Date	Variation	Number of Pairs
	″	
1935·539	+·018	92
·573	−·038	93
·619	−·018	92
·676	−·072	85
·724	−·061	94
·773	−·090	89
·819	−·040	90
·884	−·013	74
1936·023	+·097	90
·099	+·070	77
·158	+·136	80
·244	+·101	92
·310	+·145	100
·363	+·040	91
1936·397	+·178	79

From a smooth curve drawn to represent the observed results as closely as possible, the following corrections, applicable to observed north polar distances, were derived :—

	″		″
1935·0	+·01	1935·8	−·06
·1	+·07	·9	00
·2	+·11	1936·0	+·06
·3	+·12	·1	+·10
·4	+·10	·2	+·12
·5	+·04	·3	+·13
·6	−·03	1936·4	+·12
·7	−·07		

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THE PROFILE OF THE ABSORPTION LINES IN ROTATING STARS, TAKING INTO ACCOUNT THE VARIATION OF IONIZATION DUE TO CENTRIFUGAL FORCE.

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1. The investigations by Elvey * and Miss Westgate † have shown that a very great proportion of the stars of classes B and A have rotational velocities of the order of 100 km./sec., and that 5 per cent. of the early type stars have velocities larger than 200 km./sec. In several papers ‡ Struve has indicated that in certain cases the centrifugal acceleration V^2/R may be of the same order as the surface gravity g, the ratio $\epsilon = V^2/Rg$ having in quite reasonable stars a value approaching 0·5. Actually, a calculation of the ratio ϵ for many B and A type stars of large rotational velocities gives values of the order of 0·4. The importance of these facts, in view of the mechanical instability of certain rotating stars as argued by Struve, is clear.

* C. T. Elvey, *Ap. J.*, **71**, 221, 1930. † C. Westgate, *Ap. J.*, **78**, 46, 1933.
‡ O. Struve, *Ap. J.*, **72**, 1, 1930 ; *Pop. Astr.*, **43**, 492, 1935.

In a recent paper * Hynek has dealt with the effect of rotation upon the state of ionization; his work shows that, statistically, the ionization is not materially different in rotating stars. But it has seemed interesting to us to investigate theoretically the special case of the rotating stars with high values of ϵ; actually, all the numerical examples which will be considered later will correspond to $\epsilon = 0{\cdot}4$, which is a quite reasonable value for certain stars having high rotational velocities.

The gravity at a given latitude on the star is reduced by a definite amount by the centrifugal acceleration, the effect being a maximum at the equator and absent at the poles. Consequently, the ionization must vary as a function of the stellar latitude.

For high values of ϵ this effect may give rise to differences between the profiles of lines of neutral and ionized atoms. It is the object of this paper to discuss briefly this question from the theoretical side.

An investigation of this kind may be useful either for the interpretation of irregularities which should possibly be observable in accurate profiles when ϵ is known, or for the detection of stars having a high value of ϵ.

We assume the angular velocity to be constant and the star to be spherical. On the other hand, we suppose that the intrinsic width of the line is very small compared with that due to Doppler broadening; this is obviously justifiable if we consider high rotational velocities. We shall only compare the profiles of lines of neutral and ionized atoms.

2. *Standard Cases.*†—The expression "standard profiles" will mean the profiles obtained without considering the latitude effect of rotation on the ionization.

Let V be the velocity at the equator;
ν_0 the wave number of the centre of the line;
ν the wave number at any point of the profile;
v the velocity corresponding to the wave number ν;
θ the colatitude on the star.

A. *Standard profile S_{ND}, neglecting the darkening towards the limb.* It is easily seen ‡ that *the contour is an integration over flat-topped elementary*

* J. A. Hynek, *Ap. J.*, 83, 476, 1936.

† For a general summary on this subject, see Rosseland, *Theoretical Astrophysics*, pp. 202–215, 1936. We have, however, used a different method of integration, which was more suitable for our particular problem.

‡ The contribution of the region between θ and $\theta + d\theta$ is proportional to

$$R \sin \theta d\phi \,.\, \sin \phi \,.\, R \sin \theta d\theta, \qquad (a)$$

the angle ϕ being the azimuthal angle connected to ν and θ by

$$\nu = \nu_0\left(1 + \frac{V}{c} \sin \theta \cos \phi\right). \qquad (b)$$

Differentiation of (b) gives

$$d\nu = \nu_0 \frac{V}{c} \sin \theta \sin \phi d\phi \,;$$

consequently (a) is

$$I_0 \,.\, d\nu \,.\, \sin \theta d\theta.$$

contours of intensity $I_0 \, . \, d\nu \sin \theta \, . \, d\theta$, *arising between* θ *and* $\theta + d\theta$, *and extending from* θ_1 *to* $\pi/2$, θ_1 being given by $\frac{V}{c} \sin \theta_1 = \frac{\nu - \nu_0}{\nu_0}$.

We have thus

$$S_{ND} d\nu = I_0 \, . \, d\nu \left[1 - \frac{c^2}{V^2} \left(\frac{\nu - \nu_0}{\nu_0} \right)^2 \right]^{\frac{1}{2}}. \tag{1}$$

We shall use the following abbreviation :—

$$G(\nu) = \left[1 - \frac{c^2}{V^2} \left(\frac{\nu - \nu_0}{\nu_0} \right)^2 \right]^{\frac{1}{2}} = \left(1 - \frac{v^2}{V^2} \right)^{\frac{1}{2}}. \tag{2}$$

Omitting the constant factor, we write this

$$S_{ND} = G(\nu). \tag{1'}$$

B. *Standard profile* S_D, *taking into account the darkening towards the limb.* If we assume the star to be darkened towards the limb, u being the coefficient of darkening, we have now *

$$S_D = \int_{\theta_1}^{\pi/2} \sin \theta \left\{ 1 - u + u \sin \theta \left[1 - \left(\frac{\nu - \nu_0}{\nu_0} \right)^2 \cdot \frac{c^2}{V^2 \sin^2 \theta} \right]^{\frac{1}{2}} \right\} d\theta.$$

The result is

$$S_D = G(\nu) \left[(1 - u) + \frac{\pi}{4} u G(\nu) \right]. \tag{3}$$

3. *Actual Problem.*—Inside the integrals we have to introduce in each elementary contour a weighting factor, $w(\theta)$, which corresponds to the number of atoms contributing to the absorption line, this number being now a function of the angle θ.

Let us consider a single-stage ionization and let N_1 and N_0 be the proportions of ionized and neutral atoms. We use the equilibrium equation

$$\frac{N_1}{N_0} = \frac{K}{P}, \tag{4}$$

and assume the equilibrium constant K to be independent of θ.

At the angle θ the surface-gravity g is reduced to

$$g_{eff} = g \left(1 - \frac{V^2}{Rg} \cdot \sin^2 \theta \right) = g(1 - \epsilon \sin^2 \theta). \tag{5}$$

We shall consider two cases :

(*a*) P proportional to g_{eff}, thus giving †

$$P = P_0 (1 - \epsilon \sin^2 \theta) \, ; \tag{6}$$

(*b*) P proportional to $\sqrt{g_{eff}}$,

$$P = P_0 \sqrt{1 - \epsilon \sin^2 \theta}. \tag{7}$$

* We neglect the constant factor I_0.

† In this investigation, θ terms independent of ϵ are ignored and a mean value assumed for τ.

If we assume a constant optical depth τ, these two assumptions correspond respectively to the values 0 and 1 of the coefficient n introduced by Unsöld in his investigation of stellar atmospheres; * the first case is probably the most important one.

Equation (4) gives

$$\text{Case } a\begin{cases} N_1 = \dfrac{K}{K+P_0(1-\epsilon\sin^2\theta)}, \\ N_0 = \dfrac{P_0(1-\epsilon\sin^2\theta)}{K+P_0(1-\epsilon\sin^2\theta)}. \end{cases} \tag{6'}$$

$$\text{Case } b\begin{cases} N_1 = \dfrac{K}{K+P_0\,.\sqrt{1-\epsilon\sin^2\theta}}, \\ N_0 = \dfrac{P_0\sqrt{1-\epsilon\sin^2\theta}}{K+P_0\sqrt{1-\epsilon\sin^2\theta}}. \end{cases} \tag{7'}$$

We also have to assume the relation between N and w; owing to the fact that we consider here Doppler effects, we shall take w proportional † to N.

4. *First Problem.—P proportional to g_{eff}; no consideration of darkening towards the limb; K very large.*

N_1 is then nearly unity, and the contour of a line of the ionized element is evidently the standard contour $S_{ND} = G(\nu)$. For the lines of the neutral atom the contour is

$$Id\nu = I_0\,.\,d\nu\,.\,\frac{P_0}{K}\int_{\theta_1}^{\frac{\pi}{2}} \sin\theta\,.\,(1-\epsilon\sin^2\theta)d\theta.$$

Neglecting constant factors, this gives ‡

$$(\text{Contour})_0' = G(\nu)\left[1-\epsilon+\frac{\epsilon}{3}G^2(\nu)\right]. \tag{8}$$

Numerical Example.—In Table I we give the values of I as a function of v/V, in the particular case $\epsilon = 0{\cdot}4$ for the standard contour S_{ND} and for $(\text{contour})_0'$, both of them being reduced to $I = 1$ at $v/V = 0$.

Supposing that this first problem has a real physical meaning,§ an examination of Table I leads to the following conclusions:—

(1) When comparing lines of neutral and ionized atoms the widths must really differ little.

(2) Where the contour still corresponds to 44 per cent. of the central

* Unsöld, *Zs. f. Astroph.*, 1, 138, 1930; equation $\tau = \dfrac{k_0}{g}\cdot\dfrac{P^{n+1}}{n+1}$.

† The introduction of the square-root law $w \propto \sqrt{N}$ would lead to complicated integrals, but no theoretical difficulty. For numerical applications it would be possible to reduce it to the problems considered here.

‡ The notations $(\text{contour})_0$ and $(\text{contour})_1$ respectively refer to the neutral and to the ionized atoms.

§ It will be seen later on that all the other problems give similar results.

TABLE I

$\frac{v}{V}$	S_{ND}	(Contour)$_0'$	$\frac{(\text{Contour})_0'}{S_{ND}}$
0·0	1	1	1
·2	·98	·973	·99
·4	·916	·89	·971
·6	·8	·748	·935
·8	·6	·530	·883
·9	·436	·372	·853

intensity, (contour)$_0'$ is reduced by 15 per cent. compared with the standard contour (or with the contour of the ionized lines); this seems to be within the limits of photometric observations.

(3) In any case, sufficiently high dispersion and great photometric care will be required in order to show the effect conspicuously.

(4) The figures in Table I correspond to $\epsilon = 0{\cdot}4$, which is rather an extreme value, *i.e.* the effect becomes observable only for very fast rotating stars.

5. *Second Problem.—P proportional to g_{eff}; with consideration of darkening towards the limb; K very large.*

For the ionized lines the contour is of course the standard contour S_D given by formula (3). For the neutral lines the integral has to be taken over elementary contours of intensity

$$\sin\theta\left\{1 - u + u\sin\theta\left[1 - \left(\frac{\nu-\nu_0}{\nu_0}\right)^2 \cdot \frac{c^2}{V^2\sin^2\theta}\right]^{\frac{1}{2}}\right\}(1 - \epsilon\sin^2\theta)d\theta.$$

The expression obtained is

$$(\text{Contour})_0'' = (1-u)G(\nu)\left[1 - \epsilon + \frac{\epsilon}{3}G^2(\nu)\right] + u\frac{\pi}{4}G^2(\nu)\left[1 - \epsilon + \frac{\epsilon}{4}G^2(\nu)\right]. \quad (9)$$

Numerical Example.—$\epsilon = 0{\cdot}4$; $u = 0{\cdot}6$. Table II is similar to Table I.

TABLE II

$\frac{v}{V}$	S_D	(Contour)$_0''$	$\frac{(\text{Contour})_0''}{S_D}$
0·0	1	1	1
·2	·970	·963	·99
·4	·874	·854	·98
·6	·714	·672	·94
·8	·471	·422	·90
·9	·303	·263	·86

The effect of the darkening on the intensities is quite appreciable, but the ratios indicated in the last column are similar to those of Table I, leaving the conclusions unaltered.

6. *Third Problem.—P proportional to g_{eff}; no consideration of darkening; N_1 and N_0 of the same order.*

The complete formulæ (6′) have to be introduced as factors w in the integrals; as we have $N_0 = 1 - N_1$ or $w_0 = 1 - w_1$, the contour for the lines of neutral atoms is immediately obtained when we know the expression for the lines of ions. This is

$$(\text{Contour})_1 = \int_{\theta_1}^{\frac{\pi}{2}} \frac{\sin\theta \, . \, d\theta}{K + P_0(1 - \epsilon \sin^2\theta)}$$

$$= \frac{1}{\sqrt{[K + P_0(1-\epsilon)]P_0\epsilon}} \cdot \tan^{-1} G(\nu) \cdot \left[\frac{P_0\epsilon}{K + P_0(1-\epsilon)}\right]^{\frac{1}{2}}. \quad (10)$$

Numerical Example.—Let us suppose $\epsilon = 0{\cdot}4$ and $K/P_0 = 1$. Table III gives information similar to that of Tables I and II; $(\text{contour})_1$ and $(\text{contour})_0$ correspond respectively to the ionized and neutral atoms.

Table III

	$\frac{(\text{Contour})_1}{S_{ND}}$	$\frac{(\text{Contour})_0}{S_{ND}}$	$\frac{(\text{Contour})_0}{(\text{Contour})_1}$
0·0	1	1	1
·2	1	1	1
·4	1·01	0·99	0·98
·6	1·025	·975	·95
·8	1·05	·95	·90
·9	1·07	·93	·87

Table III shows that between $(\text{contour})_0$ and $(\text{contour})_1$ there would be a difference of 13 per cent. of absorption at $v/V = 0{\cdot}9$.

7. *Fourth Problem.—P proportional to g_{eff}; N_1 and N_0 of the same order; with consideration of darkening at the limb.*

The expression for $(\text{contour})_1$ is now

$$(\text{Contour})_1 = \int_{\theta_1}^{\frac{\pi}{2}} \left\{(1-u)\sin\theta + u\sin^2\theta\left[1 - \left(\frac{\nu-\nu_0}{\nu_0}\right)^2 \cdot \frac{c^2}{V^2\sin^2\theta}\right]^{\frac{1}{2}}\right\} \cdot \frac{K d\theta}{P_0(1-\epsilon\sin^2\theta)+K}.$$

Let us write

$$a^2 = \frac{P_0\epsilon}{K + P_0 - P_0\epsilon}.$$

The expression obtained for $(\text{contour})_1$ is

$$(\text{Contour})_1 = (1-u) \, . \, \frac{aK}{P_0\epsilon} \, . \, \tan^{-1}(aG(\nu)) + \frac{\pi u K}{2P_0\epsilon}(\sqrt{1 + a^2G^2(\nu)} - 1);$$

or neglecting the constant factor $\frac{K}{P_0\epsilon}(\epsilon \neq 0)$

$$(\text{Contour})_1 = (1-u) \,.\, a \tan^{-1}(aG(\nu)) + \frac{\pi}{2}u(\sqrt{1+a^2G^2(\nu)} - 1). \qquad (11)$$

The numerical results are similar to those of the preceding tables.

8. *Fifth Problem.—P proportional to square root of g_{eff}; no consideration of darkening at the limb; K very large.*

The expressions (7′) have to be used; when K is very large, $(\text{contour})_1$ is evidently $G(\nu)$. For $(\text{contour})_0$ we have the expression

$$(\text{Contour})_0 = \int_{\theta_1}^{\pi/2} \sqrt{1-\epsilon \sin^2\theta} \sin\theta \, d\theta;$$

this brings

$$(\text{Contour})_0 = \tfrac{1}{2}\left[G(\nu)\sqrt{1-\epsilon+\epsilon G^2(\nu)} + \frac{1-\epsilon}{\sqrt{\epsilon}} \,.\, \sinh^{-1} G(\nu)\sqrt{\frac{\epsilon}{1-\epsilon}}\right]. \qquad (12)$$

Of course, when ϵ is small, the expression (12) reduces to formula (8) where ϵ is replaced by $\epsilon/2$.

9. *Sixth Problem.—P proportional to square root of g_{eff}; no consideration of darkening at the limb; N_0 and N_1 of the same order.*

$(\text{Contour})_1$ is given by

$$(\text{Contour})_1 = \int_{\theta_1}^{\frac{\pi}{2}} \frac{K \sin\theta \, d\theta}{K + P_0\sqrt{1-\epsilon\sin^2\theta}}.$$

The expression obtained by integration is rather inconvenient for numerical calculations; it is

$$(\text{Contour})_1 = \frac{1}{P_0\sqrt{\epsilon}}\left[\sinh^{-1}\left(G(\nu)\sqrt{\frac{\epsilon}{1-\epsilon}}\right) + \frac{K}{\sqrt{K^2 - P_0^2(1-\epsilon)}} \,.\, \log_e \frac{mn_0}{nm_0}\right], \qquad (13)$$

with following abbreviations :—

$$m = \sqrt{\epsilon}G + \sqrt{\epsilon G^2 + 1 - \epsilon} + \frac{K}{P_0} + \frac{1}{P_0}\sqrt{K^2 - P_0^2(1-\epsilon)};$$

$$m_0 = m(G \to 0) = \sqrt{1-\epsilon} + \frac{K}{P_0} + \frac{1}{P_0}\sqrt{K^2 - P_0^2(1-\epsilon)};$$

$$n = \sqrt{\epsilon}G + \sqrt{\epsilon G^2 + 1 - \epsilon} + \frac{K}{P_0} - \frac{1}{P_0}\sqrt{K^2 - P_0^2(1-\epsilon)};$$

$$n_0 = n(G \to 0) = \sqrt{1-\epsilon} + \frac{K}{P_0} - \frac{1}{P_0}\sqrt{K^2 - P_0^2(1-\epsilon)}.$$

Similar formulæ may be obtained in case *b*, introducing the darkening towards the limb; but the expressions are complicated and have no interest with regard to numerical applications.

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