Multiqubit Symmetric States with High Geometric Entanglement

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We propose a detailed study of the geometric entanglement properties of pure symmetric N-qubit states, focusing more particularly on the identification of symmetric states with a high geometric entanglement and how their entanglement behaves asymptotically for large N. We show that much higher geometric entanglement with improved asymptotical behavior can be obtained in comparison with the highly entangled balanced Dicke states studied previously. We also derive an upper bound for the geometric measure of entanglement of symmetric states.

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Geometric measure of entanglement of an N-qubit pure state

Definition : $E_G(|\psi\rangle) = 1 - \max_{|\Phi\rangle=|\phi_1,\phi_2,\phi_3,\ldots\rangle} |\langle\psi|\Phi\rangle|^2$

where the maximum is taken over all separable states [1].

The explicit value is only known for a limited number of states because of the optimization procedure that can be of a formidable task in the general case.

Upper bound [2] : $E_G(|\psi\rangle) \leqslant 1 - \frac{1}{2^{N-1}}$

Majorana representation of an N-qubit symmetric state

Any symmetric state can be written in the form

2-qubit : $|\psi_S\rangle = \mathcal{N}(|\phi_1,\phi_2\rangle + |\phi_2,\phi_1\rangle)$

3-qubit : $|\psi_S\rangle = \mathcal{N}(|\phi_1,\phi_2,\phi_3\rangle + |\phi_1,\phi_3,\phi_2\rangle + |\phi 2,\phi_1,\phi_3\rangle + \ldots)$

N-qubit : $|\psi_S\rangle = \mathcal{N}\sum_\sigma |\phi_{\sigma(1)},\ldots,\phi_{\sigma(N)}\rangle$

where the sum is over all N! permutations.

Thus, any N-qubit symmetric state is fully determined by N single qubit states

$$|\phi_i\rangle = \cos(\theta_i/2)|0\rangle + \sin(\theta_i/2)e^{i\varphi_i}|1\rangle$$

and can be represented by N points on the Bloch sphere (Majorana points). Any symmetric separable state is of the form $|\Phi\rangle = |\phi,\ldots,\phi\rangle$ and is represented by N identical points.

Geometric measure of entanglement of a pure symmetric state

Theorem : $E_G(|\psi_S\rangle) = 1 - \max_{|\Phi\rangle=|\phi,\phi,\phi,\ldots\rangle} |\langle\psi_S|\Phi\rangle|^2$

where the maximum is only taken over all symmetric separable states (huge simplification) [3].

Upper bound [This work] : $E_G(|\psi_S\rangle) < 1 - \frac{1}{N+1}$

Illustrative examples [1]

GHZ states : $|\mathrm{GHZ}_N\rangle = \frac{1}{\sqrt{2}}(|0,\ldots,0\rangle + |1,\ldots,1\rangle)$

$$E_G(|\mathrm{GHZ}_N\rangle) = 1/2$$

Dicke states : $|D_N(k)\rangle = \frac{1}{\sqrt{C_N^k}}\sum_\sigma |\underbrace{0\ldots0}_{N-k}\underbrace{1\ldots1}_{k}\rangle$

$$E_G(|D_N(k)\rangle) = 1 - C_N^k\left(\frac{k}{N}\right)^k\left(\frac{N-k}{N}\right)^{N-k}$$

Balanced Dicke states (k = [N/2]) :

$$E_G(|D_N([N/2])\rangle) = 1 - \sqrt{\frac{2}{\pi N}} + \mathcal{O}(N^{-3/2})$$

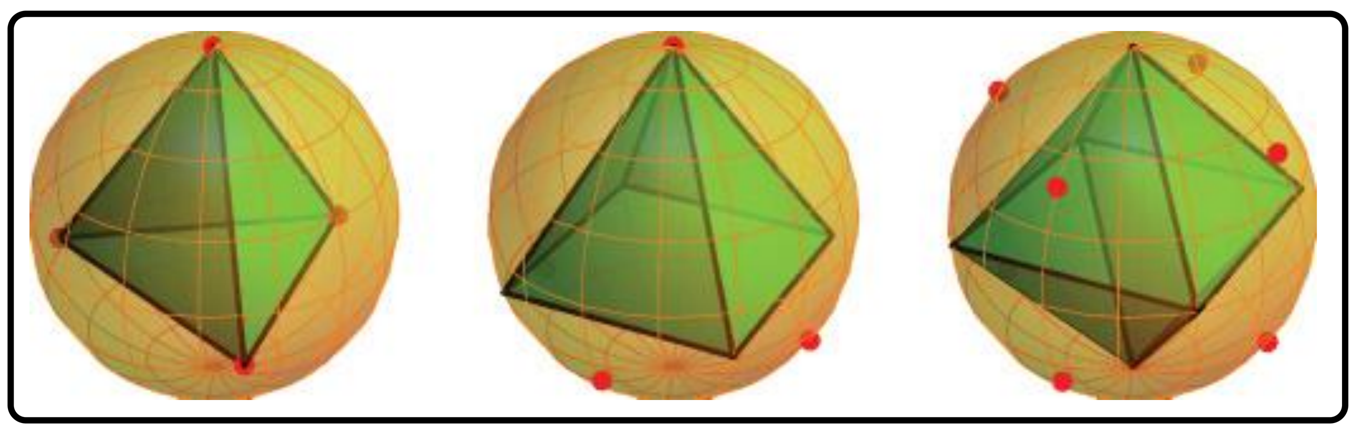
Highest geometric entanglement configurations

N = 2 [1] : $E_G|_{\max} = 1/2$ for $|\psi_S\rangle = \frac{1}{\sqrt{2}}(|0,0\rangle + |1,1\rangle)$

N = 3 [4] : $E_G|_{\max} = 5/9$ for $|\psi_S\rangle = |D_3(1)\rangle \equiv |W\rangle$

N = 4 : $E_G|_{\max} = 2/3$ N = 6 : $E_G|_{\max} = 7/9$

N = 5 : $E_G|_{\max} = 0.7011$

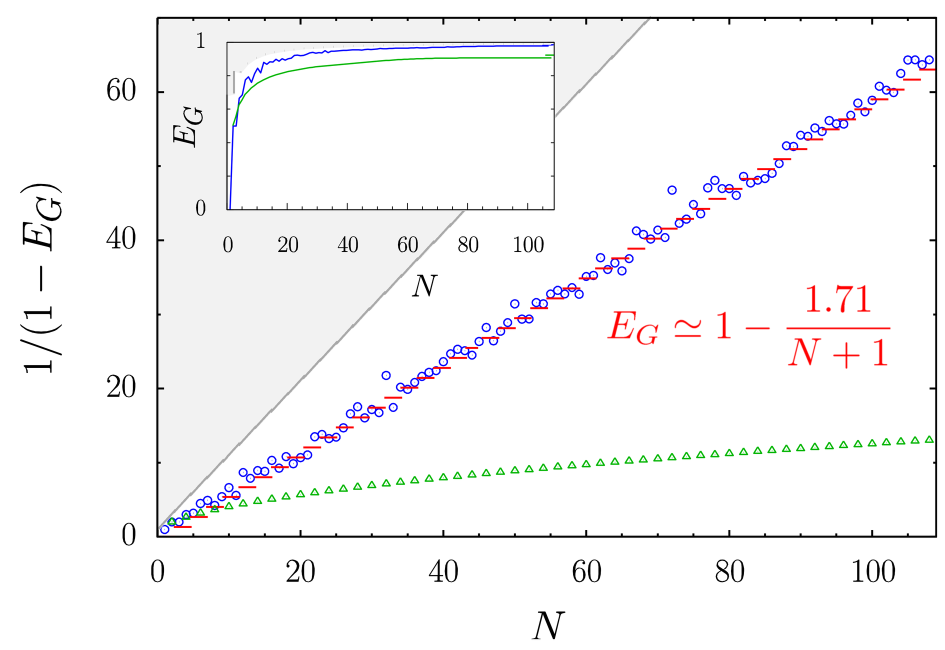


Majorana points of the maximally entangled symmetric states for N = 4 - 6 (polyhedron vertices). Red points = closest separable states.

Coulomb configurations

The highest geometric entanglement is obtained with states having points largely spread on the Bloch sphere, similar to how N equal electrical charges tend to be placed as far as possible from each other when they are constrained to a conducting sphere (Thomson problem). Though similar, the Thomson problem remains distinct from the quest of maximal entanglement as it corresponds to finding charge positions $\mathbf{r}_i$ minimizing the electrostatic energy $E = \sum_{i,j}\frac{1}{|\mathbf{r}_i - \mathbf{r}_j|}$ We nevertheless can expect high E_G :

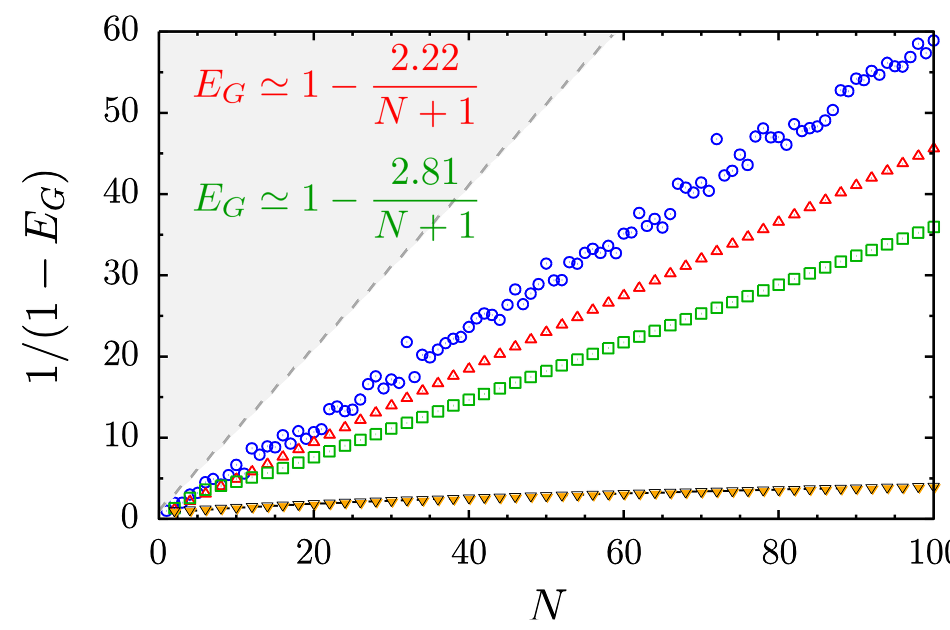
Geometric measure of entanglement of symmetric states for the Coulomb arrangement of the Majorana points for N up to 110 (blue circles). Green triangles = balanced Dicke states. Grey shaded area = domain ruled out by the upper bound. For some N (notably 4 and 6), it corresponds to the highest possible geometric entanglement.

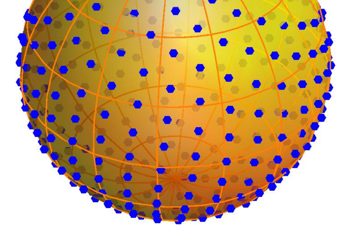


More regular behavior with respect to N is obtained when considering equally weighed superpositions of Dicke states with pseudorandom quadratic phases :

$$|\psi_\gamma(N)\rangle = \sum_{k=0}^{N}\frac{e^{i\gamma k^2}}{\sqrt{N+1}}|D_N(k)\rangle \quad (1)$$

Geometric measure of entanglement for states (1). Blue: Coulomb arrangement. Red: γ = 2/3; Green : γ = 1. Orange: equally weighed superpositions with linear phases $e^{i\gamma k}$ instead (γ = 2/3) : much less entanglement.





Majorana representation of states (1) for γ = 2/3.

References

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