(1{\cdot}0774e^{1\cdot732\tau_0} - 0{\cdot}0774e^{-1\cdot732\tau_0}) + \sqrt{\xi_2}(1{\cdot}0774e^{1\cdot732\tau_0} + 0{\cdot}0774e^{-1\cdot732\tau_0})}. \quad (20)$$

Table IV gives calculated percentage absorption as a function of ξ_2 and τ_0.

TABLE IV

Relation between ξ_2 *and* $r_\nu(\xi_2\,;\,1)$

ξ_2	Percentage Central Absorption for $\tau_0 = \frac{1}{6}$	Percentage Central Absorption for $\tau_0 = \frac{1}{3}$	Percentage Central Absorption for $\tau_0 = \frac{1}{2}$	ξ_2	Percentage Central Absorption for $\tau_0 = \frac{1}{6}$	Percentage Central Absorption for $\tau_0 = \frac{1}{3}$	Percentage Central Absorption for $\tau_0 = \frac{1}{2}$
1·1	3·2	2·4	1·8	15	53·2	39·2	29·4
1·2	6·1	4·5	3·4	20	55·3		
1·3	8·6	6·4	4·8	30	58·5		
1·5	12·9	9·6	7·1	40	60·5		
1·8	17·8	13·3	9·9	50	61·6		
2	20·5	15·3	11·5	60	62·7		
2·5	25·8	19·2	14·4	70	63·2		
3	29·8	22·2	16·4	80	63·8		
4	35·1	26·1	19·5	200	87·4		
5	39·2	28·8	21·6	1000	69·9		
10	48·6	36	27	10000	71·7		

8. *Application to the Intensities of Superposed Lines.*—Naturally, a direct application of formula (17) constitutes a very crude simplification of the real physical conditions. We have, however, seen in the first part of this paper that the absorbing elements have different distributions, but a sharp stratification of the kind assumed in formula (17) is evidently not present.* Nevertheless, we shall examine if the application of (17) to the observations of Swings and Struve leads to better agreement than the application of Eddington's formula.

(a) *The H Line of* Ca^+ *Appearing in the Wing of the Hydrogen Line* H_ϵ (*Early A-type Stars*).—Here Ca^+ is the "element 1" and hydrogen the "element 2" of §§ 6 and 7—in agreement with Table I. Application of (17), assuming $\tau_0 = 1/6$, leads to the results indicated in Table V.

In this table the second column gives the percentages of absorption which the *H* line of Ca^+ would exhibit if it were outside the wing of H_ϵ. This was done by using the measured absorption in the *K* line and applying the square-root law for the absorptions. The third column gives the

* The crude aspect of this assumption appears numerically when we want to determine the value of ξ for very strong central absorptions.

measured percentages of absorption for the wing H_ϵ—in the immediate background (in H_ϵ) of the H line of Ca^+. The fourth and fifth columns give the corresponding values of ξ_1 and ξ_2 obtained by interpolation among the values given in Tables III and IV for $\tau_0 = 1/6$. In column 6 we give the residual intensities, starting from the wing of H_ϵ and calculated according to formula (17)—*i.e.* calculating $r(\xi_1; \xi_2)$ for the values ξ_1 and ξ_2 given in columns 4 and 5. Column 7 gives the factors by which the absorptions indicated in column 2 have to be *reduced* in order to obtain the values given in column 6. The figures in column 8 are the observed reduction factors. Finally, the last column gives the reduction factors, which are obtained without assuming a stratification and by a straightforward application of Eddington's formula.

TABLE V *

Intensities of the H line of Ca^+

(1) Star	(2) Absorption 1	(3) Absorption 2	(4) ξ_1	(5) ξ_2	(6) Calculated Intensity of H	(7) Calculated Reduction Factor	(8) Observed Reduction Factor	(9) Reduction Factor Calculated Classically
α Cygni	52	25	11	2·4	45	1·15	1·02	1·28
η Leonis	30	36	4·8	4·1	22·4	1·34	1·22	1·86
α Can. Maj.	34	62	6	55	16	2·12	1·84	4·26

We see that a stratified atmosphere with $\tau_0 = 1/6$ brings out a very good agreement between the theory and the observations. When considering Table I the value for $\tau_0 = 1/6$ does not seem unlikely; the observed facts would thus indicate that the difference of distribution between the Ca^+ and hydrogen atoms in the atmospheres of the early A-type stars is perhaps roughly equivalent to a sharp separation at about $\tau_0 = 1/6$.

Any further discussion would be premature at this stage of the measures.

(b) OII *Lines Appearing in the Wings of Hγ.*—Now hydrogen would correspond to "element 1" and ionized oxygen to "element 2"—in agreement with Table II. The calculation has been made only for the OII line 4345·57 of θ Ophiuchi and for $\tau_0 = 1/6$. Application of (17) leads to a calculated central absorption of 2·2 per cent. as against the observed figure of 2 per cent., thus explaining one of the largest discrepancies observed by Swings and Struve.

9. *The Formation of Blended Absorption Lines in a Stratified Atmosphere (a more General Treatment).*—So far we have restricted ourselves to the so-called "standard case," assuming for B_ν the expression (12). This is sufficient, as is well known, for most purposes, but in view of the agreement between the "theory" and observations obtained in § 8, it is necessary to

* The star α Lyræ, for which measures had also been made, is not considered owing to the uncertainty in the determination of ξ_2; ξ_2 would have to be of the order 10,000.

examine whether a more general treatment of the formation of absorption lines in a stratified atmosphere leads to different results. We shall see that this is not the case.

Following Milne, we shall assume a "Taylor-expansion" for B_ν, retaining only the first two terms:

$$B_\nu = a_\nu + b_\nu \tau, \tag{21}$$

where

$$a_\nu = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT_0}-1}, \tag{22}$$

$$b_\nu = \frac{2h\nu^3}{c^2}\frac{e^{h\nu/kT_0}}{(e^{h\nu/kT_0}-1)^2}\frac{h\nu}{kT_0^2}\left(\frac{dT}{d\tau}\right)_{\tau=0}; \qquad \left(\frac{dT}{d\tau}\right)_0 = \tfrac{3}{8}T_0. \tag{23}$$

In (22) and (23) T_0 represents the boundary temperature.

With the form (21) for B_ν the solution of the equations proceeds exactly as in § 5. We have now (as one easily verifies),

$$J_\nu = a_\nu + b_\nu\tau + \alpha e^{-q_1\tau} + \beta e^{+q_1\tau}, \qquad (\tau < \tau_0) \tag{24'}$$

$$J_\nu = a_\nu + b_\nu\tau + \gamma e^{-q_2\tau}, \qquad (\tau \geqslant \tau_0) \tag{24''}$$

$$\tfrac{3}{4}(1+\eta_1)F_\nu = b_\nu + q_1(\beta e^{q_1\tau} - \alpha e^{-q_1\tau}), \qquad (\tau < \tau_0) \tag{25'}$$

$$\tfrac{3}{4}(1+\eta_2)F_\nu = b_\nu - q_2\gamma e^{-q_2\tau}. \qquad (\tau \geqslant \tau_0) \tag{25''}$$

The equations determining the constants α, β and γ are found to be:

$$\left.\begin{aligned} &\alpha(1+\eta_1+\tfrac{2}{3}q_1)+\beta(1+\eta_1-\tfrac{2}{3}q_1)+(1+\eta_1)a_\nu-\tfrac{2}{3}b_\nu=0,\\ &\alpha e^{-q_1\tau_0}+\beta e^{q_1\tau_0}-\gamma e^{-q_2\tau_0}=0,\\ &\alpha e^{-q_1\tau_0}(1+\eta_2)q_1-\beta e^{q_1\tau_0}(1+\eta_2)q_1-\gamma e^{-q_2\tau_0}(1+\eta_1)q_2+b_\nu(\eta_1-\eta_2)=0.\end{aligned}\right\} \tag{26}$$

The above equations can be solved for α and β, and we find for the emergent flux the expression

$$F_\nu(0)=\frac{4}{3}\frac{1}{\xi_1}\left\{b_\nu+q_1\frac{2\xi_1 b_\nu(\xi_1-\xi_2)+(\xi_1 a_\nu-\frac{2}{3}b_\nu)[(\xi_2 q_1+\xi_1 q_2)e^{q_1\tau_0}-(\xi_2 q_1-\xi_1 q_2)e^{-q_1\tau_0}]}{(\xi_2 q_1-\xi_1 q_2)(\xi_1-\frac{2}{3}q_1)e^{-q_1\tau_0}+(\xi_2 q_1+\xi_1 q_2)(\xi_1+\frac{2}{3}q_1)e^{q_1\tau_0}}\right\}. \tag{27}$$

The emergent flux $F(0)$ in the immediate neighbourhood of the line (but in the continuous background) is obtained from (27) by putting $\eta_1=\eta_2=0$ and $q_1=q_2=\sqrt{3}$. We obtain

$$F(0)=\frac{4a_\nu}{3}\frac{3+\frac{b_\nu}{a_\nu}\sqrt{3}}{2+\sqrt{3}}. \tag{28}$$

The residual intensity then is given by

$$r_\nu(\xi_1;\ \xi_2)=\frac{F_\nu(0)}{F(0)}=\frac{2+\sqrt{3}}{\xi_1\left(3+\frac{b_\nu}{a_\nu}\sqrt{3}\right)}$$

$$\times\left\{\frac{b_\nu}{a_\nu}+q_1\frac{2\frac{b_\nu}{a_\nu}\xi_1(\xi_1-\xi_2)+\left(\xi_1-\frac{2}{3}\frac{b_\nu}{a_\nu}\right)[(\xi_2 q_1+\xi_1 q_2)e^{q_1\tau_0}-(\xi_2 q_1-\xi_1 q_2)e^{-q_1\tau_0}]}{(\xi_2 q_1-\xi_1 q_2)(\xi_1-\frac{2}{3}q_1)e^{-q_1\tau_0}+(\xi_2 q_1+\xi_1 q_2)(\xi_1+\frac{2}{3}q_1)e^{q_1\tau_0}}\right\}. \tag{29}$$

When $\xi_1 = \xi_2$, (29) reduces to

$$r_\nu(\xi_1 = \xi_2) = \frac{b_\nu + a_\nu q}{b_\nu + a_\nu \sqrt{3}} \cdot \frac{1 + \frac{2}{\sqrt{3}}}{1 + \eta + \frac{2}{3}q}. \tag{30}$$

From (29) we see that

$$r_\nu(\xi_1;\ 1) = \frac{2 + \sqrt{3}}{\xi_1\left(3 + \frac{b_\nu}{a_\nu}\sqrt{3}\right)}$$
$$\times \left\{ \frac{b_\nu}{a_\nu} + q_1 \frac{2\frac{b_\nu}{a_\nu}\xi_1(\xi_1 - 1) + \left(\xi_1 - \frac{2}{3}\frac{b_\nu}{a_\nu}\right)[(q_1 + \sqrt{3}\xi_1)e^{q_1\tau_0} - (q_1 - \sqrt{3}\xi_1)e^{-q_1\tau_0}]}{(q_1 + \sqrt{3}\xi_1)(\frac{2}{3}q_1 + \xi_1)e^{q_1\tau_0} - (q_1 - \sqrt{3}\xi_1)(\frac{2}{3}q_1 - \xi_1)e^{-q_1\tau_0}} \right\}. \tag{31}$$

$$r_\nu(1;\ \xi_2) = \frac{2 + \sqrt{3}}{\left(3 + \frac{b_\nu}{a_\nu}\sqrt{3}\right)}$$
$$\times \left\{ \frac{b_\nu}{a_\nu} + \frac{\frac{b_\nu}{a_\nu}(1 - \xi_2) + \left(\frac{1}{2} - \frac{1}{3}\frac{b_\nu}{a_\nu}\right)[(\sqrt{3}\xi_2 + q_2)e^{\sqrt{3}\tau_0} - (\sqrt{3}\xi_2 - q_2)e^{-\sqrt{3}\tau_0}]}{\xi_2(1\cdot0774e^{\sqrt{3}\tau_0} - 0\cdot0774e^{-\sqrt{3}\tau_0}) + \sqrt{\xi_2}(1\cdot0774e^{\sqrt{3}\tau_0} + 0\cdot0774e^{-\sqrt{3}\tau_0})} \right\}. \tag{32}$$

Formulæ (29), (31) and (32) correspond to our earlier formulæ (17), (19) and (20). One sees that these newer formulæ reduce to our earlier ones for the case $b_\nu/a_\nu = 3/2$. The formal discussion on the basis of these equations is of course the same as before.

10. *Applications.*—In § 8 (9) we applied formula (17) to the case of the *H* line of Ca^+ appearing in the wing of H_ϵ. For the early A-type stars one easily verifies that the appropriate value for b_ν/a_ν is almost exactly 3/2. But when $b_\nu/a_\nu = 3/2$, as we have already pointed out, (29) is *identical* with (17), and consequently the earlier discussion of this problem requires no modification.

However, when one considers the problem of the *O*II lines appearing in the wing of H_γ our earlier discussion requires some reconsideration. For, in the case of B-type stars, at about 4350 A. the value of b_ν/a_ν is about 0·75. Hence for this case we have to use equations (29), (31) and (32) with $b_\nu/a_\nu = 0\cdot75$. Owing to the lengthy nature of the calculations required, a rough estimation has been made only for the *O*II line 4345·57 of θ Ophiuchi, assuming $\tau_0 = 1/6$. The roughly calculated central absorption is in fairly good agreement with the observations. Owing to the uncertainties of the measured intensities no further attempt has been made.

Summary

1. The distribution of the absorbing atoms of specified ionization and excitation is investigated as a function of the optical thickness τ of the stellar atmosphere. This problem may be important in spectral classification, in the dissociation equilibria of molecular compounds, in the

investigation of stars of the ξ Aurigæ type, and especially in the interpretation of an effect observed by Swings and Struve. These authors have found that absorption lines appearing in the wings of other lines have peculiar intensities which cannot be interpreted by the usual formulæ for the absorption lines.

2. The cases of H and O^+ in B-type stars and of Ca^+ and H in A-type stars are investigated numerically; differences of distribution appear conspicuously; the effect of surface gravity is important.

3. The conditions of radiative equilibrium in a sharply stratified atmosphere are investigated; an adequate formula is provided for the intensities of the corresponding absorption lines.

4. This formula explains the measures by Swings and Struve, when it is assumed that:

(*a*) in the examined A-type stars, the Ca^+ absorbing atoms occupy the upper part of the reversing layers, until approximately $\tau_0 = \frac{1}{6}$;

(*b*) in the B-type stars, for which the calculation has been made, the absorbing hydrogen atoms are in the upper part of the atmosphere until $\tau_0 = \frac{1}{6}$; the absorbing ionized oxygen atoms would be below $\tau_0 = \frac{1}{6}$.

Roughly speaking, this would mean that the effect of the different distributions of absorbing atoms is somewhat similar to that of a sharp stratification.

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1936 *July* 29.

DYNAMICS OF RADIATION PRESSURE FOR A DIFFUSE NEBULA.

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1. *Introduction and Summary.*—The importance of L_α radiation pressure for nebulæ was first pointed out by Ambarzumian.* He gave a complete solution of the radiative equilibrium in a stationary nebula, using the Schwarzschild-Schuster approximation. A closer approximation allowing the intensity to vary with direction was presented by Chandrasekhar.†

According to well-known mechanisms, line spectra are produced in a nebula under the action of the exciting star. Among these the first line L_α of the Lyman series of hydrogen is most vigorously absorbed, and the resulting radiation pressure tends to "blow up" the nebula, as Ambarzumian showed. As soon as the nebula starts expanding, however, the radiation pressure is greatly reduced, since the Doppler effect of the moving material makes it permeable to L_α radiation from the more inward parts. The author,‡ modifying Ambarzumian's methods, worked out the case of radiative

* V. A. Ambarzumian, *M.N.*, **93**, 50, 1932.
† S. Chandrasekhar, *Zeitschr. f. Astrophys.*, **9**, 266, 1935.
‡ H. Zanstra, *M.N.*, **95**, 84, 1934 (Comm. Univ. Obs. Oxford, No. 2).