An analysis of mixed integer linear sets based on lattice point free convex sets

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A maximal lattice free polyhedron L has max-facet-width equal to w if $\max_{x\in L}\pi^T x - \min_{x\in L}\pi^T x \leq w$ for all facets $\pi^T x \leq \pi_0$ of L, and $\max_{x\in L}\pi^T x - \min_{x\in L}\pi^T x = w$ for some facet $\pi^T x \leq \pi_0$ of L. The set obtained by adding all cuts whose validity follows from a maximal lattice free polyhedron with max-facet-width at most w is called the w^{th} split closure. We show the w^{th} split closure is a polyhedron. This generalizes a previous result showing this to be true when $w = 1$. We also consider the design of finite cutting plane proofs for the validity of an inequality. Given a measure of "size" of a maximal lattice free polyhedron, a natural question is how large a size s^* of a maximal lattice free polyhedron is required to design a finite cutting plane proof for the validity of an inequality. We characterize s^* based on the faces of the linear relaxation of the mixed integer linear set.

Key words: mixed integer set ; lattice point free convex set ; cutting plane ; split closure

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1. Introduction. We consider a polyhedron in $\mathbb{R}^n$ of the form

$$P := \text{conv}(\{v^i\}_{i\in V}) + \text{cone}(\{r^j\}_{j\in E}), \tag{1}$$

where V and E are finite index sets, $\{v^i\}_{i\in V}$ denotes the vertices of P and $\{r^j\}_{j\in E}$ denotes the extreme rays of P. We assume P is rational, *i.e.*, we assume $\{r^j\}_{j\in E} \subset \mathbb{Z}^n$ and $\{v^i\}_{i\in V} \subset \mathbb{Q}^n$.

We are interested in points in P that have integer values on certain coordinates. For simplicity assume the first $p > 0$ coordinates must have integer values, and let $q := n - p$. The set $N_I := \{1, 2, \ldots, p\}$ is used to index the integer constrained variables and the set $P_I := \{x \in P : x_j \in \mathbb{Z} \text{ for all } j \in N_I\}$ denotes the mixed integer points in P.

The following concepts from convex analysis are needed (see [**?**] for a presentation of the theory of convex analysis). For a convex set $C \subseteq \mathbb{R}^n$, the interior of C is denoted $\text{int}(C)$, and the relative interior of C is denoted $\text{ri}(C)$ (where $\text{ri}(C) = \text{int}(C)$ when C is full dimensional).

We consider the generalization of *split sets* (see [**?**]) to lattice point free rational polyhedra (see [**?**]). A split set is of the form $S^{(\pi,\pi_0)} := \{x \in \mathbb{R}^p : \pi_0 \leq \pi^T x \leq \pi_0 + 1\}$, where $(\pi, \pi_0) \in \mathbb{Z}^{p+1}$ and $\pi \neq 0$. Clearly a split set does not have integer points in its interior. In general, a lattice point free convex set is a convex set that does not contain integer points in its relative interior. Lattice point free convex sets that are maximal wrt. inclusion are known to be polyhedra. We call lattice point free rational polyhedra that are maximal wrt. inclusion for *split polyhedra*. A split polyhedron is full dimensional and can be written as the sum of a polytope $\mathcal{P}$ and a linear space $\mathcal{L}$.

A lattice point free convex set is an object that assumes integrality of *all* coordinates. For *mixed* integrality in $\mathbb{R}^{p+q}$, we use a lattice point free convex set $C^x \subset \mathbb{R}^p$ to form a *mixed* integer lattice point free convex set $C \subset \mathbb{R}^n$ of the form $C := \{(x, y) \in \mathbb{R}^p \times \mathbb{R}^q : x \in C^x\}$. A *mixed integer split polyhedron* is then a polyhedron of the form $L := \{(x, y) \in \mathbb{R}^p \times \mathbb{R}^q : x \in L^x\}$, where L^x is a split polyhedron in $\mathbb{R}^p$.

An important measure in this paper of the size of a mixed integer split polyhedron L is the *facet width* of L. The facet width measures how wide a mixed integer split polyhedron is parallel to a given facet. Specifically, given any facet $\pi^T x \geq \pi_0$ of a mixed integer split polyhedron L, the width of L along π is defined to be the number $w(L,\pi) := \max_{x\in L}\pi^T x - \min_{x\in L}\pi^T x$. The *max-facet-width* of a mixed integer split polyhedron L measures how wide L is along any facet of L, *i.e.*, the max-facet-width $w_f(L)$ of L is defined to be the largest of the numbers $w(L,\pi)$ over *all* facet defining inequalities $\pi^T x \geq \pi_0$ for L.

Any mixed integer lattice point free convex set $C \subseteq \mathbb{R}^n$ gives a relaxation of $\mathrm{conv}(P_I)$

$$R(C,P) := \mathrm{conv}(P \setminus \mathrm{ri}(C))$$

that satisfies $\mathrm{conv}(P_I) \subseteq R(C,P) \subseteq P$. The set $R(C,P)$ might exclude fractional points in $\mathrm{ri}(C)\cap P$ and give a tighter approximation of $\mathrm{conv}(P_I)$ than P.

Mixed integer split polyhedra L give as tight relaxations of P_I of the form above as possible. Specifically, if $C, C' \subseteq \mathbb{R}^n$ are mixed integer lattice point free convex sets that satisfy $C \subseteq C'$, then $R(C',P) \subseteq R(C,P)$. For a general mixed integer lattice point free convex set C, the set $R(C,P)$ may not be a polyhedron. However, it is sufficient to consider mixed integer split polyhedra, and we show $R(L,P)$ is a polyhedron when L is a mixed integer split polyhedron (Lemma 2.3).

Observe that the set of mixed integer split polyhedra with max-facet-width equal to one are exactly the split sets $S^{(\pi,\pi_0)} = \{x \in \mathbb{R}^n : \pi_0 \leq \pi^T x \leq \pi_0 + 1\}$, where $(\pi,\pi_0) \in \mathbb{Z}^{n+1}$, $\pi_j = 0$ for $j > p$ and $\pi \neq 0$. In [**?**], Cook et. al. considered the set of split sets

$$\mathcal{L}^1 := \{L \subseteq \mathbb{R}^n : L \text{ is a mixed integer split polyhedron satisfying } w_f(L) \leq 1\}$$

and showed that the *split closure*

$$\mathrm{SC}^1 := \cap_{L\in\mathcal{L}^1} R(L,P)$$

is a polyhedron. A natural generalization of the split closure is to allow for mixed integer split polyhedra that have max-facet-width larger than one. For any $w > 0$, define the set of mixed integer split polyhedra

$$\mathcal{L}^w := \{L \subseteq \mathbb{R}^n : L \text{ is a mixed integer split polyhedron satisfying } w_f(L) \leq w\}$$

with max-facet-width at most w. We define the w^{th} *split closure* to be the set

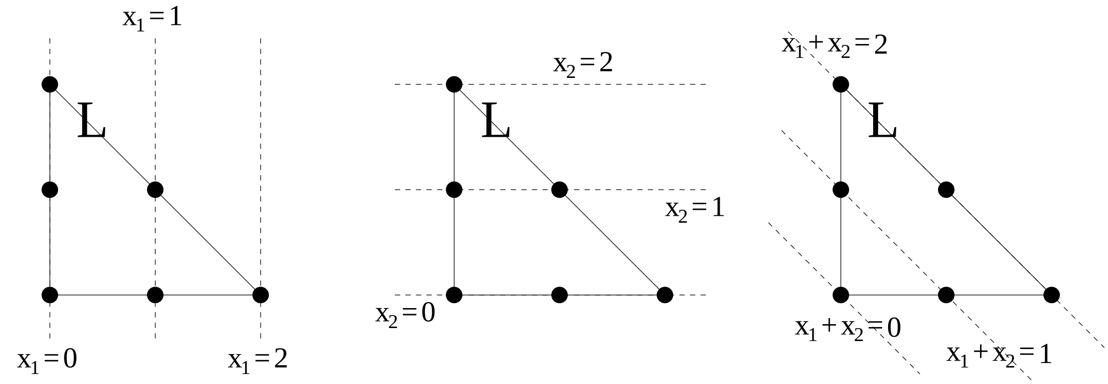
$$\mathrm{SC}^w := \cap_{L\in\mathcal{L}^w} R(L,P).$$

We prove that for any family $\bar{\mathcal{L}} \subseteq \mathcal{L}^w$ of mixed integer split polyhedra with bounded max-facet-width $w > 0$, the set $\cap_{L\in\bar{\mathcal{L}}} R(L,P)$ is a polyhedron (Theorem 4.3). The proof is an application of a more general result (Theorem 4.2) that gives a sufficient condition for the set $\cap_{L\in\bar{\mathcal{L}}} R(L,P)$ to be a polyhedron for *any* set $\bar{\mathcal{L}}$ of mixed integer split polyhedra. Many of our arguments are obtained by generalizing results of Andersen et. al. [**?**] from the first split closure to the w^{th} split closure.

Given a family $\{(\delta^l)^T x \geq \delta_0^l\}_{i\in I}$ of rational cutting planes, we also provide a sufficient condition for the set $\{x \in P : (\delta^l)^T x \geq \delta_0^l \text{ for all } l \in I\}$ to be a polyhedron (Theorem 3.1). This condition (Assumption 3.1) concerns the number of intersection points between hyperplanes defined from the cuts $\{(\delta^l)^T x \geq \delta_0^l\}_{i\in I}$ and line segments either of the form $\{v^i + \alpha r^j : \alpha \geq 0\}$, or of the form $\{\beta v^i + (1-\beta)v^k : \beta \in [0,1]\}$, where $i,k \in V$ denote two vertices of P and $j \in E$ denotes an extreme ray of P.

Finite cutting plane proofs of validity of an inequality for P_I can be designed by using mixed integer split polyhedra. Given a measure size(L) of the "size" or "complexity" of a mixed integer split polyhedron, a measure of the size of a finite cutting plane proof is the largest size s^* of a mixed integer split polyhedron used in the proof. Possible measures could be the max-lattice-width or the lattice width of L [**?**]. In fact, the function size(L) could also estimate the time complexity involved in using the mixed integer split polyhedron L in an algorithm. A measure of the size of a valid inequality $\delta^T x \geq \delta_0$ for P_I is then the smallest number $s_{(\delta,\delta_0)}$ for which there exists a finite cutting plane proof for the validity of $\delta^T x \geq \delta_0$ for P_I only using mixed integer split polyhedra of size at most $s_{(\delta,\delta_0)}$. We give a formula for $s_{(\delta,\delta_0)}$ (Theorem 5.1) that explains geometrically why mixed integer split polyhedra of large size can be necessary.

The remainder of the paper is organized as follows. In Sect. 2 we present the main results on lattice point free convex sets needed in the remainder of the paper. We also present the construction of polyhedral relaxations of P_I from mixed integer split polyhedra. Most results in Sect. 2 can also be found in a paper of Lovász [**?**]. In Sect. 3 we discuss cutting planes from the viewpoint of an inner representation of P. The main result in Sect. 3 is a sufficient condition for a set obtained by adding an infinite family of cutting planes to be a polyhedron. The structure of the relaxation $R(L,P)$ of P_I obtained from a given mixed integer split polyhedron L is characterized in Sect. 4. The main outcome is a sufficient condition for the set $\cap_{L\in\bar{\mathcal{L}}} R(L,P)$ to be a polyhedron, where $\bar{\mathcal{L}}$ is a family of mixed integer split polyhedra. We also apply this sufficient condition to show that the w^{th} split closure is a polyhedron. Finally, in Sect. 5, we discuss the complexity of finite cutting plane proofs for the validity of an inequality for P_I.



(a) L and lines $x_1 = \{0, 1, 2\}$ (b) L and lines $x_2 = \{0, 1, 2\}$ (c) L and lines $x_1 + x_2 = \{0, 1, 2\}$

Figure 1: The split polyhedron $L = \{x \in \mathbb{R}^2 : x_1, x_2 \geq 0 \text{ and } x_1 + x_2 \leq 2\}$

2. Lattice point free convex sets and polyhedral relaxations We now discuss the main object of this paper, namely lattice point free convex sets, which are defined as follows

Definition 2.1 *(Lattice point free convex sets)*
Let $L \subseteq \mathbb{R}^p$ be a convex set. If $\mathrm{ri}(L) \cap \mathbb{Z}^p = \emptyset$*, then L is called* lattice point free.

The discussion of lattice point free convex sets in this section is based on a paper of Lovász [**?**]. We are mainly interested in lattice point free convex sets that are maximal wrt. inclusion. Our point of departure is the following characterization of maximal lattice point free convex sets.

Lemma 2.1 *Every maximal lattice point free convex set $L \subseteq \mathbb{R}^p$ is a polyhedron.*

As mentioned in the introduction, we call maximal lattice point free rational polyhedra for *split polyhedra*. Figure 1 gives an example of a split polyhedron L. Maximal lattice point free polyhedra are not necessarily rational polyhedra. The polyhedron $Q := \{(x_1, x_2) : x_2 = x_1\sqrt{2}, x_1 \geq 0\}$ is an example of a maximal lattice point free set which is *not* a rational polyhedron. However, we will only use maximal lattice point free convex sets to describe (mixed) integer points in rational polyhedra, and for this purpose split polyhedra suffice.

We next argue that the recession cone $0^+(L)$ of a split polyhedron L must be a linear space. This fact follows from the following operation to enlarge any lattice point free convex set $C \subseteq \mathbb{R}^p$. Let $r \in 0^+(C) \cap \mathbb{Q}^p$ be a rational vector in the recession cone of C. We claim that also $C' = C + \mathrm{span}(\{r\})$ is lattice point free. Indeed, if $\bar{x} - \mu r \in \mathrm{ri}(C')$ is integer with $\mu > 0$ and $\bar{x} \in \mathrm{ri}(C)$, then there exists a positive integer $\mu^I > \mu$ such that $\bar{x} - \mu r + \mu^I r = \bar{x} + (\mu^I - \mu)r \in \mathrm{ri}(C) \cap \mathbb{Z}^p$, which contradicts that C is lattice point free. Since the recession cone of a split polyhedron is rational, we therefore have

Lemma 2.2 *Let $L \subseteq \mathbb{R}^p$ be a split polyhedron. Then L can be written in the form $L = \mathcal{P} + \mathcal{L}$, where $\mathcal{P} \subseteq \mathbb{R}^p$ is a rational polytope and $\mathcal{L} \subseteq \mathbb{R}^p$ is a linear space with an integer basis.*

Observe that Lemma 2.2 implies that every split polyhedron $L \subseteq \mathbb{R}^p$ is full dimensional. Indeed, if this was not the case, then we would have $L \subseteq \{x : \mathbb{R}^p : \pi^T x = \pi_0\}$ for some $(\pi, \pi_0) \in \mathbb{Z}^{p+1}$ which implies $L \subseteq \{x : \mathbb{R}^p : \pi_0 \leq \pi^T x \leq \pi_0 + 1\}$, and this contradicts that L is maximal lattice point free.

Observation 2.1 *Every split polyhedron L in $\mathbb{R}^p$ is full dimensional.*

We are interested in using split polyhedra to characterize *mixed* integer sets. Let $L^x \subseteq \mathbb{R}^p$ be a split polyhedron. We can then use the set $L := \{(x, y) \in \mathbb{R}^p \times \mathbb{R}^q : x \in L^x\}$ for mixed integer sets, since L does not contain any *mixed* integer points $(x, y) \in P_I$. We call L a *mixed integer split polyhedron*. Let

$\Pi := \{(\pi, \pi_0) \in \mathbb{Z}^{n+1} : \pi_j = 0 \text{ for } j \notin N_I\}$. Every mixed integer split polyhedron $L \subseteq \mathbb{R}^n$ can be written in the form

$$L := \{x \in \mathbb{R}^n : (\pi^k)^T x \geq \pi_0^k \text{ for } k \in F(L)\},$$

where $F(L)$ is a finite index set for the facets of L, $(\pi^k, \pi_0^k) \in \Pi$ and $\gcd(\pi^k, \pi_0^k) = 1$. Note that, since L is full dimensional, this representation of L is unique.

We now consider how to measure the size of a mixed integer split polyhedron. Given a vector $\pi \in \mathbb{Z}^n$ satisfying $\pi_j = 0$ for $j \notin N_I$, the number of parallel hyperplanes $\pi^T x = \pi_0$ that intersect a mixed integer split polyhedron $L \subseteq \mathbb{R}^n$ for varying $\pi_0 \in \mathbb{R}$ gives a measure of how wide L is along the vector π. The *width* of L along a vector π is defined to be the number

$$w(L, \pi) := \max_{x \in L} \pi^T x - \min_{x \in L} \pi^T x.$$

By considering the width of L along all the facets of L, and choosing the largest of these numbers, we obtain a measure of how wide L is.

Definition 2.2 *(The max-facet-width of a mixed integer split polyhedron).*
Let $L = \{x \in \mathbb{R}^n : (\pi^k)^T x \geq \pi_0^k$ for $k \in F(L)\}$ be a mixed integer split polyhedron, where $F(L)$ is an index set for the facets of L, $(\pi^k, \pi_0^k) \in \Pi$ and gcd$(\pi^k, \pi_0^k) = 1$. The max-facet-width *of L is the number:*

$$w_f(L) := \max\{w(L, \pi^k) : k \in F(L)\}.$$

Example 2.1 *Figure 1 gives an example that demonstrates how to compute the max-facet-width of the split polyhedron $L = \{x \in \mathbb{R}^2 : x_1, x_2 \geq 0$ and $x_1 + x_2 \leq 2\}$. The split polyhedron L has three facets $(\pi^k)^T x \geq \pi_0^k$ for $k \in F(L) = \{1, 2, 3\}$ given by $(\pi^1, \pi_0^1) = (1, 0, 0)$, $(\pi^2, \pi_0^2) = (0, 1, 0)$ and $(\pi^3, \pi_0^3) = (-1, -1, -2)$.*

The width of L along π^1 is given by $w(L, \pi^1) = \max_{x \in L} (\pi^1)^T x - \min_{x \in L} (\pi^1)^T x = \max_{x \in L} x_1 - \min_{x \in L} x_1 = 2 - 0 = 2$. As can be seen from Figure 1(a), the optimal solutions to the problem $\max_{x \in L} (\pi^1)^T x = \max_{x \in L} x_1$ are given by the intersection of L with the hyperplane $x_1 = 2$, and the optimal solutions to the problem $\min_{x \in L} (\pi^1)^T x = \min_{x \in L} x_1$ are given by the intersection of L with the hyperplane $x_1 = 0$. In general, the width $w(L, \pi^1)$ of L along π^1 is determined by the parallel hyperplanes $(\pi^1)^T x = k$ for varying values of $k \in \mathbb{R}$.

With similar computations, and by considering Figure 1(b) and Figure 1(c), we obtain that $w(L, \pi^2) = w(L, \pi^3) = 2$. Since the max-facet-width $w_f(L)$ of L is the largest of the numbers $w(L, \pi^1)$, $w(L, \pi^2)$ and $w(L, \pi^3)$, we obtain $w_f(L) = 2$.

2.1 Polyhedral relaxations from mixed integer split polyhedra As mentioned in the introduction, any mixed integer lattice point free convex set $C \subseteq \mathbb{R}^n$ gives a relaxation of $\text{conv}(P_I)$

$$R(C, P) := \text{conv}(P \setminus \text{ri}(C))$$

that satisfies $\text{conv}(P_I) \subseteq R(C, P) \subseteq P$. Since mixed integer split polyhedra L are maximal wrt. inclusion, the sets $R(L, P)$ for mixed integer split polyhedra L are as tight relaxations as possible wrt. this operation.

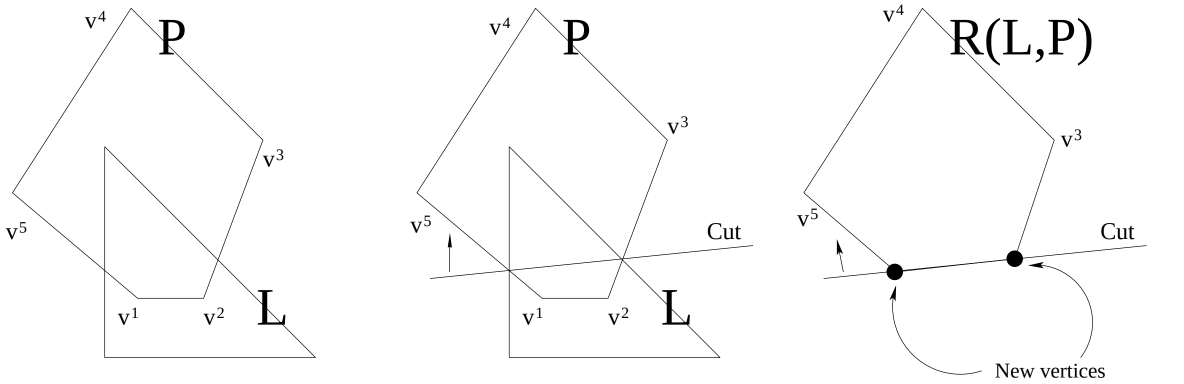
Figure 2 demonstrates the operation $R(L, P)$ for a polytope P with five vertices and a split polyhedron L. Observe that the set of points in $P \cap \text{int}(L)$ that are below the cut in Figure 2(b) are exactly those points in $P \cap \text{int}(L)$ that can not be expressed as a convex combination of points in $P \setminus \text{int}(L)$.

For the example in Figure 2, the set $R(L, P)$ is a polyhedron. We now show that, in general, mixed integer split polyhedra give polyhedral relaxations $R(L, P)$ of P_I.

Lemma 2.3 *Let $L \subseteq \mathbb{R}^n$ be a full dimensional rational polyhedron whose recession cone $0^+(L)$ is a linear space, and let P be a rational polyhedron. Then the following set $R(L, P)$ is a polyhedron.*

$$R(L, P) := \text{conv}(P \setminus \text{int}(L)).$$

Proof. We assume $L = \{x \in \mathbb{R}^n : (\pi^k)^T x \geq \pi_0^k \text{ for } k \in F\}$ and $P = \{x \in \mathbb{R}^n : Dx \leq d\}$, where F is an index set for the facets of L, $(\pi^k, \pi_0^k) \in \mathbb{Z}^{n+1}$ and $\gcd(\pi^k, \pi_0^k) = 1$ for $k \in F$, $D \in \mathbb{Q}^{m \times n}$ and $d \in \mathbb{Q}^m$. Observe that L has the property that, if $y \in 0^+(L)$, then $(\pi^k)^T y = 0$ for all $k \in F$. This follows from



(a) The linear relaxation P of P_I and a split polyhedron L

(b) The only cut that can be derived from L and P

(c) The strengthened relaxation $R(L,P)$ of P_I

Figure 2: Strengthening the linear relaxation P of P_I by using a split polyhedron L

the fact that the recession cone $0^+(L)$ of L is a linear space. We claim $R(L,P)$ is the projection of the following polyhedron onto the space of x-variables.

$$x = \sum_{k \in F} x^k, \tag{2}$$

$$Dx^k \leq \lambda^k d, \quad \text{for } k \in F, \tag{3}$$

$$(\pi^k)^T x^k \leq \lambda^k \pi_0^k, \quad \text{for } k \in F, \tag{4}$$

$$\sum_{i \in F} \lambda^k = 1, \tag{5}$$

$$\lambda^k \geq 0, \qquad \text{for } k \in F. \tag{6}$$

The above construction was also used by Balas for disjunctive programming [?]. Let $S(L,P)$ denote the set of $x \in \mathbb{R}^n$ that can be represented in the form (2)-(6) above. We need to prove $R(L,P) = S(L,P)$. A result in Cornuéjols [?] shows that $\mathrm{cl}(\mathrm{conv}(\cup_{k\in F} P^k)) = \mathrm{cl}(R(L,P)) = S(L,P)$, where P^k is defined by $P^k := \{x \in P : (\pi^k)^T x \leq \pi_0^k\}$ for $k \in F$. It follows that $R(L,P) \subseteq S(L,P)$.

We now show $S(L,P) \subseteq R(L,P)$. Let $\bar{x} \in S(L,P)$. By definition this means there exists $\bar{F} \subseteq F$, $\{\bar{x}^k\}_{k\in\bar{F}}$ and $\{\bar{\lambda}^k\}_{k\in\bar{F}}$ s.t. $\bar{x}$, $\{\bar{x}^k\}_{k\in\bar{F}}$ and $\{\bar{\lambda}^k\}_{k\in\bar{F}}$ satisfy (2)-(6). We can assume $|\bar{F}| \geq 2$, and that $|\bar{F}|$ is chosen as small as possible. Define $\bar{F}^0 := \{k \in \bar{F} : \bar{\lambda}^k = 0\}$. We prove $\bar{x} \in R(L,P)$ in three steps.

(a) For all $k_0 \in \bar{F}^0$, there exists $k \in F$ such that $(\pi^k)^T \bar{x}^{k_0} < 0$:
Let $k_0 \in \bar{F}^0$ be arbitrary. Suppose, for a contradiction, that $(\pi^k)^T \bar{x}^{k_0} \geq 0$ for all $k \in F$. This implies $\bar{x}^{k_0} \in 0^+(L)$, and therefore $(\pi^k)^T \bar{x}^{k_0} = 0$ for all $k \in F$. We now show this contradicts the assumption that $|\bar{F}|$ is chosen as small as possible. Indeed, choose $\bar{k} \in \bar{F} \setminus \{k_0\}$ arbitrarily. Define $y^{\bar{k}} := \bar{x}^{k_0} + \bar{x}^{\bar{k}}$ and $y^k := \bar{x}^k$ for $k \in \bar{F} \setminus \{k_0, \bar{k}\}$. We have that $\bar{x}$, $\{y^k\}_{k\in\bar{F}\setminus\{k_0\}}$ and $\{\bar{\lambda}^k\}_{k\in\bar{F}\setminus\{k_0\}}$ satisfy (2)-(6), which contradicts the minimality of $|\bar{F}|$. Therefore there exists $k \in F$ such that $(\pi^k)^T \bar{x}^{k_0} < 0$.

(b) We have $\bar{x}^{k_0} \in 0^+(R(L,P))$ for all $k_0 \in \bar{F}^0$:
Let $k_0 \in \bar{F}^0$ and $r \in R(L,P)$ be arbitrary. Consider the points $x(\alpha) := r + \alpha \bar{x}^{k_0}$ on the halfline $\{x(\alpha) : \alpha \geq 0\}$ starting from r in the direction $\bar{x}^{k_0}$. From (1) it follows that there exists $k \in F$ such that $(\pi^k)^T \bar{x}^{k_0} < 0$. Since $(\pi^k)^T \bar{x}^{k_0} < 0$, there exists $\bar{\alpha} > 0$ such that $(\pi^k)^T x(\alpha) \leq \pi_0^k$ for all $\alpha \geq \bar{\alpha}$. This implies $\bar{x}^{k_0}(\alpha) \in P^k \subseteq R(L,P)$ for all $\alpha \geq \bar{\alpha}$, and therefore $\bar{x}^{k_0} \in 0^+(R(L,P))$.

(c) $\bar{x} \in R(L,P)$:
We can write $\bar{x} = \sum_{k\in\bar{F}^+} \bar{\lambda}^k \frac{\bar{x}^k}{\bar{\lambda}^k} + \sum_{k\in\bar{F}^0} \bar{x}^k$, where $\bar{F}^+ := \{k \in \bar{F} : \bar{\lambda}^k > 0\}$. Since $\frac{\bar{x}^k}{\bar{\lambda}^k} \in P^k \subseteq R(L,P)$ for $k \in \bar{F}^+$, $\bar{x}^k \in 0^+(R(L,P))$ for $k \in \bar{F}^0$ (from (b)) and $\sum_{k\in\bar{F}^+} \bar{\lambda}^k = 1$, we have $\bar{x} \in R(L,P)$. □

Lemma 2.3 implies that for every *finite* collection $\mathcal{L}$ of mixed integer split polyhedra, the set

$$\mathrm{Cl}(P,\mathcal{L}) := \cap_{L\in\mathcal{L}} R(L,P),$$

is a polyhedron. A next natural question is under which conditions the same is true for an *infinite* collection of mixed integer split polyhedra. As mentioned, in Sect. 4 we will show that a sufficient condition for this to be the case is that it is possible to provide an upper bound w^* on the max-facet-width of the mixed integer split polyhedra in the infinite collection $\mathcal{L}$ of mixed integer split polyhedra. We therefore consider the family of *all* mixed integer split polyhedra whose max-facet-width is bounded by a given constant $w > 0$

$$\mathcal{L}^w := \{L \subseteq \mathbb{R}^n : L \text{ is a mixed integer split polyhedron satisfying } w_f(L) \leq w\}.$$

An extension of the (first) split closure can now be defined.

Definition 2.3 *(The w^{th} split closure).*
Given $w > 0$, the w^{th} split closure *of P is defined to be the set*

$$Cl_w(P, \mathcal{L}^w) := \cap_{L \in \mathcal{L}^w} R(L, P).$$

An important property of mixed integer split polyhedra, which will be used heavily in the remainder of the paper, is that the extreme rays of $R(L, P)$ are the same as the extreme rays of P.

Lemma 2.4 *Let L be a mixed integer split polyhedron, and assume $R(L, P) \neq \emptyset$. The extreme rays of $R(L, P)$ are the same as the extreme rays of P, i.e., we have $0^+(R(L, P)) = \text{cone}(\{r^j\}_{j \in E})$.*

Proof. Let $j \in E$ and $\bar{x} \in P \setminus \text{int}(L)$ be arbitrary. If $\bar{x} + \alpha r^j \notin \text{int}(L)$ for all $\alpha \geq 0$, we are done. Therefore assume there exists $\alpha' > 0$ such that $\bar{x} + \alpha' r^j \in \text{int}(L)$. We cannot have that $\{\bar{x} + \alpha r^j : \alpha \geq \alpha'\} \subseteq \text{int}(L)$. Indeed, this would imply $r^j \in 0^+(L)$, and since $0^+(L)$ is a linear space, this implies $\{\bar{x} + \alpha r^j : \alpha \in \mathbb{R}\} \subseteq \text{int}(L)$, which contradicts $\bar{x} \notin \text{int}(L)$. Hence there exists $\alpha'' > \alpha'$ such that $\bar{x} + \alpha r^j \notin \text{int}(L)$ for all $\alpha \geq \alpha''$. Now, any point on the line segment $\{\bar{x} + \alpha r^j : 0 \leq \alpha \leq \alpha''\}$ is a convex combination of $\bar{x} \notin \text{int}(L)$ and $\bar{x} + \alpha'' r^j \notin \text{int}(L)$, and therefore $\{\bar{x} + \alpha r^j : \alpha \geq 0\} \subseteq R(L, P)$. □

Not all lattice point free rational polyhedra L have the property that $R(L, P)$ has the same extreme rays as P. Consider the example with $P = \{\binom{0}{\frac{1}{2}} + \alpha\binom{1}{0} : \alpha \geq 0\}$ and $L = \{x \in \mathbb{R}^2 : x \geq 0 \text{ and } x_2 \leq 1\}$. In this case $0^+(R(L, P)) = \{0\}$ and $0^+(P) = \text{cone}(\{(\binom{1}{0})\})$. The reason for the difference between $0^+(R(L, P))$ and $0^+(P)$ in this example is that L is not maximal lattice point free.

Another question is which condition a mixed integer split polyhedron L must satisfy in order to have $R(L, P) \neq P$. The following lemma shows that $R(L, P) \neq P$ exactly when there is a vertex of P in the interior of L. For the example in Figure 2, we have $v^1, v^2 \in \text{int}(L)$, and therefore $R(L, P) \neq P$.

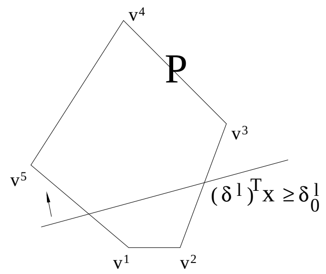
Lemma 2.5 *Let $L \subset \mathbb{R}^n$ be a mixed integer split polyhedron. Then $R(L, P) \neq P$ if and only if there is a vertex of P in the interior of L.*

Proof. If v^i is a vertex of P in the interior of L, where $i \in V$, then v^i can not be expressed as a convex combination of points in P that are not in the interior of L, and therefore $v^i \notin R(L, P)$. Conversely, when L does *not* contain a vertex of P in its interior, then $\delta^T v^i \geq \delta_0$ for every valid inequality $\delta^T x \geq \delta_0$ for $R(L, P)$ and $i \in V$. Since the extreme rays of $R(L, P)$ are the same as the extreme rays of P, we have $\delta^T r^j \geq 0$ for every extreme ray $j \in E$. □

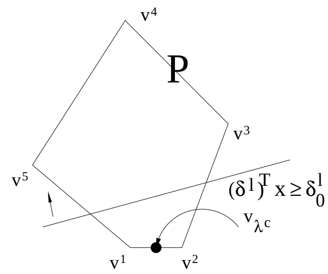
3. Cutting planes and inner representations of polyhedra The focus in this section is on analyzing the effect of adding cutting planes (or cuts) to the linear relaxation P of P_I from the viewpoint of an inner representation of P. We define cuts to be inequalities that cut off some vertices of P. In other words, we say an inequality $\delta^T x \geq \delta_0$ is a cut for P if $\delta^T v^i < \delta_0$ for some $i \in V$.

A cut $\delta^T x \geq \delta_0$ is called *non-negative* if $\delta^T r^j \geq 0$ for all extreme rays $j \in E$. Throughout this section we only consider non-negative cuts. Observe that non-negativity is necessary for valid cuts for the mixed integer set P_I. Indeed, if $\delta^T x \geq \delta_0$ is valid for P_I, and $j \in E$ is an extreme ray of P, then given a mixed integer point $x^I \in P_I$, the halfline $\{x^I + \mu r^j : \mu \geq 0\}$ contains an infinite number of mixed integer points belonging to P_I. Therefore, if we had $\delta^T r^j < 0$, this would contradict the validity of $\delta^T x \geq \delta_0$ for P_I.

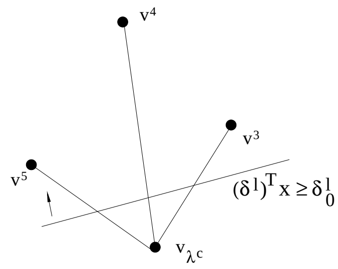
We will use the following notation. The set $\{(\delta^l)^T x \geq \delta_0^l\}_{l \in I}$ denotes an arbitrary family of non-negative cutting planes for P. Given a cut $l \in I$, the set $V_l^c := \{i \in V : (\delta^l)^T v^i < \delta_0^l\}$ is used to index



(a) The polytope P from Figure 2 and a cut with $V_l^c = \{1,2\}$



(b) The convex combination v_{λ^c} of v^1 and v^2 for $\lambda_1^c = \lambda_2^c = \frac{1}{2}$



(c) The line segments that determine $\beta'_{k,l}(\lambda^c)$ for $k = 3,4,5$

Figure 3: Determining the intersection points from a polytope P and a cut $(\delta^l)^T x \geq \delta_0^l$

the vertices of P that are cut off by $(\delta^l)^T x \geq \delta_0^l$ (where the superscript "c" is an abbreviation of the word "cut"), and $V_l^s := \{i \in V : (\delta^l)^T v^i \geq \delta_0^l\}$ is used to index the vertices of P that satisfy the cut $(\delta^l)^T x \geq \delta_0^l$ (where the superscript "s" is an abbreviation of the word "satisfied"). For the example in Figure 3(a), we have $V_l^c = \{1,2\}$ and $V_l^s = \{3,4,5\}$.

3.1 The new vertices created by the addition of a cut Adding a non-negative cut $(\delta^l)^T x \geq \delta_0^l$ to the linear relaxation P of P_I creates a polyhedron with different vertices than P. We now describe these new vertices that are created. We let $\Lambda := \{\lambda \in \mathbb{R}_+^{|V|} : \sum_{i\in V} \lambda_i = 1\}$ and $\Lambda_l^c := \{\lambda \in \Lambda : \sum_{i\in V_l^c} \lambda_i = 1\}$ denote the multipliers that are used when forming convex combinations of the vertices of P, and the multipliers that are used when forming convex combinations of vertices that are cut off by $(\delta^l)^T x \geq \delta_0^l$ respectively. Also, for any $\lambda \in \Lambda$, define $v_\lambda := \sum_{i\in V} \lambda_i v^i$, and for any $\mu \in \mathbb{R}_+^{|E|}$, define $r_\mu := \sum_{j\in E} \mu_j r^j$. We first argue that the new vertices that are created by adding $(\delta^l)^T x \geq \delta_0^l$ to P are *intersection points* [**?**]. Intersection points are crucial for the polyhedrality result we present in Sect. 3.3 (Theorem 3.1). Intersection points are defined as follows. Given an extreme ray $j \in E$ that satisfies $(\delta^l)^T r^j > 0$, and a convex combination $\lambda^c \in \Lambda_l^c$ of the vertices that are cut off by $(\delta^l)^T x \geq \delta_0^l$, the halfline $\{v_{\lambda^c} + \alpha r^j : \alpha \geq 0\}$ intersects the hyperplane $\{x \in \mathbb{R}^n : (\delta^l)^T x = \delta_0^l\}$. For $j \in E$ and $\lambda^c \in \Lambda_l^c$, define

$$\alpha'_{j,l}(\lambda^c) := \begin{cases} \frac{\delta_0^l - (\delta^l)^T v_{\lambda^c}}{(\delta^l)^T r^j} & \text{if } (\delta^l)^T r^j > 0, \\ +\infty & \text{otherwise.} \end{cases} \tag{7}$$

The number $\alpha'_{j,l}(\lambda^c)$ is the value of α for which $v_{\lambda^c} + \alpha r^j$ is on the hyperplane $(\delta^l)^T x = \delta_0^l$. When there is no such point, we define $\alpha'_{j,l}(\lambda^c) = +\infty$. If $\alpha'_{j,l}(\lambda^c) < +\infty$, the point $v_{\lambda^c} + \alpha'_{j,l}(\lambda^c) r^j$ is called the intersection point associated with the convex combination $\lambda^c \in \Lambda_l^c$ and the extreme ray r^j of P.

Given a convex combination $\lambda^c \in \Lambda_l^c$, and a vertex $k \in V_l^s$ that satisfies the inequality $(\delta^l)^T x \geq \delta_0^l$, the line segment between v_{λ^c} and v^k intersects the hyperplane $\{x \in \mathbb{R}^n : (\delta^l)^T x = \delta_0^l\}$. For $k \in V_l^s$ and $\lambda^c \in \Lambda_l^c$, define

$$\beta'_{k,l}(\lambda^c) := \frac{\delta_0^l - (\delta^l)^T v_{\lambda^c}}{(\delta^l)^T (v^k - v_{\lambda^c})}. \tag{8}$$

The number $\beta'_{k,l}(\lambda^c)$ denotes the value of β for which the point $v_{\lambda^c} + \beta(v^k - v_{\lambda^c})$ is on the hyperplane $(\delta^l)^T x = \delta_0^l$. Observe that $\beta'_{k,l}(\lambda^c) \in]0,1]$. The point $v_{\lambda^c} + \beta'_{k,l}(\lambda^c)(v^k - v_{\lambda^c})$ is called the intersection point associated with the convex combination $\lambda^c \in \Lambda_l^c$ and the vertex v^k of P. For the polytope P of Figure 2 and a cut $(\delta^l)^T x \geq \delta_0^l$, Figure 3 gives an example of how to compute the intersection points for a given convex combination $\lambda^c = (\frac{1}{2}, \frac{1}{2}, 0, 0, 0)$.

It is not all convex combinations $\lambda^c \in \Lambda_l^c$ and vertices $k \in V_l^s$ that lead to interesting intersection points. Specifically, some intersection points may be convex combinations of other intersection points. The following lemma shows that, in this sense, the only vectors $\lambda^c \in \Lambda_l^c$ for which intersection points of the type $v_{\lambda^c} + \beta'_{k,l}(\lambda^c)(v^k - v_{\lambda^c})$ are interesting are those for which λ^c is a unit vector.

LEMMA 3.1 *Let $(\delta^l)^T x \geq \delta_0^l$ be a cut, where $l \in I$, and let $k \in V_l^s$. For every $\lambda^c \in \Lambda_l^c$, the intersection point $v_{\lambda^c} + \beta'_{k,l}(\lambda^c)(v^k - v_{\lambda^c})$ is a convex combination of the intersection points $v^i + \beta'_{k,l}(e^i)(v^k - v^i)$ for $i \in V_l^c$, where e^i denotes the unit vector in $\mathbb{R}^{|V|}$ corresponding to i.*

PROOF. Define $C := \text{conv}(\{v^k\} \cup \{v^i\}_{i \in V_l^c})$. Trivially we have $v_{\lambda^c} + \beta'_{k,l}(\lambda^c)(v^k - v_{\lambda^c}) \in C$. We will show the vertices of the polytope $\{x \in C : (\delta^l)^T x = \delta_0^l\}$ are given by the points $v^i + \beta'_{k,l}(e^i)(v^k - v^i)$ for $i \in V_l^c$ from which the result follows. If $(\delta^l)^T v^k = \delta_0^l$, the result is trivial, so we assume $(\delta^l)^T v^k > \delta_0^l$.

Therefore suppose $\bar{x}$ is a vertex of $\{x \in C : (\delta^l)^T x = \delta_0^l\}$. We may write $\bar{x} = \lambda_0 v^k + \sum_{i \in V_l^c} \lambda_i v^i$, where $\lambda_0 + \sum_{i \in V_l^c} \lambda_i = 1$ and $\lambda_i \geq 0$ for all $i \in \{0\} \cup V_l^c$. Using $\lambda_0 = 1 - \sum_{i \in V_l^c} \lambda_i$, we can write $\bar{x} = v^k + \sum_{i \in V_l^c} \lambda_i (v^i - v^k)$. Multiplying with δ^l on both sides gives $\sum_{i \in V_l^c} \frac{\lambda_i}{\eta_{i,k}} = 1$, where $\eta_{i,k} := \frac{(\delta^l)^T v^k - \delta_0^l}{(\delta^l)^T (v^k - v^i)}$. We can now write $\bar{x} = v^k + \sum_{i \in V_l^c} \lambda_i (v^i - v^k) = \sum_{i \in V_l^c} \frac{\lambda_i}{\eta_{i,k}} v^k + \sum_{i \in V_l^c} \lambda_i (v^i - v^k) = \sum_{i \in V_l^c} \frac{\lambda_i}{\eta_{i,k}} (v^k + \eta_{i,k}(v^i - v^k))$. Since $v^k + \eta_{i,k}(v^i - v^k) = v^i + \beta'_{k,l}(e^i)(v^k - v^i)$ for $i \in V_l^c$, the result follows. $\square$

We next give a representation of $\{x \in P : (\delta^l)^T x \geq \delta_0^l\}$ in a higher dimensional space. Note that any point that is a convex combination of the vertices of P can be written as a convex combination of a point that satisfies $(\delta^l)^T x \geq \delta_0^l$, and the a point that is cut off by $(\delta^l)^T x \geq \delta_0^l$. We may write P in the form

$$P = \{x \in \mathbb{R}^n : x = v_{\lambda^c} + \sum_{k \in V_l^s} \epsilon_k (v^k - v_{\lambda^c}) + r_\mu, \text{ where } \epsilon, \mu \geq 0, \ \lambda^c \in \Lambda_l^c \text{ and } \sum_{k \in V_l^s} \epsilon_k \leq 1\}.$$

Consider the set obtained from P by fixing the convex combination $\lambda^c \in \Lambda_l^c$

$$P(\lambda^c) = \{x \in \mathbb{R}^n : x = v_{\lambda^c} + \sum_{k \in V_l^s} \epsilon_k (v^k - v_{\lambda^c}) + r_\mu, \text{ where } \epsilon, \mu \geq 0 \text{ and } \sum_{k \in V_l^s} \epsilon_k \leq 1\}.$$

Observe that the set $\tilde{P}(\lambda^c)$ obtained from $P(\lambda^c)$ by deleting the inequality $\sum_{k \in V_l^s} \epsilon_k \leq 1$ from the above description of $P(\lambda^c)$ is a translate of a polyhedral cone. Furthermore, if λ^c is a unit vector, then $\tilde{P}(\lambda^c)$ is a relaxation of $P(\lambda^c)$. Now consider the set $P^H(\lambda^c)$ obtained from $P(\lambda^c)$ by including in the description the multipliers on the vertices of P indexed by V_l^s, and the multipliers on the extreme rays of P

$$P^H(\lambda^c) := \{(x, \epsilon, \mu) \in \mathbb{R}^{n+|V_l^s|+|E|} : x = v_{\lambda^c} + \sum_{k \in V_l^s} \epsilon_k (v^k - v_{\lambda^c}) + r_\mu, \text{ where } \epsilon, \mu \geq 0 \text{ and } \sum_{k \in V_l^s} \epsilon_k \leq 1\}.$$

The letter "H" is used to emphasize that $P^H(\lambda^c)$ is an image of $P(\lambda^c)$ in a higher dimensional space. The scalars $\alpha'_{j,l}(\lambda^c)$ and $\beta'_{k,l}(\lambda^c)$ for $(j,k) \in E \times V_l^s$ give an alternative description of $\{x \in P(\lambda^c) : (\delta^l)^T x \geq \delta_0^l\}$ in this higher dimensional space.

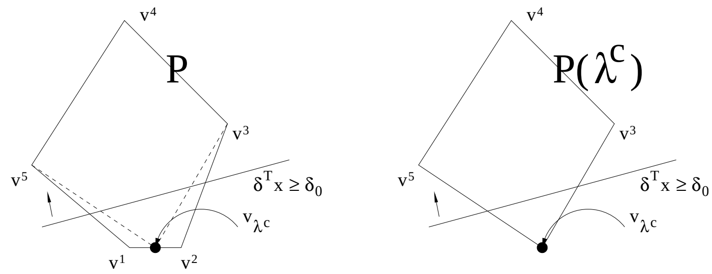
LEMMA 3.2 *([?, Lemma 2]). Let $l \in I$ be a non-negative cut for P. For any $\lambda^c \in \Lambda_l^c$, we have*

$$\{(x, \epsilon, \mu) \in P^H(\lambda^c) : \delta^T x \geq \delta_0\} = \{(x, \epsilon, \mu) \in P^H(\lambda^c) : \sum_{j \in E} \frac{\mu_j}{\alpha'_{j,l}(\lambda^c)} + \sum_{k \in V_l^s} \frac{\epsilon_k}{\beta'_{k,l}(\lambda^c)} \geq 1\}.$$

PROOF. We have $(\bar{x}, \bar{\epsilon}, \bar{\mu}) \in P^H(\lambda^c)$ and $(\delta^l)^T \bar{x} \geq \delta_0^l \iff \bar{x} = v_{\lambda^c} + \sum_{k \in V_l^s} \bar{\epsilon}_k (v^k - v_{\lambda^c}) + r_{\bar{\mu}}$, where $\bar{\epsilon}, \bar{\mu} \geq 0$, $\sum_{k \in V_l^s} \bar{\epsilon}_k \leq 1$ and $(\delta^l)^T \bar{x} \geq \delta_0^l \iff \bar{x} = v_{\lambda^c} + \sum_{k \in V_l^s} \bar{\epsilon}_k (v^k - v_{\lambda^c}) + r_{\bar{\mu}}$, $\bar{\epsilon}, \bar{\mu} \geq 0$, $\sum_{k \in V_l^s} \bar{\epsilon}_k \leq 1$ and $\sum_{k \in V_l^s} \bar{\epsilon}_k (\delta^l)^T (v^k - v_{\lambda^c}) + \sum_{j \in E} \bar{\mu}_j ((\delta^l)^T r^j) \geq (\delta_0^l - (\delta^l)^T v_{\lambda^c}) \iff (\bar{x}, \bar{\epsilon}, \bar{\mu}) \in P^H(\lambda^c)$ and $\sum_{k \in V_l^s} \bar{\epsilon}_k / \beta'_{k,l}(\lambda^c) + \sum_{j \in E} \bar{\mu}_j / \alpha'_{j,l}(\lambda^c) \geq 1$. $\square$

Observe that the intersection points $v_{\lambda^c} + \alpha'_{j,l}(\lambda^c) r^j$ for $j \in E$, and the intersection points $v_{\lambda^c} + \beta'_{j,l}(\lambda^c)(v^k - v_{\lambda^c})$ for $k \in V_l^s$, satisfy the inequality $\sum_{k \in V_l^s} \frac{\epsilon_k}{\beta'_{k,l}(\lambda^c)} + \sum_{j \in E} \frac{\mu_j}{\alpha'_{j,l}(\lambda^c)} \geq 1$ with equality.

Also observe that the inequality $\sum_{k \in V_l^s} \frac{\epsilon_k}{\beta'_{k,l}(\lambda^c)} + \sum_{j \in E} \frac{\mu_j}{\alpha'_{j,l}(\lambda^c)} \geq 1$ may be viewed as an intersection cut [?]. Specifically, suppose C is a full-dimensional convex set satisfying : (i) v_{λ^c} is in the interior of C, (ii) the intersection points $v_{\lambda^c} + \alpha'_{j,l}(\lambda^c) r^j$ for $j \in E$ are on the boundary of C and (iii) the intersection points $v_{\lambda^c} + \beta'_{k,l}(\lambda^c)(v^k - v_{\lambda^c})$ for $k \in V_l^k$ are on the boundary of C. Then $\sum_{k \in V_l^s} \frac{\epsilon_k}{\beta'_{k,l}(\lambda^c)} + \sum_{j \in E} \frac{\mu_j}{\alpha'_{j,l}(\lambda^c)} \geq 1$ is the intersection cut obtained from C and the translated cone $\tilde{P}(\lambda^c)$ defined above.



(a) v_{λ^c} and line segments from v_λ to the vertices satisfying $(\delta^l)^T x \geq \delta_0^l$ (b) The set $P(\lambda^c)$ constructed from λ^c

Figure 4: Constructing the set $P(\lambda^c)$ from the convex combination v_{λ^c}

Based on the above result, we can now characterize the vertices of $\{x \in P : (\delta^l)^T x \geq \delta_0^l\}$. Specifically we show that every vertex of $\{x \in P : (\delta^l)^T x \geq \delta_0^l\}$ is either a vertex of P that satisfies $(\delta^l)^T x \geq \delta_0^l$, or an intersection point obtained from a vertex of P that violates $(\delta^l)^T x \geq \delta_0^l$.

Lemma 3.3 *Let $l \in I$ be a non-negative cut for P. The vertices of $\{x \in P : (\delta^l)^T x \geq \delta_0^l\}$ are:*

(i) vertices v^k of P with $k \in V_l^s$,

(ii) intersection points $v^i + \beta'_{k,l}(e^i)(v^k - v^i)$, where $i \in V_l^c$ and $k \in V_l^s$, and

(iii) intersection points $v^i + \alpha'_{j,l}(e^i)r^j$, where $i \in V_l^c$ and $j \in E$ satisfies $(\delta^l)^T r^j > 0$.

Proof. Let $\bar{x}$ be a vertex of $\{x \in P : (\delta^l)^T x \geq \delta_0^l\}$. Also let $\lambda^c \in \Lambda_l^c$ and $(\bar{\epsilon}, \bar{\mu})$ be such that $(\bar{x}, \bar{\epsilon}, \bar{\mu}) \in P^H(\lambda^c)$. Since $P(\lambda^c) \subseteq P$, we must have that $\bar{x}$ is a vertex of $\{x \in P(\lambda^c) : (\delta^l)^T x \geq \delta_0^l\}$. We first show $\bar{x}$ must be either: (a) a vertex v^k of P with $k \in V_l^s$, (b) an intersection point $v_{\lambda^c} + \beta'_{k,l}(\lambda^c)(v^k - v_{\lambda^c})$ with $k \in V_l^s$, or (c) an intersection point $v_{\lambda^c} + \alpha'_{j,l}(\lambda^c)r^j$ with $j \in E$ satisfying $(\delta^l)^T r^j > 0$.

Clearly, if $\bar{x}$ is a vertex of $\{x \in P(\lambda^c) : (\delta^l)^T x \geq \delta_0^l\}$ which is *not* a vertex of P, then $\bar{x}$ satisfies $(\delta^l)^T x \geq \delta_0^l$ with equality. From $(\delta^l)^T \bar{x} = \delta_0^l$, it follows from Lemma 3.2 that $(\bar{x}, \bar{\epsilon}, \bar{\mu}) \in P^H(\lambda^c)$ and

$$\sum_{j \in E} \frac{\bar{\mu}_j}{\alpha'_{j,l}(\lambda^c)} + \sum_{k \in V_l^s} \frac{\bar{\epsilon}_k}{\beta'_{k,l}(\lambda^c)} = 1.$$

We can now write

$$\bar{x} = \sum_{j \in E \setminus E^0} \eta_j (v_{\lambda^c} + \alpha'_{j,l}(\lambda^c) r^j) + \sum_{k \in V_l^s} \gamma_k (v_{\lambda^c} + \beta'_{k,l}(\lambda^c)(v^k - v_{\lambda^c})) + \sum_{j \in E^0} \bar{\mu}_j r^j,$$

where $E^0 := \{j \in E : (\delta^l)^T r^j = 0\}$, $\eta_j := \frac{\bar{\mu}_j}{\alpha'_{j,l}(\lambda^c)}$ for $j \in E \setminus E^0$, $\gamma_k := \frac{\bar{\epsilon}_k}{\beta'_{k,l}(\lambda^c)}$ for $k \in V_l^s$ and $\sum_{j \in E \setminus E^0} \eta_j + \sum_{k \in V_l^s} \gamma_k = 1$. Hence $\bar{x}$ must be of one of the forms (a)-(c) above.

We now show (i)-(iii). If $\bar{x}$ is a vertex v^k of P, where $k \in V_l^s$, we are done, so we may assume that either $\bar{x} = v_{\lambda^c} + \beta'_{k,l}(\lambda^c)(v^k - v_{\lambda^c})$, where $k \in V_l^s$, or $\bar{x} = v_{\lambda^c} + \alpha'_{j,l}(\lambda^c) r^j$, where $j \in E$ satisfies $\alpha'_{j,l}(\lambda^c) < +\infty$. If $\bar{x}$ is of the form $\bar{x} = v_{\lambda^c} + \alpha'_{j,l}(\lambda^c) r^j$, we may write $\bar{x} = v_{\lambda^c} + \alpha'_{j,l}(\lambda^c) r^j = v_{\lambda^c} + \frac{\delta_0^l - (\delta^l)^T v_{\lambda^c}}{(\delta^l)^T r^j} r^j = \sum_{i \in V_l^c} \lambda_i \left(v^i + \frac{\delta_0^l - (\delta^l)^T v^i}{(\delta^l)^T r^j} r^j\right)$. Since $\alpha'_{j,l}(e^i) = \frac{\delta_0^l - (\delta^l)^T v^i}{(\delta^l)^T r^j}$ and $\bar{x}$ is a vertex of $\{x \in P : (\delta^l)^T x \geq \delta_0^l\}$, this implies $\lambda_{\bar{i}} = 1$ for some $\bar{i} \in V_l^c$. Finally, if $\bar{x}$ is of the form $\bar{x} = v_{\lambda^c} + \beta'_{k,l}(\lambda^c)(v^k - v_{\lambda^c})$, then Lemma 3.1 shows that $\bar{x}$ is of the form $v^{\bar{i}} + \beta'_{k,l}(e^{\bar{i}})(v^k - v^{\bar{i}})$ for some $\bar{i} \in V_l^c$ and $k \in V_l^s$. □

Lemma 3.3 motivates the following notation for those intersection points $v_{\lambda^c} + \alpha'_{j,l}(\lambda^c) r^j$ and $v_{\lambda^c} + \beta'_{k,l}(\lambda^c)(v^k - v_{\lambda^c})$, where λ^c is a unit vector. This notation will be used heavily in the following sections.

Notation 3.1 *Given $(i,j) \in V_l^c \times E$, define $\alpha'_{i,j,l} := \alpha'_{j,l}(e^i)$, and given $(i,k) \in V_l^c \times V_l^s$, define $\beta'_{i,k,l} := \beta'_{k,l}(e^i)$, where e^i denotes the unit vector in $\mathbb{R}^{|V_l^c|}$ corresponding to v^i.*

3.2 Dominance and equivalence between cuts Given two non-negative cuts $(\delta^{l_1})^T x \geq \delta_0^{l_1}$ and $(\delta^{l_2})^T x \geq \delta_0^{l_2}$ for P, where $l_1, l_2 \in I$, it is not clear how to compare them in the space of the x variables. By including in the description the multipliers on the extreme rays, and on the vertices that are satisfied by the inequalities, such a comparison is possible. We assume all non-negative cuts considered in this section all cut off exactly the same set of vertices $V^c \subseteq V$ of P. Our notion of dominance is the following.

Definition 3.1 *Let $(\delta^{l_1})^T x \geq \delta_0^{l_1}$ and $(\delta^{l_2})^T x \geq \delta_0^{l_2}$ be two non-negative cuts for P that cut off the same set of vertices $V^c \subseteq V$ of P.*

(i) $(\delta^{l_1})^T x \geq \delta_0^{l_1}$ dominates $(\delta^{l_2})^T x \geq \delta_0^{l_2}$ *on P iff* $\{x \in P : (\delta^{l_1})^T x \geq \delta_0^{l_1}\} \subseteq \{x \in P : (\delta^{l_2})^T x \geq \delta_0^{l_2}\}$.

(ii) If $(\delta^1)^T x \geq \delta_0^1$ dominates $(\delta^2)^T x \geq \delta_0^2$ on P, and $(\delta^2)^T x \geq \delta_0^2$ dominates $(\delta^1)^T x \geq \delta_0^1$ on P, we say $(\delta^1)^T x \geq \delta_0^1$ and $(\delta^2)^T x \geq \delta_0^2$ are equivalent *on P.*

We now show that an equivalent definition of dominance between a pair of non-negative cuts is possible, which is based on intersection points.

Lemma 3.4 *Let $(\delta^{l_1})^T x \geq \delta_0^{l_1}$ and $(\delta^{l_2})^T x \geq \delta_0^{l_2}$ be non-negative cuts for P satisfying $V^c := V^c_{l_1} = V^c_{l_2}$. Then $(\delta^{l_1})^T x \geq \delta_0^{l_1}$ dominates $(\delta^{l_2})^T x \geq \delta_0^{l_2}$ on P if and only if*

(i) The inequality $\frac{1}{\alpha'_{i,j,l_1}} \leq \frac{1}{\alpha'_{i,j,l_2}}$ holds for all $(i,j) \in V^c \times E$.
(The halfline $\{v^i + \alpha r^j : \alpha \geq 0\}$ is intersected later by $(\delta^{l_1})^T x \geq \delta_0^{l_1}$ than $(\delta^{l_2})^T x \geq \delta_0^{l_2}$)

(ii) The inequality $\frac{1}{\beta'_{i,k,l_1}} \leq \frac{1}{\beta'_{i,k,l_2}}$ holds for all $(i,k) \in V^c \times (V \setminus V^c)$.
(The halfline $\{v^i + \beta(v^k - v^i) : \beta \geq 0\}$ is intersected later by $(\delta^{l_1})^T x \geq \delta_0^{l_1}$ than $(\delta^{l_2})^T x \geq \delta_0^{l_2}$)

Proof. Define $Q^1 := \{x \in P : (\delta^{l_1})^T x \geq \delta_0^{l_1}\}$ and $Q^2 := \{x \in P : (\delta^{l_2})^T x \geq \delta_0^{l_2}\}$. First suppose $(\delta^{l_1})^T x \geq \delta_0^{l_1}$ dominates $(\delta^{l_2})^T x \geq \delta_0^{l_2}$ on P, *i.e.*, suppose $Q^1 \subseteq Q^2$. We will verify that (i) and (ii) are satisfied. First let $(i,j) \in V^c \times E$ be arbitrary. If $\alpha'_{i,j,l_1} = +\infty$, clearly $0 = \frac{1}{\alpha'_{i,j,l_1}} \leq \frac{1}{\alpha'_{i,j,l_2}}$. If $\alpha'_{i,j,l_1} < +\infty$, then the intersection point $\bar{y} := v^i + \alpha'_{i,j,l_1} r^j$ satisfies $\bar{y} \in Q^1 \subseteq Q^2$. Hence we have $(\delta^{l_2})^T \bar{y} = (\delta^{l_2})^T v^i + \alpha'_{i,j,l_1} (\delta^{l_2})^T r^j \geq \delta_0^{l_2}$, which implies $\frac{1}{\alpha'_{i,j,l_1}} \leq \frac{1}{\alpha'_{i,j,l_2}}$.

Now let $(i,k) \in V^c \times (V \setminus V^c)$ be arbitrary. The intersection point $\bar{z} := v^i + \beta'_{i,k,l_1}(v^k - v^i)$ satisfies $\bar{z} \in Q^1 \subseteq Q^2$, and therefore $(\delta^{l_2})^T \bar{z} = (\delta^{l_2})^T v^i + \beta'_{i,k,l_1} (\delta^{l_2})^T (v^k - v^i) \geq \delta_0^{l_2}$, which implies $\frac{1}{\beta'_{i,k,l_1}} \leq \frac{1}{\beta'_{i,k,l_2}}$.

Conversely suppose (i) and (ii) are satisfied. Since $V^c = V^c_{l_1} = V^c_{l_2}$, every vertex v^k of P with $k \in V \setminus V^c$ is a vertex of both Q^1 and Q^2. Furthermore, (i) ensures that every vertex of Q^1 of the form $v^i + \alpha'_{i,j,l_1} r^j$ belongs to Q^2, where $(i,j) \in V^c \times E$ and $\alpha'_{i,j,l_1} < +\infty$. Finally, (ii) ensures every vertex of Q^1 of the form $v^i + \beta'_{i,k,l_1}(v^k - v^i)$ belongs to Q^2, where $(i,k) \in V^c \times (V \setminus V^c)$. We therefore have that every vertex of Q^1 belongs to Q^2. Since Q^1 and Q^2 have the same extreme rays $\{r^j\}_{j \in E}$, we have $Q^1 \subseteq Q^2$. □

Let $V^c \subseteq V$ be arbitrary, and let $I^f \subseteq I$ index a finite set of non-negative cuts from the family $\{(\delta^l)^T x \geq \delta_0^l\}_{l \in I}$. We assume $V^c_l = V^c$ for all $l \in I^f$. Consider the following polyhedron $X(I^f)$

$$X(I^f) := \{x \in P : (\delta^l)^T x \geq \delta_0^l \text{ for all } l \in I^f\}.$$

Given a non-negative cut $(\delta^{l^*})^T x \geq \delta_0^{l^*}$ with $l^* \in I \setminus I^f$ which is valid for $X(I^f)$, there might not exist an inequality in the family $\{(\delta^l)^T x \geq \delta_0^l\}_{l \in I^f}$ which dominates $(\delta^{l^*})^T x \geq \delta_0^{l^*}$ on $X(I^f)$. However, even though such an inequality does not exist, it may be possible to construct a non-negative combination of the inequalities $\{(\delta^l)^T x \geq \delta_0^l\}_{l \in I^f}$ which dominates $(\delta^{l^*})^T x \geq \delta_0^{l^*}$ on $X(I^f)$. Indeed, the following lemma shows that, if $(\delta^{l^*})^T x \geq \delta_0^{l^*}$ is valid for $X(I^f)$, then there exists an inequality $\delta^T x \geq \delta_0$, which is a convex combination of the inequalities $\{(\delta^l)^T x \geq \delta_0^l\}_{l \in I^f}$, and which dominates $(\delta^{l^*})^T x \geq \delta_0^{l^*}$ on $X(I^f)$.

Lemma 3.5 *(This lemma is a generalization of [**?**, Lemma 3])*
Assume $X(I^f) \neq \emptyset$. Let $l^* \in I \setminus I^f$ be a non-negative cut for P satisfying $V^c_{l^*} = V^c = V^c_l$ for all $l \in I^f$. Then $(\delta^{l^*})^T x \geq \delta_0^{l^*}$ is valid for $X(I^f)$ iff there exists a non-negative cut $\delta^T x \geq \delta_0$ for P that satisfies

(i) $\delta^T x \geq \delta_0$ *is a convex combination of the inequalities* $\{(\delta^l)^T x \geq \delta_0^l\}_{l\in I^f}$, *and*

(ii) $\delta^T x \geq \delta_0$ *dominates* $(\delta^{l^*})^T x \geq \delta_0^{l^*}$ *on* P.

PROOF. We only need to show one direction. Suppose $(\delta^{l^*})^T x \geq \delta_0^{l^*}$ is valid for $X(I^f)$. Consider the linear program (LP) given by $\min\{(\delta^{l^*})^T x : x \in X(I^f)\}$. The assumption $X(I^f) \neq \emptyset$ and the validity of $(\delta^{l^*})^T x \geq \delta_0^{l^*}$ for $X(I^f)$ implies that (LP) is feasible and bounded. We can formulate (LP) as follows.

$$\begin{aligned}
&\min\ (\delta^{l^*})^T x \\
&x = \sum_{i\in V^c}\lambda_i v^i + \sum_{i\in V^c}\sum_{k\in V\setminus V^c}\epsilon_k^i (v^k - v^i) + \sum_{i\in V^c}\sum_{j\in E}\mu_j^i r^j, && (u)\\
&(\delta^l)^T x \geq \delta_0^l \text{ for all } l \in I^f, && (w_l)\\
&\sum_{k\in V\setminus V^c}\epsilon_k^i \leq \lambda_i \text{ for all } i \in V^c, && (z_i)\\
&\sum_{i\in V^c}\lambda_i = 1, && (u_0)\\
&\epsilon^i, \mu^i, \lambda \geq 0 \text{ for all } i \in V^c.
\end{aligned}$$

From the dual of (LP), we obtain $\bar{u} \in \mathbb{R}^n$, $\bar{w} \in \mathbb{R}^{|I^f|}$, $\bar{z} \in \mathbb{R}^{|V^c|}$ and $\bar{u}_0 \in \mathbb{R}$ that satisfy

$$\begin{array}{llcll}
\text{(i)} & \bar{u}_0 + \sum_{l\in I^f}\bar{w}_l\delta_0^l & \geq & \delta_0^{l^*}, & \\
\text{(ii)} & -\bar{u} & = & \delta^{l^*}, & (x)\\
\text{(iii)} & \bar{u}^T v^i + \sum_{l\in I^f}\bar{w}_l(\delta^l)^T v^i + \bar{z}_i + \bar{u}_0 & \leq & 0, \quad \text{for all } i \in V^c, & (\lambda_i)\\
\text{(iv)} & \bar{u}^T(v^k - v^i) + \sum_{l\in I^f}\bar{w}_l(\delta^l)^T(v^k - v^i) - \bar{z}_i & \leq & 0, \quad \text{for all } (i,k) \in V^c \times (V\setminus V^c), & (\epsilon_k^i)\\
\text{(v)} & \bar{u}^T r^j + \sum_{l\in I^f}\bar{w}_l(\delta^l)^T r^j & \leq & 0, \quad \text{for all } j \in E, & (\mu_j)\\
\text{(vi)} & \bar{w} \geq 0 \text{ and } \bar{z} \geq 0. & & &
\end{array}$$

Let $\delta := \sum_{l\in I^f}\bar{w}_l\delta^l$ and $\delta_0 := \sum_{l\in I^f}\delta_0^l\bar{w}_l$. Since $\delta^T x \geq \delta_0$ is a non-negative combination of the inequalities $\{(\delta^l)^T x \geq \delta_0^l\}_{l\in I^f}$, $\delta^T x \geq \delta_0$ is valid for $X(I^f)$. Furthermore, $\delta^T x \geq \delta_0$ is a non-negative combination of non-negative cuts for P, and therefore $\delta^T x \geq \delta_0$ is also a non-negative cut for P. Observe that the vertices of P that are cut of $\delta^T x \geq \delta_0$ are indexed by V^c. Given $(i,j) \in V^c \times E$, let $\alpha'_{i,j}(\delta,\delta_0)$ denote the value of $\alpha > 0$ for which $v^i + \alpha r^j$ is on the hyperplane $\delta^T x = \delta_0$ ($\alpha'_{i,j}(\delta,\delta_0) = +\infty$ if no such point exists). Finally, given $(i,k) \in V^c \times (V\setminus V^c)$, let $\beta'_{i,k}(\delta,\delta_0)$ denote the value of $\beta > 0$ for which $v^i + \beta(v^k - v^i)$ is on the hyperplane $\delta^T x = \delta_0$.

We will show $\delta^T x \geq \delta_0$ dominates $(\delta^{l^*})^T x \geq \delta_0^{l^*}$ on P. We show this by showing that the conditions of Lemma 3.4 are satisfied. The system (i)-(vi) above implies the following inequalities.

(a) $\bar{u}_0 + \delta_0 \geq \delta_0^{l^*}$.
(b) $-(\delta^{l^*})^T v^i + \delta^T v^i + \bar{z}_i + \bar{u}_0 \leq 0$ for all $i \in V^c$.
(c) $-(\delta^{l^*})^T(v^k - v^i) + \delta^T(v^k - v^i) - \bar{z}_i \leq 0$ for all $(i,k) \in V^c \times (V\setminus V^c)$.
(d) $-(\delta^{l^*})^T r^j + \delta^T r^j \leq 0$ for all $j \in E$.

We first show $\frac{1}{\alpha'_{i,j}(\delta,\delta_0)} \leq \frac{1}{\alpha'_{i,j,l^*}}$ for all $(i,j) \in V^c \times E$. Therefore let $(\bar{i},\bar{j}) \in V^c \times E$. If $\alpha'_{\bar{i},\bar{j},l^*} = +\infty$, then $(\delta^{l^*})^T r^{\bar{j}} = 0$, which by (d) implies that also $\delta^T r^{\bar{j}} = 0$, and therefore $0 = \frac{1}{\alpha'_{\bar{i},\bar{j}}(\delta,\delta_0)} = \frac{1}{\alpha'_{\bar{i},\bar{j},l^*}}$. We can therefore assume $\alpha'_{\bar{i},\bar{j},l^*} < +\infty$. Multiplying the inequality of (d) corresponding to $\bar{j}$ with $\alpha'_{\bar{i},\bar{j},l^*}$ and adding the result to the inequality of (b) corresponding to $\bar{i}$ gives $-(\delta^{l^*})^T(v^{\bar{i}} + \alpha'_{\bar{i},\bar{j},l^*}r^{\bar{j}}) + \delta^T(v^{\bar{i}} + \alpha'_{\bar{i},\bar{j},l^*}r^{\bar{j}}) \leq -\bar{u}_0 - \bar{z}_{\bar{i}} \leq \delta_0 - \delta_0^{l^*}$. Since $(\delta^{l^*})^T\ (v^{\bar{i}} + \alpha'_{\bar{i},\bar{j},l^*}r^{\bar{j}}) = \delta_0^{l^*}$, we get $\delta^T(v^{\bar{i}} + \alpha'_{\bar{i},\bar{j},l^*}r^{\bar{j}}) \leq \delta_0$. Now, $\alpha'_{\bar{i},\bar{j}}(\delta,\delta_0)$ is the smallest value of α such that $\delta^T(v^{\bar{i}} + \alpha r^{\bar{j}}) = \delta_0$. Since $\delta^T(v^{\bar{i}} + \alpha'_{\bar{i},\bar{j},l^*}r^{\bar{j}}) \leq$

δ_0, this implies $\alpha'_{\bar{i},\bar{j}}(\delta,\delta_0) \geq \alpha'_{\bar{i},\bar{j},l^*}$, and therefore $\frac{1}{\alpha'_{\bar{i},\bar{j}}(\delta,\delta_0)} \leq \frac{1}{\alpha'_{\bar{i},\bar{j},l^*}}$. Hence condition (i) of Lemma 3.4 is satisfied.

We now show $\frac{1}{\beta'_{i,k}(\delta,\delta_0)} \leq \frac{1}{\beta'_{i,k,l^*}}$ for all $(i,k) \in V^c \times (V \setminus V^c)$. Let $(\bar{i},\bar{k}) \in V^c \times (V \times \setminus V^c)$. Multiplying the inequality of (c) corresponding to $(\bar{i},\bar{k})$ with $\beta'_{\bar{i},\bar{k},l^*}$ and adding the result to the inequality of (b) corresponding to $\bar{i}$ gives $-(\delta^{l^*})^T(v^{\bar{i}}+\ \beta'_{\bar{i},\bar{k},l^*}(v^{\bar{k}}-v^{\bar{i}}))+\ \delta^T(v^{\bar{i}}+\ \beta'_{\bar{i},\bar{k},l^*}(v^{\bar{k}}-v^{\bar{i}})) \leq -\bar{u}_0 - \bar{z}_{\bar{i}} \leq \delta_0 - \delta_0^{l^*}$. Since $(\delta^{l^*})^T(v^{\bar{i}}+\beta'_{\bar{i},\bar{k},l^*}(v^{\bar{k}}-v^{\bar{i}})) = \delta_0^{l^*}$, this implies $\delta^T\ (v^{\bar{i}}+\beta'_{\bar{i},\bar{k},l^*}(v^{\bar{k}}-v^{\bar{i}})) \leq \delta_0$. We have that $\beta'_{\bar{i},\bar{k}}(\delta,\delta_0)$ is the smallest value of β s.t. $\delta^T(v^{\bar{i}}+\beta(v^{\bar{k}}-v^{\bar{i}})) = \delta_0$, and since $\delta^T(v^{\bar{i}}+\ \beta'_{\bar{i},\bar{j},l^*}(v^{\bar{k}}-v^{\bar{i}}) \leq \delta_0$, this implies $\beta'_{\bar{i},\bar{k}}(\delta,\delta_0) \geq \beta'_{\bar{i},\bar{k},l^*}$. It follows that $\frac{1}{\beta'_{\bar{i},\bar{k}}(\delta,\delta_0)} \leq \frac{1}{\beta'_{\bar{i},\bar{k},l^*}}$. Hence condition (ii) of Lemma 3.4 is also satisfied, and therefore $\delta^T x \geq \delta_0$ dominates $(\delta^{l^*})^T x \geq \delta_0^{l^*}$ on P.

To finish the proof, we argue that we can choose $\delta^T x \geq \delta_0$ to be a convex combination of the inequalities $\{(\delta^l)^T x \geq \delta_0^l\}_{l\in I^f}$. Observe that if $\sum_{l\in I^f}\bar{w}_l \neq 0$, then the inequality $(\delta')^T x \geq \delta'_0$ defined by $(\delta',\delta'_0) := \frac{1}{\sum_{l\in I^f}\bar{w}_l}(\delta,\delta_0)$ is a convex combination of the inequalities $\{(\delta^l)^T x \geq \delta_0^l\}_{l\in I^f}$ and $(\delta')^T x \geq \delta'_0$ is equivalent to $\delta^T x \geq \delta_0$ on P. We therefore only have to show $\sum_{l\in I^f}\bar{w}_l \neq 0$. If $\sum_{l\in I^f}\bar{w}_l = 0$, then (i)-(iii) give $\bar{u}_0 \geq \delta_0^{l^*}$ and $-(\delta^{l^*})^T v^i + \bar{z}_i + \bar{u}_0 \leq 0$ for all $i \in V^c$, which implies $(\delta^{l^*})^T v^i \geq \delta_0^{l^*}$ for all $i \in V^c$, and this contradicts that $(\delta^{l^*})^T x \geq \delta_0^{l^*}$ is a cut for P with $V^c_{l^*} = V^c$. □

3.3 A sufficient condition for polyhedrality We now consider the addition of an *infinite* family of non-negative cuts to the polyhedron P. Specifically, consider the convex set

$$X := \{x \in P : (\delta^l)^T x \geq \delta_0^l \text{ for } l \in I\},$$

where I now can be an *infinite* index set. The goal in this section is to provide a sufficient condition for X to be a polyhedron. For this purpose we can assume $V_l^c = V^c$ for all $l \in I$, *i.e.*, we can assume all cuts cut off the same vertices. Indeed, if the cuts $\{(\delta^l)^T x \geq \delta_0^l\}_{l\in I}$ do not cut off the same of vertices, then define

$$I^c(S) := \{l \in I : V_l^c = S\}$$

for every $S \subseteq V$, and let $\mathcal{S} := \{S \subseteq V : I^c(S) \neq \emptyset\}$. We can then write

$$X = \cap_{S\in\mathcal{S}}\{x \in P : (\delta^l)^T x \geq \delta_0^l \text{ for all } l \in S\}$$

Since $\mathcal{S}$ is finite, we have that X is a polyhedron if and only if X is a polyhedron under the assumption that $V_l^c = V^c$ for all $l \in I$.

We will show that X is a polyhedron under the following assumption.

Assumption 3.1

(i) For all $(i,j) \in V^c \times E$ and $\alpha^* > 0$, the set $\{\alpha'_{i,j,l} \geq \alpha^* : l \in I\}$ is finite
(There is only a finite number of intersection points between the cuts $\{(\delta^l)^T x \geq \delta_0^l\}_{l\in I}$ and the halfline $\{v^i + \alpha r^j : \alpha \geq \alpha^*\}$).

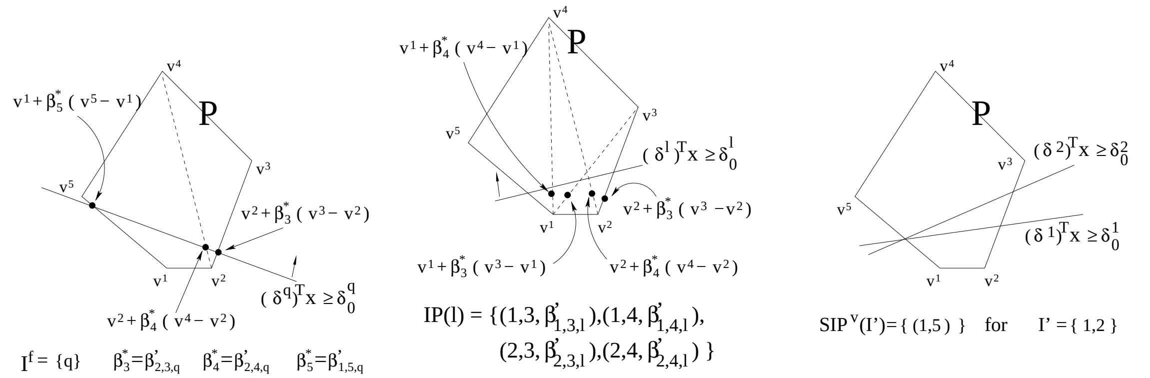
(ii) For all $(i,k) \in V^c \times V \setminus V^c$ and $\beta^* \in]0,1]$, the set $\{\beta'_{i,k,l} \geq \beta^* : l \in I\}$ is finite
(There is only a finite number of intersection points between the cuts $\{(\delta^l)^T x \geq \delta_0^l\}_{l\in I}$ and the line segment $\{v^i + \beta(v^k - v^i) : \beta^* \leq \beta \leq 1\}$).

The main theorem is the following.

Theorem 3.1 *Suppose $\{(\delta^l)^T x \geq \delta_0^l\}_{l\in I}$ is a family of non-negative cuts for P that satisfies Assumption 3.1, and suppose $V^c = V_l^c$ for all $l \in I$. Then X is a polyhedron.*

The idea of the proof is based on counting the number of intersection points that are shared by *all* cuts in a family $I' \subseteq I$ of cuts. This number is given by $\mathrm{s}(I') := |\mathrm{SIP}^e(I')| + |\mathrm{SIP}^v(I')|$, where the sets $\mathrm{SIP}^e(I')$ and $\mathrm{SIP}^v(I')$ are defined by

$$\mathrm{SIP}^e(I') := \{(i,j) \in V^c \times E : \alpha'_{i,j,l_1} = \alpha'_{i,j,l_2} \text{ for all } l_1, l_2 \in I'\}, \text{ and}$$
$$\mathrm{SIP}^v(I') := \{(i,k) \in V^c \times (V \setminus V^c) : \beta'_{i,k,l_1} = \beta'_{i,k,l_2} \text{ for all } l_1, l_2 \in I'\}.$$



(a) Determining the values β_3^*,β_4^* and β_5^* from a set $I^f = \{q\}$ (b) Determining the set IP(l) for a cut $(\delta^l)^T x \geq \delta_0^l$ (c) Two cuts sharing intersection point on the line between v^1 and v^5

Figure 5: The sets $\mathrm{SIP}^v(I')$ and $\mathrm{IP}^v(l)$

Figure 5(c) gives an example of the set $\mathrm{SIP}^v(I')$ for a polytope P, where I' consists of two cuts. Since P is a polytope, we have $\mathrm{SIP}^e(I') = \emptyset$ for any set I'. Both cuts $(\delta^{l_1})^T x \geq \delta_0^{l_1}$ and $(\delta^{l_1})^T x \geq \delta_0^{l_1}$ intersect the line segment between v^1 and v^5 at the same point, and therefore $(1,5) \in \mathrm{SIP}^v(I')$. No other line segment in 5(c) is intersected by both $(\delta^{l_1})^T x \geq \delta_0^{l_1}$ and $(\delta^{l_1})^T x \geq \delta_0^{l_1}$ at the same point, and therefore $\mathrm{SIP}^v(I') = \{(1,5)\}$.

Clearly $0 \leq \mathrm{s}(I') \leq |V^c \times E| + |V^c \times (V \setminus V^c)|$ for all $I' \subseteq I$. Also, if $\mathrm{s}(I') = |V^c \times E| + |V^c \times (V \setminus V^c)|$, then *all* cuts indexed by I' share *all* intersection points with the halflines $\{v^i + \alpha r^j : \alpha \geq 0\}$ and $\{v^i + \beta(v^k - v^i) : \beta \geq 0\}$ for *all* $i \in V^c$, $j \in E$ and $k \in V \setminus V^c$. This implies that all cuts indexed by I' are equivalent on P (Lemma 3.4). Therefore, if $\mathrm{s}(I') = |V^c \times E| + |V^c \times (V \setminus V^c)|$, then the set $\{x \in P : (\delta^l)^T x \geq \delta_0^l \text{ for all } l \in I'\}$ is a polyhedron that can be described with exactly one cut from I'.

We will partition I into a number of subsets:

$$S^1, S^2, \ldots, S^{\mathrm{ns}} \subseteq I, \cup_{m=1}^{\mathrm{ns}} S^m = I \text{ and } S^{m_1} \cap S^{m_2} = \emptyset \text{ for all } m_1 \neq m_2$$

such that for every $m \in \{1, 2, \ldots, \mathrm{ns}\}$, either $X(S^m) := \{x \in P : (\delta^l)^T x \geq \delta_0 \text{ for all } l \in S^m\}$ is a polyhedron, or $s(S^m) > s(I)$ (the letters "ns" are an abbreviation for "number of subsets"). Figure 6 illustrates the construction. The fact that $\{S^m\}_{m=1}^{\mathrm{ns}}$ is a partitioning of I implies

$$X = \cap_{m=1}^{\mathrm{ns}} X(S^m).$$

Therefore X is a polyhedron if $X(S^m)$ is a polyhedron for all $m \in \{1, 2, \ldots, \mathrm{ns}\}$. Given $m \in \{1, 2, \ldots, \mathrm{ns}\}$, if $X(S^m)$ is not a polyhedron, then $s(S^m) > s(I)$. Hence, recursively applying this construction to the nodes in Figure 6 will create a tree of subcases. The fact that $s(T)$ is bounded from above for all $T \subseteq I$, and the fact that $s(T)$ increases strictly with the depth of the tree, will ensure that the tree is bounded in size. Therefore, if we can construct a partitioning $\{S^m\}_{m=1}^{\mathrm{ns}}$ of I such that for every $m \in \{1, 2, \ldots, \mathrm{ns}\}$, either $X(S^m)$ is a polyhedron, or $s(S^m) > s(I)$, then the proof of Theorem 3.1 is complete.

We now construct the partitioning $\{S^m\}_{m=1}^{\mathrm{ns}}$ of I. Choose an arbitrary finite and non-empty subset $I^f \subseteq I$. The partitioning of I is based on the following positive numbers $\{\alpha_j^*\}_{j \in E}$ and $\{\beta_k^*\}_{k \in V \setminus V^c}$, which we use to create halflines of the form $\{v^i + \alpha r^j : \alpha \geq \alpha_j^*\}$, and line segments of the form $\{v^i + \beta(v^k - v^i) : \beta_k^* \leq \beta \leq 1\}$ for all $i \in V^c$, $k \in V \setminus V^c$ and $j \in E$. The numbers $\{\alpha_j^*\}_{j \in E}$ and $\{\beta_k^*\}_{k \in V \setminus V^c}$ are defined as follows.

$$\alpha_j^* := \min\{\alpha'_{i,j,l} : i \in V^c \text{ and } l \in I^f\} \text{ for } j \in E, \text{ and}$$
$$\beta_k^* := \min\{\beta'_{i,k,l} : i \in V^c \text{ and } l \in I^f\} \text{ for } k \in V \setminus V^c.$$

Given $\bar{j} \in E$, $\alpha_{\bar{j}}^*$ is determined from a vertex $\bar{i} \in V^c$ and a cut $\bar{l} \in I^f$ for which the intersection point $v^{\bar{i}} + \alpha'_{\bar{i},\bar{j},\bar{l}} r^{\bar{j}}$ is as close to $v^{\bar{i}}$ as possible. Similarly, given $\bar{k} \in V \setminus V^c$, $\beta_{\bar{k}}^*$ is determined from a vertex $\bar{i} \in V^c$ and a cut $\bar{l} \in I^f$ for which the intersection point $v^{\bar{i}} + \beta'_{\bar{i},\bar{k},\bar{l}}(v^{\bar{k}} - v^{\bar{i}})$ is as close to $v^{\bar{i}}$ as possible. Figure 5(a) illustrates the computation of the numbers $\{\beta_k^*\}_{k \in V \setminus V^c}$ for a polytope with 5 vertices, $V^c = \{1, 2\}$,

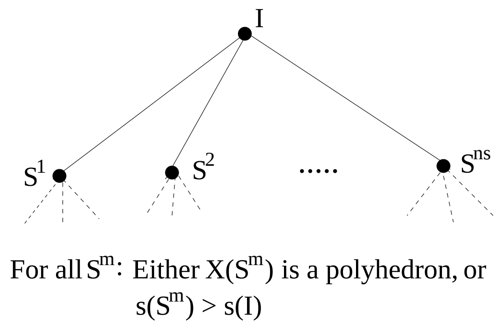


Figure 6: Partitioning the set I into the subsets $\{S^m\}_{m=1}^{\mathrm{ns}}$

and a chosen subset $I^f \subseteq I$ with one inequality $I^f = \{q\}$. In this example, the numbers β_3^*, β_4^* and β_5^* are determined by the intersection points $v^2 + \beta'_{2,3,q}(v^3 - v^2)$, $v^2 + \beta'_{2,4,q}(v^4 - v^2)$ and $v^1 + \beta'_{1,5,q}(v^5 - v^1)$ respectively.

Assumption 3.1 implies that the cuts $\{(\delta^l)^T x \geq \delta_0^l\}_{l \in I}$ only intersect the halflines $\{v^i + \alpha r^j : \alpha \geq \alpha_j^*\}$ and the line segments $\{v^i + \beta(v^k - v^i) : \beta_k^* \leq \beta \leq 1\}$ at a finite number of points. We will use this fact to define an equivalence relation on the cuts $\{(\delta^l)^T x \geq \delta_0^l\}_{l \in I}$. Given a cut $(\delta^l)^T x \geq \delta_0^l$, let

$$\mathrm{IP}^e(l) := \{(i, j, \alpha'_{i,j,l}) \in V^c \times E \times \mathbb{R}_+ : \alpha'_{i,j,l} \geq \alpha_j^*\} \text{ and}$$
$$\mathrm{IP}^v(l) := \{(i, k, \beta'_{i,k,l}) \in V^c \times (V \setminus V^c) \times \mathbb{R}_+ : \beta'_{i,k,l} \geq \beta_k^*\}$$

denote the intersection points between $(\delta^l)^T x \geq \delta_0^l$ and the halflines $\{v^i + \alpha r^j : \alpha \geq \alpha_j^*\}$ for $(i, j) \in V^c \times E$, and the line segments $\{v^i + \beta(v^k - v^i) : \beta_k^* \leq \beta \leq 1\}$ for $(i, k) \in V^c \times (V \setminus V^c)$ respectively. Observe that $\mathrm{IP}^e(l) = \emptyset$ whenever $\alpha'_{i,j,l} < \alpha_j^*$ for all $(i, j) \in V^c \times E$, and that $\mathrm{IP}^v(l) = \emptyset$ whenever $\beta'_{i,k,l} < \beta_k^*$ for all $(i, k) \in V^c \times (V \setminus V^c)$.

Figure 5(b) gives an example of the set $\mathrm{IP}^v(l)$ for a cut $(\delta^l)^T x \geq \delta_0^l$. Since P is a polytope in this example, we necessarily have $\mathrm{IP}^e(l) = \emptyset$. The numbers β_3^*, β_4^* and β_5^* were computed in 5(a). The numbers $\beta'_{1,5,l}$ and $\beta'_{2,5,l}$ satisfy $\beta'_{1,5,l} < \beta_5^*$ and $\beta'_{2,5,l} < \beta_5^*$, and therefore $(1, 5, \beta'_{1,5,l}), (2, 5, \beta'_{2,5,l}) \notin \mathrm{IP}^v(l)$.

We can use the sets $\mathrm{IP}^e(l)$ and $\mathrm{IP}^v(l)$ for $l \in I$ to define the following equivalence relation on the cuts $\{(\delta^l)^T x \geq \delta_0^l\}_{l \in I}$.

$$\text{For all } l_1, l_2 \in I : l_1 \equiv l_2 \iff \mathrm{IP}^v(l_1) = \mathrm{IP}^v(l_2) \text{ and } \mathrm{IP}^e(l_1) = \mathrm{IP}^e(l_2).$$

Observe that Assumption 3.1 implies that there is only a finite number of equivalence classes corresponding to this equivalence relation. Let $S^1, S^2, \ldots, S^{\mathrm{ns}} \subseteq I$ denote the sets of inequalities corresponding to each equivalence class, where ns denotes the number of equivalence classes. Clearly the sets $\{S^m\}_{m=1}^{\mathrm{ns}}$ partition I. We first make some observations on each equivalence class $m \in \{1, 2, \ldots, \mathrm{ns}\}$.

(a) For every $(i, j) \in V^c \times E$, if there exists $l \in S^m$ such that $\alpha'_{i,j,l} \geq \alpha_j^*$, then $(i, j) \in \mathrm{SIP}^e(S^m)$:
Suppose $\bar{l} \in S^m$ satisfies $\alpha'_{i,j,\bar{l}} \geq \alpha_j^*$ for some $(i, j) \in V^c \times E$. This implies $(i, j, \alpha'_{i,j,\bar{l}}) \in \mathrm{IP}^e(\bar{l})$. By definition of the equivalence class S^m, this implies $(i, j, \alpha'_{i,j,\bar{l}}) \in \mathrm{IP}^e(l)$ for all $l \in S^m$, and therefore $\alpha'_{i,j,l} = \alpha'_{i,j,\bar{l}}$ for all $l \in S^m$. Hence $(i, j) \in \mathrm{SIP}^e(S^m)$.

(b) For every $(i, k) \in V^c \times (V \setminus V^c)$, if there exists $l \in S^m$ such that $\beta'_{i,k,l} \geq \beta_k^*$, then $(i, k) \in \mathrm{SIP}^v(S^m)$:
The argument is identical to the argument for (a).

(c) If $S^m \cap I^f \neq \emptyset$, then $\mathrm{SIP}^e(S^m) = V^c \times E$ and $\mathrm{SIP}^v(S^m) = V^c \times (V \setminus V^c)$:
Suppose $\bar{l} \in S^m \cap I^f$. This implies $\alpha_j^* = \min\{\alpha'_{i,j,l} : i \in V^c \text{ and } l \in I^f\} \leq \alpha'_{i,j,\bar{l}}$ for all $(i, j) \in V^c \times E$. Since $\alpha'_{i,j,\bar{l}} \geq \alpha_j^*$ for all $(i, j) \in V^c \times E$, (1) shows that $\mathrm{SIP}^e(S^m) = V^c \times E$. With a similar argument, (b) shows that $\mathrm{SIP}^v(S^m) = V^c \times (V \setminus V^c)$.

We now complete the proof of Theorem 3.1. Observation (c) implies we can assume $S^m \cap I^f = \emptyset$. To finish the proof, we show that either $s(S^m) > s(I)$, or all inequalities in S^m are dominated by an inequality in I^f. Clearly $\mathrm{SIP}^e(I) \subseteq \mathrm{SIP}^e(S^m)$ and $\mathrm{SIP}^v(I) \subseteq \mathrm{SIP}^v(S^m)$, which implies $s(S^m) \geq s(I)$.

Suppose $s(S^m) = s(I)$, *i.e.*, suppose $\mathrm{SIP}^e(I) = \mathrm{SIP}^e(S^m)$ and $\mathrm{SIP}^v(I) = \mathrm{SIP}^v(S^m)$. Also let $\bar{l} \in I^f$ be arbitrary. We will show $\alpha'_{i,j,l} \leq \alpha'_{i,j,\bar{l}}$ for all $(i,j,l) \in V^c \times E \times S^m$ and $\beta'_{i,k,l} \leq \beta'_{i,k,\bar{l}}$ for all $(i,k,l) \in V^c \times (V \setminus V^c) \times S^m$. This shows $\bar{l}$ dominates all inequalities in S^m and completes the proof of Theorem 3.1. The argument that shows $\beta'_{i,k,l} \leq \beta'_{i,k,\bar{l}}$ for all $(i,k,l) \in V^c \times (V \setminus V^c) \times S^m$ is the same as the argument that shows $\alpha'_{i,j,l} \leq \alpha'_{i,j,\bar{l}}$ for all $(i,j,l) \in V^c \times E \times S^m$. We therefore only show $\alpha'_{i,j,l} \leq \alpha'_{i,j,\bar{l}}$ for all $(i,j,l) \in V^c \times E \times S^m$. Let $(i',j',l') \in V^c \times E \times S^m$ be arbitrary. There are two cases.

(i) $(i',j') \notin \mathrm{SIP}^e(S^m)$: Then (a) implies $\alpha'_{i',j',l'} < \alpha^*_{j'} = \min\{\alpha'_{i,j',l} : i \in V^c \text{ and } l \in I^f\} \leq \alpha'_{i',j',\bar{l}}$.

(ii) $(i',j') \in \mathrm{SIP}^e(S^m)$: Since $\mathrm{SIP}^e(S^m) = \mathrm{SIP}^e(I)$, this implies $\alpha'_{i',j',l'} = \alpha'_{i',j',\bar{l}}$.

Hence $(\delta^{\bar{l}})^T x \geq \delta_0^{\bar{l}}$ dominates $(\delta^l)^T x \geq \delta_0^l$ for all $l \in S^m$, and the proof is complete.

4. The structure of polyhedral relaxations obtained from mixed integer split polyhedra

We now describe the polyhedral structure of the polyhedron $R(L,P)$ for a given mixed integer split polyhedron L. Throughout this section, L denotes an arbitrary mixed integer split polyhedron. Also, $V^{\mathrm{in}}(L) := \{i \in V : v^i \in \mathrm{int}(L)\}$ denotes the vertices of P in the interior of L and $V^{\mathrm{out}}(L) := V \setminus V^{\mathrm{in}}(L)$ denotes the vertices of P that are *not* in the interior of L. We assume $V^{\mathrm{in}}(L) \neq \emptyset$, since otherwise $R(L,P) = P$ (Lemma 2.5). The set $\Lambda := \{\lambda \in \mathbb{R}^{|V|} : \lambda \geq 0 \text{ and } \sum_{i \in V} \lambda_i = 1\}$ is used to form convex combinations of the vertices of P, and the set $\Lambda^{\mathrm{in}}(L) := \{\lambda \in \Lambda : \sum_{i \in V^{\mathrm{in}}(L)} \lambda_i = 1\}$ is used to form convex combinations of the vertices in $V^{\mathrm{in}}(L)$.

4.1 Intersection points Now consider possible intersection points between a halfline of the form $\{v_{\lambda^{\mathrm{in}}} + \alpha r^j : \alpha \geq 0\}$ and the boundary of L, where $\lambda^{\mathrm{in}} \in \Lambda^{\mathrm{in}}(L)$ and $j \in E$. Given $\lambda^{\mathrm{in}} \in \Lambda^{\mathrm{in}}(L)$ and $j \in E$, define:

$$\alpha_j(L, \lambda^{\mathrm{in}}) := \sup\{\alpha : v_{\lambda^{\mathrm{in}}} + \alpha r^j \in L\}. \tag{9}$$

The number $\alpha_j(L,\lambda^{\mathrm{in}}) > 0$ determines the closest point $v_{\lambda^{\mathrm{in}}} + \alpha_j(L,\lambda^{\mathrm{in}}) r^j$ (if any) to $v_{\lambda^{\mathrm{in}}}$ on the halfline $\{v_{\lambda^{\mathrm{in}}} + \alpha r^j : \alpha \geq 0\}$ which is *not* in the interior of L. Observe that if $\{v_{\lambda^{\mathrm{in}}} + \alpha r^j : \alpha \geq 0\} \subseteq \mathrm{int}(L)$, then $\alpha_j(L,\lambda^{\mathrm{in}}) = +\infty$. When $\alpha_j(L,\lambda^{\mathrm{in}}) < +\infty$, the point $v_{\lambda^{\mathrm{in}}} + \alpha_j(L,\lambda^{\mathrm{in}}) r^j$ is called an *intersection point*.

The value $\alpha_j(L,\lambda^{\mathrm{in}})$ is a function of λ^{in}. This function has the following important property. Given any convex set $C \subseteq \mathbb{R}^{n+1}$, it is well known (see Rockafellar [**?**]) that the function $f : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(x) := \sup\{\mu : (x,\mu) \in C\}$$

is a concave function. Now, given any $\lambda^{\mathrm{in}} \in \Lambda^{\mathrm{in}}(L)$ and $j \in E$, we may write

$$\alpha_j(L,\lambda^{\mathrm{in}}) = \sup\{\alpha : (\lambda^{\mathrm{in}}, \alpha) \in \tilde{P}(L)\},$$

where $\tilde{P}(L)$ is the convex polyhedron $\tilde{P}(L) := \{(\lambda^{\mathrm{in}},\alpha) \in \mathbb{R}^{|V^{\mathrm{in}}(L)|+1} : v_{\lambda^{\mathrm{in}}} + \alpha r^j \in L\}$. We therefore have that the function $\alpha_j(L,\lambda^{\mathrm{in}})$ has the following property.

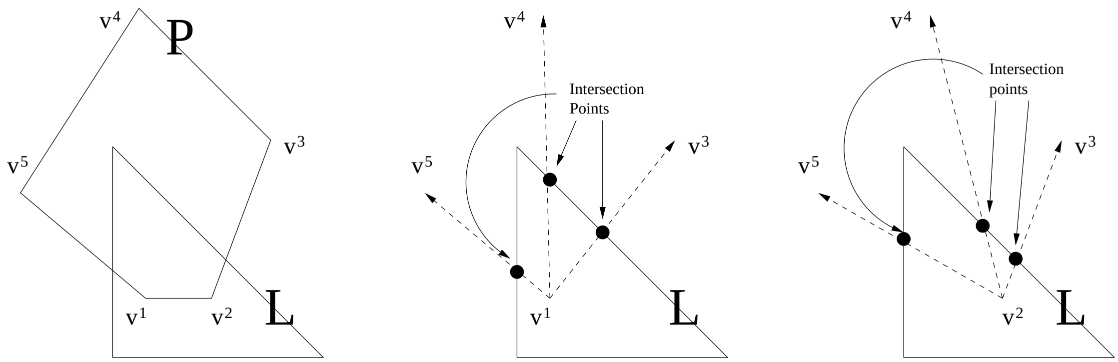
Observation 4.1 *Let L be a mixed integer split polyhedron satisfying $V^{in}(L) \neq \emptyset$, and let $j \in E$. The function $\alpha_j(L,\lambda^{in})$ is concave in λ^{in}, i.e., for every $\lambda^1, \lambda^2 \in \Lambda^{in}(L)$ and $\mu \in [0,1]$, we have that $\alpha_j(L, \mu\lambda^1 + (1-\mu)\lambda^2) \geq \mu\alpha_j(L,\lambda^1) + (1-\mu)\alpha_j(L,\lambda^2)$.*

Given a convex combination $\lambda^{\mathrm{in}} \in \Lambda^{\mathrm{in}}(L)$, and a vertex $k \in V^{\mathrm{out}}(L)$, the line between $v_{\lambda^{\mathrm{in}}}$ and v^k intersects the boundary of L. For $k \in V^{\mathrm{out}}(L)$ and $\lambda^{\mathrm{in}} \in \Lambda^{\mathrm{in}}(L)$, define

$$\beta_k(L,\lambda^{\mathrm{in}}) := \sup\{\beta : v_{\lambda^{\mathrm{in}}} + \beta(v^k - v_{\lambda^{\mathrm{in}}}) \in L\}. \tag{10}$$

The number $\beta_k(L,\lambda^{\mathrm{in}})$ denotes the value of β for which the point $v_{\lambda^{\mathrm{in}}} + \beta(v^k - v_{\lambda^{\mathrm{in}}})$ is on the boundary of L. The point $v_{\lambda^{\mathrm{in}}} + \beta(v^k - v_{\lambda^{\mathrm{in}}})$ is also called an intersection point, and we observe that $\beta_k(L,\lambda^{\mathrm{in}}) \in]0,1]$.

Some intersection points are less interesting than others in the sense that some intersection points can be written as a convex combinations of other points in $R(L,P)$. Such intersection points can therefore not be vertices of $R(L,P)$. The following lemma shows that the only intersection points of the form $v_{\lambda^{\mathrm{in}}} + \beta_k(L,\lambda^{\mathrm{in}})(v^k - v_{\lambda^{\mathrm{in}}})$ with $k \in V^{\mathrm{out}}(L)$ and $\lambda^{\mathrm{in}} \in \Lambda^{\mathrm{in}}(L)$ which can *not* be written as a convex combination of other points in $R(L,P)$ must be such that λ^{in} is a unit vector.



(a) The polytope P and the split polyhedron L from Figure 2

(b) Intersection points from v^1

(c) Intersection points from v^2

Figure 7: Determining the intersection points from a linear relaxation P and a split polyhedron L

Lemma 4.1 *Let L be a mixed integer split polyhedron satisfying $V^{in}(L) \neq \emptyset$, and let $k \in V^{out}(L)$. For every $\lambda^{in} \in \Lambda^{in}(L)$, the intersection point $v_{\lambda^{in}} + \beta_k(L, \lambda^{in})(v^k - v_{\lambda^{in}})$ is a convex combination of v^k and the intersection points $v^i + \beta_k(L, e^i)(v^k - v^i)$ for $i \in V^{in}(L)$.*

Proof. Define $C := \mathrm{conv}(\{v^i + \beta_k(L, e^i)(v^k - v^i)\}_{i \in V^{\mathrm{in}}(L)})$. We first show that the halfline given by $\{v_{\lambda^{\mathrm{in}}} + \beta(v^k - v_{\lambda^{\mathrm{in}}}) : \beta \geq 0\}$ intersects C for some $\beta^* \geq 0$. We have that $\{v_{\lambda^{\mathrm{in}}} + \beta(v^k - v_{\lambda^{\mathrm{in}}}) : \beta \geq 0\} \cap C \neq \emptyset$ if and only if the following LP is feasible.

$$\min\ 0$$

$$\sum_{i \in V^{\mathrm{in}}(L)} \eta_i(v^i + \beta_k(L, e^i)(v^k - v^i)) + \beta(v_{\lambda^{\mathrm{in}}} - v^k) = v_{\lambda^{\mathrm{in}}}, \tag{11}$$

$$\sum_{i \in V^{\mathrm{in}}(L)} \eta_i = 1, \tag{12}$$

$$\eta, \beta \geq 0. \tag{13}$$

The dual of this LP is given by

$$\max\ \delta^T v_{\lambda^{\mathrm{in}}} - \delta_0$$

$$\delta^T(v_{\lambda^{\mathrm{in}}} - v^k) \leq 0, \tag{14}$$

$$\delta^T(v^i + \beta_k(L, e^i)(v^k - v^i)) - \delta_0 \leq 0, \text{ for all } i \in V^{\mathrm{in}}(L). \tag{15}$$

Let $(\bar{\delta}, \bar{\delta}_0)$ be a solution to (14)-(15). Suppose, for a contradiction, that $\bar{\delta}^T v_{\lambda^{\mathrm{in}}} - \bar{\delta}_0 > 0$. Adding (14) to the inequality of (15) corresponding to $\bar{i} \in V^{\mathrm{in}}(L)$ gives $\bar{\delta}^T v_{\lambda^{\mathrm{in}}} - \bar{\delta}_0 + (1 - \beta_k(L, e^{\bar{i}}))\bar{\delta}^T(v^{\bar{i}} - v^k) \leq 0$. Since by assumption $\bar{\delta}^T v_{\lambda^{\mathrm{in}}} - \bar{\delta}_0 > 0$, this implies $\bar{\delta}^T(v^{\bar{i}} - v^k) < 0$. Hence we have $\bar{\delta}^T(v^i - v^k) < 0$ for all $i \in V^{\mathrm{in}}(L)$. Now, for all $i \in V^{\mathrm{in}}(L)$, inequality (15) gives $\bar{\delta}_0 - \bar{\delta}^T v^i \geq \beta_k(L, e^i)\bar{\delta}^T(v^k - v^i)$. Since $\bar{\delta}^T(v^k - v^i) > 0$ for all $i \in V^{\mathrm{in}}(L)$, this implies $\bar{\delta}_0 - \bar{\delta}^T v^i > 0$ for all $i \in V^{\mathrm{in}}(L)$. Multiplying each of the inequalities $\bar{\delta}_0 - \bar{\delta}^T v^i > 0$ for $i \in V^{\mathrm{in}}(L)$ with λ_i^{in} and adding the resulting inequalities together then gives $\bar{\delta}_0 - \bar{\delta}^T v_{\lambda^{\mathrm{in}}} > 0$. This contradicts our initial assumption that $\bar{\delta}^T v_{\lambda^{\mathrm{in}}} - \bar{\delta}_0 > 0$.

Therefore there exists $\beta^* \geq 0$ s.t. $v_{\lambda^{\mathrm{in}}} + \beta^*(v^k - v_{\lambda^{\mathrm{in}}}) \in C$. Observe that, since $v^i + \beta_k(L, e^i)(v^k - v^i) \in L$ for all $i \in V^{\mathrm{in}}(L)$, we have $v_{\lambda^{\mathrm{in}}} + \beta^*(v^k - v_{\lambda^{\mathrm{in}}}) \in L$. If $v_{\lambda^{\mathrm{in}}} + \beta^*(v^k - v_{\lambda^{\mathrm{in}}}) \in \mathrm{int}(L)$, then $\beta_k(L, \lambda^{\mathrm{in}}) > \beta^*$, and therefore $v_{\lambda^{\mathrm{in}}} + \beta_k(L, \lambda^{\mathrm{in}})(v^k - v_{\lambda^{\mathrm{in}}}) \in \mathrm{conv}(C \cup \{v^k\})$. If $v_{\lambda^{\mathrm{in}}} + \beta^*(v^k - v_{\lambda^{\mathrm{in}}})$ is on the boundary of L, then $\beta_k(L, \lambda^{\mathrm{in}}) = \beta^*$, which implies $v_{\lambda^{\mathrm{in}}} + \beta_k(L, \lambda^{\mathrm{in}})(v^k - v_{\lambda^{\mathrm{in}}}) \in \mathrm{conv}(C \cup \{v^k\})$. □

Figure 7 gives all the intersection points which can potentially be vertices of $R(L, P)$ for the example of Figure 2.

4.2 The intersection cut In [?], Balas considered a mixed integer set defined from the translate of a polyhedral cone, and a mixed integer split polyhedron was used to derive a valid inequality the *intersection* *cut*. We now consider a subset $P(\lambda^{\text{in}})$ of P defined from a fixed convex combination $\lambda^{\text{in}} \in \Lambda^{\text{in}}(L)$ of the vertices in the interior of L, and we show that the intersection cut gives a complete description of the set $R(L, P(\lambda^{\text{in}}))$ in a higher dimensional space. Specifically, given any fixed convex combination $\lambda^{\text{in}} \in \Lambda^{\text{in}}(L)$, we have the following subset $P(\lambda^{\text{in}})$ of P

$$P(\lambda^{\text{in}}) := \{x \in \mathbb{R}^n : x = v_{\lambda^{\text{in}}} + \sum_{k\in V^{\text{out}}(L)} \epsilon_k(v^k - v_{\lambda^{\text{in}}}) + \sum_{j\in E} \mu_j r^j, \epsilon, \mu \geq 0 \text{ and } \sum_{k\in V^{\text{out}}} \epsilon_k \leq 1\}.$$

The corresponding image $P^H(\lambda^{\text{in}})$ of $P(\lambda^{\text{in}})$ in (x, ϵ, μ) space is given by

$$P^H(\lambda^{\text{in}}) := \{(x, \epsilon, \mu) : x = v_{\lambda^{\text{in}}} + \sum_{k\in V^{\text{out}}(L)} \epsilon_k(v^k - v_{\lambda^{\text{in}}}) + \sum_{j\in E} \mu_j r^j, \epsilon, \mu \geq 0 \text{ and } \sum_{k\in V^{\text{out}}} \epsilon_k \leq 1\}.$$

As in Sect. 3.1, we use the superscript "H" to emphasize that $P^H(\lambda^{\text{in}})$ is an image of $P(\lambda^{\text{in}})$ in a higher dimensional space. The set $P(\lambda^{\text{in}})$ and the mixed integer split polyhedron L gives a relaxation $R(L, P(\lambda^{\text{in}}))$ of the set of mixed integer points in $P(\lambda^{\text{in}})$

$$R(L, P(\lambda^{\text{in}})) = \text{conv}(P(\lambda^{\text{in}}) \setminus \text{int}(L)).$$

The lifted version $R^H(L, P(\lambda^{\text{in}}))$ of $R(L, P(\lambda^{\text{in}}))$ in (x, ϵ, μ) space is then defined to be the set $R^H(L, P(\lambda^{\text{in}})) := \text{conv}(\{(x, \epsilon, \mu) \in P^H(\lambda^{\text{in}}) : x \notin \text{int}(L)\})$. Given $\lambda^{\text{in}} \in \Lambda^{\text{in}}(L)$, and the corresponding intersection points, Balas [**?**] derived the *intersection cut*

$$\sum_{j\in E} \frac{\mu_j}{\alpha_j(L, \lambda^{\text{in}})} + \sum_{k\in V^{\text{out}}(L)} \frac{\epsilon_k}{\beta_k(L, \lambda^{\text{in}})} \geq 1 \tag{16}$$

and showed that the intersection cut is valid for $R^H(L, P(\lambda^{\text{in}}))$. We now show that, in fact, the intersection cut gives a complete description of $R^H(L, P(\lambda^{\text{in}}))$.

THEOREM 4.1 *Let L be a mixed integer split polyhedron satisfying $V^{in}(L) \neq \emptyset$, and let $\lambda^{in} \in \Lambda^{in}(L)$.*

$$R^H(L, P(\lambda^{in})) = \{(x, \epsilon, \mu) \in P^H(\lambda^{in}) : \sum_{j\in E} \frac{\mu_j}{\alpha_j(L, \lambda^{in})} + \sum_{k\in V^{out}(L)} \frac{\epsilon_k}{\beta_k(L, \lambda^{in})} \geq 1\}.$$

PROOF. Since the theorem concerns a fixed mixed integer split polyhedron L and a fixed convex combination $\lambda^{\text{in}} \in \Lambda^{\text{in}}(L)$, we abbreviate $\alpha_j := \alpha_j(L, \lambda^{\text{in}})$, $\beta_j := \beta_j(L, \lambda^{\text{in}})$ and $V^{\text{out}} := V^{\text{out}}(L)$.

Since (16) is valid for $R^H(L, P(\lambda^{\text{in}}))$, we have

$$R^H(L, P(\lambda^{\text{in}})) \subseteq \{(x, \epsilon, \mu) \in P^H(\lambda^{\text{in}}) : \sum_{j\in E} \frac{\mu_j}{\alpha_j} + \sum_{k\in V^{\text{out}}} \frac{\epsilon_k}{\beta_k} \geq 1\}.$$

Conversely suppose $(\bar{x}, \bar{\epsilon}, \bar{\mu}) \in P^H(\lambda^{\text{in}})$ and $\sum_{j\in E} \frac{\bar{\mu}_j}{\alpha_j} + \sum_{k\in V^{\text{out}}} \frac{\bar{\epsilon}_k}{\beta_k} \geq 1$. We will show that $(\bar{x}, \bar{\epsilon}, \bar{\mu}) \in R^H(L, P(\lambda^{\text{in}}))$. Define $E^\infty := \{j \in E : \alpha_j = +\infty\}$. We distinguish three cases.

(a) First suppose $\sum_{j\in E} \frac{\bar{\mu}_j}{\alpha_j} + \sum_{k\in V^{\text{out}}} \frac{\bar{\epsilon}_k}{\beta_k} = 1$. We can write

$$\begin{pmatrix} \bar{x} \\ \bar{\epsilon} \\ \bar{\mu} \end{pmatrix} = \sum_{k\in V^{\text{out}}} \frac{\bar{\epsilon}_k}{\beta_k} \begin{pmatrix} v_{\lambda^{\text{in}}} + \beta_k(v^k - v_{\lambda^{\text{in}}}) \\ \beta_k e^k \\ 0 \end{pmatrix} + \sum_{j\in E\setminus E^\infty} \frac{\bar{\mu}_j}{\alpha_j} \begin{pmatrix} v_{\lambda^{\text{in}}} + \alpha_j r^j \\ 0 \\ \alpha_j e^j \end{pmatrix} + \sum_{j\in E^\infty} \bar{\mu}_j \begin{pmatrix} r^j \\ 0 \\ e^j \end{pmatrix}.$$

Since $v_{\lambda^{\text{in}}} + \alpha_j r^j \notin \text{int}(L)$ for $j \in E \setminus E^\infty$, $v_{\lambda^{\text{in}}} + \beta_k(v^k - v_{\lambda^{\text{in}}}) \notin \text{int}(L)$ for $k \in V^{\text{out}}$ and $(r^j, 0, e^j)$ is in the recession cone of $R^H(L, P(\lambda^{\text{in}}))$ for $j \in E^\infty$, we have $(\bar{x}, \bar{\epsilon}, \bar{\mu}) \in R^H(L, P(\lambda^{\text{in}}))$.

(b) Next suppose $\sum_{j\in E} \frac{\bar{\mu}_j}{\alpha_j} + \sum_{k\in V^{\text{out}}} \frac{\bar{\epsilon}_k}{\beta_k} > 1$ and $0 < \sum_{k\in V^{\text{out}}} \bar{\epsilon}_k \leq 1$. We will construct two points (x^1, ϵ^1, μ^1), $(x^2, \epsilon^2, \mu^2) \in R^H(L, P(\lambda^{\text{in}}))$ such that $(\bar{x}, \bar{\epsilon}, \bar{\mu})$ is on the line between (x^1, ϵ^1, μ^1) and (x^2, ϵ^2, μ^2). Let $\bar{\delta} \in]0, 1[$ be such that $(x^1, \epsilon^1, \mu^1) := \bar{\delta}(v_{\lambda^{\text{in}}}, 0, 0) + (1 - \bar{\delta})(\bar{x}, \bar{\epsilon}, \bar{\mu})$ satisfies $\sum_{j\in E} \frac{\mu^1_j}{\alpha_j} + \sum_{k\in V^{\text{out}}} \frac{\epsilon^1_k}{\beta_k} = 1$. It follows from (a) that $(x^1, \epsilon^1, \mu^1) \in R^H(L, P(\lambda^{\text{in}}))$. Consider the halfline $\{v_{\lambda^{\text{in}}} + \alpha(x^1 - v_{\lambda^{\text{in}}}) : \alpha \geq 0\}$. For $\alpha_{x^1} := 1$, we have $v_{\lambda^{\text{in}}} + \alpha_{x^1}(x^1 - v_{\lambda^{\text{in}}}) = x^1$, and for $\alpha_{\bar{x}} := \frac{1}{1-\bar{\delta}}$, we have $v_{\lambda^{\text{in}}} + \alpha_{\bar{x}}(x^1 - v_{\lambda^{\text{in}}}) = \bar{x}$. Consider the point $x^2 := v_{\lambda^{\text{in}}} + \alpha_{x^2}(x^1 - v_{\lambda^{\text{in}}})$, where $\alpha_{x^2} := \frac{1}{\sum_{k\in V^{\text{out}}} \epsilon^1_k}$. Since $0 < \sum_{k\in V^{\text{out}}} \epsilon^1_k \leq 1 - \bar{\delta}$, we have $\alpha_{x^1} < \alpha_{\bar{x}} \leq \alpha_{x^2} < +\infty$.

Hence $\bar{x}$ is on the line between x^1 and x^2. Defining $\epsilon^2 := \alpha_{x^2}\epsilon^1$ and $\mu^2 := \alpha_{x^2}\mu^1$, we may write $x^2 = v_{\lambda^{\text{in}}} + \sum_{k\in V^{\text{out}}} \epsilon^2_k(v^k - v_{\lambda^{\text{in}}}) + \sum_{j\in E}\mu^2_j r^j$. Observe that $\sum_{k\in V^{\text{out}}}\epsilon^2_k = 1$. Hence we can write $x^2 = \sum_{k\in V^{\text{out}}}\epsilon^2_k v^k + \sum_{j\in E}\mu^2_j r^j$, where $\sum_{k\in V^{\text{out}}}\epsilon^2_k = 1$. Since $(r^j, 0, e^j)$ is in the recession cone of $R^H(L, P(\lambda^{\text{in}}))$ for $j \in E$, and since $(v^k, e^k, 0) \in R^H(L, P(\lambda^{\text{in}}))$ for $k \in V^{\text{out}}$, we have $(x^2, \epsilon^2, \mu^2) \in R^H(L, P(\lambda^{\text{in}}))$. Finally, since $(\bar{x}, \bar{\epsilon}, \bar{\mu})$ is on the line between (x^1, ϵ^1, μ^1) and (x^2, ϵ^2, μ^2), we have $(\bar{x}, \bar{\epsilon}, \bar{\mu}) \in R^H(L, P(\lambda^{\text{in}}))$.

(c) Finally suppose $\sum_{k\in V^{\text{out}}}\bar{\epsilon}_k = 0$ and $\sum_{j\in E}\frac{\bar{\mu}_j}{\alpha_j} > 1$. As in (b), let $\bar{\delta} \in]0,1[$ be s.t. $(x^1, \epsilon^1, \mu^1) := \bar{\delta}(v_{\lambda^{\text{in}}}, 0, 0) + (1-\bar{\delta})(\bar{x}, 0, \bar{\mu})$ satisfies $\sum_{j\in E}\frac{\mu^1_j}{\alpha_j} = 1$. From (a) we have $(x^1, \epsilon^1, \mu^1) \in R^H(L, P(\lambda^{\text{in}}))$, and $(\bar{x}, 0, \bar{\mu}) = (v_{\lambda^{\text{in}}}, 0, 0) + \frac{1}{1-\bar{\delta}}(x^1 - v_{\lambda^{\text{in}}}, \epsilon^1, \mu^1)$. Since $(x^1, 0, \mu^1) \in R^H(L, P(\lambda^{\text{in}}))$ satisfies $\sum_{j\in E}\frac{\mu^1_j}{\alpha_j} = 1$, (a) shows

$$\begin{pmatrix} x^1 \\ 0 \\ \mu^1 \end{pmatrix} = \sum_{j\in E\setminus E^\infty}\frac{\mu^1_j}{\alpha_j}\begin{pmatrix} v_{\lambda^{\text{in}}} + \alpha_j r^j \\ 0 \\ \alpha_j e^j \end{pmatrix} + \sum_{j\in E^\infty}\mu^1_j\begin{pmatrix} r^j \\ 0 \\ e^j \end{pmatrix}.$$

Using $(\bar{x}, 0, \bar{\mu}) = (v_{\lambda^{\text{in}}}, 0, 0) + \frac{1}{1-\bar{\delta}}(x^1 - v_{\lambda^{\text{in}}}, \epsilon^1, \mu^1)$, we can now write

$$\begin{pmatrix} \bar{x} \\ 0 \\ \bar{\mu} \end{pmatrix} = \sum_{j\in E\setminus E^\infty}\frac{\mu^1_j}{\alpha_j}\begin{pmatrix} v_{\lambda^{\text{in}}} + \alpha_j r^j \\ 0 \\ \alpha_j e^j \end{pmatrix} + \sum_{j\in E\setminus E^\infty}\frac{\mu^1_j\bar{\delta}}{1-\bar{\delta}}\begin{pmatrix} r^j \\ 0 \\ e^j \end{pmatrix} + \sum_{j\in E^\infty}\frac{\mu^1_j}{1-\bar{\delta}}\begin{pmatrix} r^j \\ 0 \\ e^j \end{pmatrix}.$$

Since $(v_{\lambda^{\text{in}}} + \alpha_j r^j, 0, \alpha_j e^j) \in R^H(L, P(\lambda^{\text{in}}))$ for $j \in E \setminus E^\infty$, $\sum_{j\in E}\frac{\mu^1_j}{\alpha_j} = 1$ and the vectors $(r^j, 0, e^j)$ for $j \in E$ are the extreme rays of $R^H(L, P(\lambda^{\text{in}}))$, this completes the proof.

□

4.3 The vertices of $R(L, P)$ The proof of Theorem 4.1 allows us to characterize the vertices of $R(L, P)$. Observe that in the proof of Theorem 4.1, every point in $R(L, P(\lambda^{\text{in}}))$ is expressed in terms of intersection points, vertices of P that are not in the interior of L and the extreme rays r^j of $R(L, P(\lambda^{\text{in}}))$ for $j \in E$. Hence the proof of Theorem 4.1 provides a characterization of the vertices of $R(L, P(\lambda^{\text{in}}))$.

Corollary 4.1 *Let L be a mixed integer split polyhedron satisfying $V^{in}(L) \neq \emptyset$, and let $\lambda^{in} \in \Lambda^{in}(L)$. Define $E^\infty(\lambda^{in}) := \{j \in E : \alpha_j(L, \lambda^{in}) = +\infty\}$. A vertex of $R(L, P(\lambda^{in}))$ is either*

(i) A vertex v^k of P, where $k \in V^{out}(L)$,

(ii) An intersection point $v_{\lambda^{in}} + \beta_k(L, \lambda^{in})(v^k - v_{\lambda^{in}})$, where $k \in V^{out}(L)$, or

(iii) An intersection point $v_{\lambda^{in}} + \alpha_j(L, \lambda^{in})r^j$, where $j \in E \setminus E^\infty(\lambda^{in})$.

By using the properties of $\alpha_j(L, \lambda^{\text{in}})$ and $\beta_k(L, \lambda^{\text{in}})$ for $\lambda^{\text{in}} \in \Lambda^{\text{in}}(L)$ given in Lemma 4.1 and Lemma 4.1, we can use Corollary 4.1 to characterize the vertices of $R(L, P)$. In the following, for simplicity let $\alpha_{i,j}(L) := \alpha_j(L, e^i)$ and $\beta_{i,k}(L) := \beta_k(L, e^i)$ for $i \in V^{\text{in}}(L)$, $j \in E$ and $k \in V^{\text{out}}(L)$. Also let $E^\infty(L) := \{j \in E : \alpha_{i,j}(L) = +\infty \text{ for some } i \in V^{\text{in}}(L)\}$ denote those extreme rays of P that are also rays of L.

Lemma 4.2 *Let L be a mixed integer split polyhedron satisfying $V^{in}(L) \neq \emptyset$. Every vertex of $R(L, P)$ is of one of the following the forms.*

(i) A vertex v^k of P, where $k \in V^{out}(L)$,

(ii) An intersection point $v^i + \beta_{i,k}(L)(v^k - v^i)$, where $i \in V^{in}(L)$ and $k \in V^{out}(L)$, or

(iii) An intersection point $v^i + \alpha_{i,j}(L)r^j$, where $i \in V^{in}(L)$ and $j \in E \setminus E^\infty(L)$.

Proof. Let $\bar{x} \in R(L, P)$ be a vertex of $R(L, P)$, and let $(\bar{\lambda}^{\text{in}}, \bar{\epsilon}, \bar{\mu}) \in \mathbb{R}^{|V|+|E|}$ satisfy $\bar{x} = v_{\bar{\lambda}^{\text{in}}} + \sum_{k\in V^{\text{out}}(L)}\bar{\epsilon}_k(v^k - v_{\bar{\lambda}^{\text{in}}}) + \sum_{j\in E}\bar{\mu}_j r^j$, $\bar{\epsilon} \geq 0$, $\bar{\mu} \geq 0$, $\bar{\lambda}^{\text{in}} \in \Lambda^{\text{in}}(L)$ and $\sum_{k\in V^{\text{out}}(L)}\bar{\epsilon}_k \leq 1$. Now, we have $\bar{x} \in R(L, P(\bar{\lambda}^{\text{in}}))$, and since $R(L, P(\bar{\lambda}^{\text{in}})) \subseteq R(L, P)$, we must have that $\bar{x}$ is a vertex of $R(L, P(\bar{\lambda}^{\text{in}}))$. It follows that $\bar{x}$ is of one of the forms Corollary 4.1.(i)-(iii). If $\bar{x}$ is of the form $\bar{x} = v^k$ for some $k \in V^{\text{out}}(L)$, we are done. Furthermore, if $\bar{x} = v_{\bar{\lambda}^{\text{in}}} + \beta_k(L, \bar{\lambda}^{\text{in}})(v^k - v_{\bar{\lambda}^{\text{in}}})$ for some $k \in V^{\text{out}}(L)$, then Lemma 4.1 shows that either $\bar{x} = v^k$, or $\bar{x} = v^{\bar{i}} + \beta_{\bar{i},k}(L)(v^k - v^{\bar{i}})$ for some $\bar{i} \in V^{\text{out}}(L)$.

Finally consider the case when $\bar{x}$ is of the form $\bar{x} = v_{\bar{\lambda}^{\text{in}}} + \alpha_{\bar{j}}(L, \bar{\lambda}^{\text{in}})r^{\bar{j}}$ for some $\bar{j} \in E \setminus E^\infty(L)$. Since $\alpha_{\bar{j}}(L, \bar{\lambda}^{\text{in}})$ is concave in $\bar{\lambda}^{\text{in}}$, we have $\alpha_{\bar{j}}(L, \bar{\lambda}^{\text{in}}) \geq \sum_{i \in V^{\text{in}}(L)} \bar{\lambda}_i^{\text{in}} \alpha_{i,\bar{j}}(L)$. Let $\delta \geq 0$ satisfy $\alpha_{\bar{j}}(L, \bar{\lambda}^{\text{in}}) = \sum_{i \in V^{\text{in}}(L)} \bar{\lambda}_i^{\text{in}}(\alpha_{i,\bar{j}}(L) + \delta)$. We can now write $\bar{x}$ in the form $\bar{x} = v_{\bar{\lambda}^{\text{in}}} + \alpha_{\bar{j}}(L, \bar{\lambda}^{\text{in}})r^{\bar{j}} = \sum_{i \in V^{\text{in}}(L)} \bar{\lambda}_i^{\text{in}}(v_i + (\alpha_{i,\bar{j}}(L) + \delta)r^{\bar{j}})$. Since $v_i + (\alpha_{i,\bar{j}}(L) + \delta)r^{\bar{j}} \notin \text{int}(L)$ for all $i \in V^{\text{in}}(L)$, and $\bar{x}$ is a vertex of $R(L, P)$, we must have $\delta = 0$ and $\bar{\lambda}_{\bar{i}}^{\text{in}} = 1$ for some $\bar{i} \in V^{\text{in}}(L)$. □

An important consequence of Lemma 4.2 is the following dominance result. For two mixed integer split polyhedra L^1 and L^2, if $V^{\text{in}}(L^1) = V^{\text{in}}(L^2)$, and if all the halflines $\{v^i + \alpha r^j : \alpha \geq 0\}$ and $\{v^i + \beta(v^k - v^i)\}$ for $i \in V^{\text{in}}(L^1) = V^{\text{in}}(L^2)$, $j \in E$ and $k \in V^{\text{out}}(L^1) = V^{\text{out}}(L^2)$ all intersect the boundary of L^1 later than the boundary of L^2, then $R(L^1, P) \subseteq R(L^2, P)$. In other words, the relaxation of P_I obtained from L^1 is stronger than the relaxation of P_I obtained from L^2.

COROLLARY 4.2 *Let L^1 and L^2 be mixed integer split polyhedra satisfying $V^{in} := V^{in}(L^1) = V^{in}(L^2)$. If*

(i) $\alpha_{i,j}(L^1) \geq \alpha_{i,j}(L^2)$ *for all $i \in V^{in}$ and $j \in E$, and*

(ii) $\beta_{i,k}(L^1) \geq \beta_{i,k}(L^2)$ *for all $i \in V^{in}$ and $k \in V \setminus V^{in}$,*

then $R(L^1, P) \subseteq R(L^2, P)$, and we say L^1 dominates *L^2 on P. Furthermore, if L^1 dominates L^2 on P, and L^2 dominates L^1 on P, then $R(L^1, P) = R(L^2, P)$, and we say L^1 and L^2 are* equivalent *on P.*

Another consequence of Lemma 4.2 is that it is possible to write $R(L, P)$ as the convex hull of the union of the polyhedra $R(L, P(e^i))$ for $i \in V^{\text{in}}(L)$.

COROLLARY 4.3 *Let L be a mixed integer split polyhedron satisfying $V^{in}(L) \neq \emptyset$. We have*

$$R(L, P) = \text{conv}(\cup_{i \in V^{in}(L)} R(L, P(e^i))).$$

PROOF. Lemma 4.2 shows that every vertex of $R(L, P)$ is a vertex of a set $R(L, P(e^i))$ for some $i \in V^{\text{in}}(L)$. Furthermore, the union of the vertices of the sets $R(L, P(e^i))$ over all $i \in V^{\text{in}}(L)$ is exactly the set of vertices of $R(L, P)$. Since the extreme rays of $R(L, P)$ and the sets $R(L, P(e^i))$ for $i \in V^{\text{in}}(L)$ are the same, namely the vectors $\{r^j\}_{j \in E}$, the result follows. □

Figure 8 illustrates Corollary 4.3 on the example of Figure 2. The sets $P(e^1)$ and $P(e^2)$ corresponding to the two vertices v^1 and v^2 of P that are in the interior of L are shown in Figure 8.(b) and Figure 8.(c). Observe that the sets $R(L, P(e^1))$ and $R(L, P(e^2))$ are both described by adding exactly one cut to $P(e^1)$ and $P(e^2)$ respectively. Corollary 4.3 then shows that $R(L, P)$ can be obtained by taking the convex hull of the union of the sets $R(L, P(e^1))$ and $R(L, P(e^2))$.

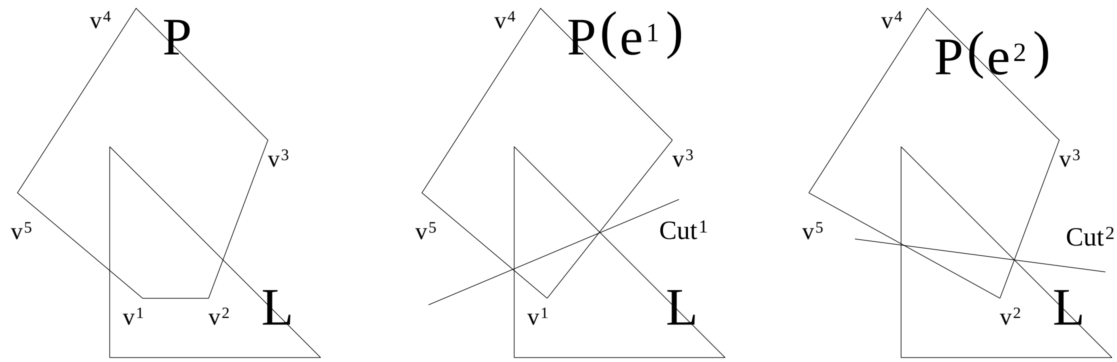
4.4 Polyhedrality of split closures We now derive a result for split closures of the polyhedron P. Let $\bar{\mathcal{L}}$ denote an arbitrary family of mixed integer split polyhedra. We call the set $\text{Cl}(P, \bar{\mathcal{L}}) = \cap_{L \in \bar{\mathcal{L}}} R(L, P)$ a *split closure* of P. We will provide a sufficient condition for $\text{Cl}(P, \bar{\mathcal{L}})$ to be a polyhedron. This result is then used to show that a bound on the max-facet-width on the mixed integer split polyhedra in $\bar{\mathcal{L}}$ satisfies this sufficient condition. The main result in this section is the following.

THEOREM 4.2 *Let $\bar{\mathcal{L}}$ be an arbitrary family of mixed integer split polyhedra that all have the same vertices of P in their interior, i.e., we have $V^{in} := V^{in}(L^1) = V^{in}(L^2)$ for all $L^1, L^2 \in \bar{\mathcal{L}}$. If*

(i) For all $(i, j) \in V^{in} \times E$ and $\alpha^* > 0$, the set $\{\alpha_{i,j}(L) \geq \alpha^* : L \in \bar{\mathcal{L}}\}$ is finite
(There is only a finite number of intersection points between the the halfline $\{v^i + \alpha r^j : \alpha \geq \alpha^*\}$ and the boundaries of the mixed integer split polyhedra $L \in \bar{\mathcal{L}}$),

(ii) For all $(i, k) \in V^{in} \times (V \setminus V^{in})$ and $\beta^* \in]0, 1]$, the set $\{\beta_{i,k}(L) \geq \beta^* : L \in \bar{\mathcal{L}}\}$ is finite
(There is only a finite number of intersection points between the line segment $\{v^i + \beta(v^k - v^i) : \beta^* \leq \beta \leq 1\}$ and the boundaries of the mixed integer split polyhedra $L \in \bar{\mathcal{L}}$),

then $Cl(P, \bar{\mathcal{L}})$ is a polyhedron.



(a) The polytope P and the split polyhedron L from Figure 2 (b) The set $P(e^1)$ constructed from v^1 (c) The set $P(e^2)$ constructed from v^2

Figure 8: Constructing $R(L,P)$ as the convex hull of the union of polyhedra

Our proof of Theorem 4.2 is essentially the same as the proof of Theorem 3.1. We therefore only sketch the proof. Similar to the proof of Theorem 3.1, the proof is based on counting the number of intersection points shared by all mixed integer split polyhedra in a family $\mathcal{L}' \subseteq \bar{\mathcal{L}}$ of mixed integer split polyhedra. Define $\mathrm{s}(\mathcal{L}') := |\mathrm{SIP}^e(\mathcal{L}')|+|\mathrm{SIP}^v(\mathcal{L}')|$, where $\mathrm{SIP}^e(\mathcal{L}')$ and $\mathrm{SIP}^v(\mathcal{L}')$ are given by

$$\mathrm{SIP}^e(\mathcal{L}') := \{(i,j) \in V^{\mathrm{in}} \times E : \alpha_{i,j}(L^1) = \alpha_{i,j}(L^2) \text{ for all } L^1, L^2 \in \mathcal{L}'\}, \text{ and}$$
$$\mathrm{SIP}^v(\mathcal{L}') := \{(i,k) \in V^{\mathrm{in}} \times (V \setminus V^{\mathrm{in}}) : \beta_{i,k}(L^1) = \beta_{i,k}(L^2) \text{ for all } L^1, L^2 \in \mathcal{L}'\}.$$

We have $0 \leq \mathrm{s}(\mathcal{L}') \leq |V^{\mathrm{in}} \times E|+|V^{\mathrm{in}} \times (V \setminus V^{\mathrm{in}})|$ for all families $\mathcal{L}' \subseteq \bar{\mathcal{L}}$ of mixed integer split polyhedra. If $\mathrm{s}(\mathcal{L}') = |V^{\mathrm{in}} \times E| + |V^{\mathrm{in}} \times (V \setminus V^{\mathrm{in}})|$, then Corollary 4.2 shows that all split polyhedra indexed by $\mathcal{L}'$ are equivalent on P. Therefore, if $\mathrm{s}(\mathcal{L}') = |V^{\mathrm{in}} \times E| + |V^{\mathrm{in}} \times (V \setminus V^{\mathrm{in}})|$, then $\mathrm{Cl}(P,\mathcal{L}')$ is a polyhedron that can be described with exactly one mixed integer split polyhedron, *i.e.*, $\mathrm{Cl}(P,\mathcal{L}') = R(L,P)$, where $L \in \mathcal{L}'$ is arbitrary.

We next partition $\bar{\mathcal{L}}$ into subsets:

$$S^1, S^2, \ldots, S^{\mathrm{ns}} \subseteq \bar{\mathcal{L}}, \cup_{m=1}^{\mathrm{ns}} S^m = \bar{\mathcal{L}} \text{ and } S^{m_1} \cap S^{m_2} = \emptyset \text{ for all } m_1 \neq m_2$$

such that for every $m \in \{1,2,\ldots,\mathrm{ns}\}$, either $\mathrm{Cl}(P,S^m)$ is a polyhedron, or $s(S^m) > s(\bar{\mathcal{L}})$. (Again, "ns" is an abbreviation of "number of subsets"). Since $\{S^m\}_{m=1}^{\mathrm{ns}}$ is a partitioning of $\bar{\mathcal{L}}$, we have

$$\mathrm{Cl}(P,\bar{\mathcal{L}}) = \cap_{m=1}^{\mathrm{ns}} \mathrm{Cl}(P,S^m),$$

and $\mathrm{Cl}(P,\bar{\mathcal{L}})$ is a polyhedron if $\mathrm{Cl}(P,S^m)$ is a polyhedron for all $m \in \{1,2,\ldots,\mathrm{ns}\}$. To prove Theorem 4.2, it suffices to construct a partitioning $\{S^m\}_{m=1}^{\mathrm{ns}}$ of $\bar{\mathcal{L}}$ such that for every $m \in \{1,2,\ldots,\mathrm{ns}\}$, either $\mathrm{Cl}(P,S^m)$ is a polyhedron, or $s(S^m) > s(\bar{\mathcal{L}})$.

The partitioning $\{S^m\}_{m=1}^{\mathrm{ns}}$ of $\bar{\mathcal{L}}$ is now constructed. Let $\mathcal{L}^f \subseteq \bar{\mathcal{L}}$ be an arbitrary non-empty and finite subset of $\bar{\mathcal{L}}$. The partitioning of $\bar{\mathcal{L}}$ is based on the following positive numbers.

$$\alpha_j^* := \min\{\alpha_{i,j}(L) : i \in V^{\mathrm{in}} \text{ and } L \in \mathcal{L}^f\} \text{ for } j \in E, \text{ and}$$
$$\beta_k^* := \min\{\beta_{i,k}(L) : i \in V^{\mathrm{in}} \text{ and } L \in \mathcal{L}^f\} \text{ for } k \in V \setminus V^{\mathrm{in}}.$$

Properties (i) and (ii) of Theorem 4.2 imply that the sets $\{\alpha_{i,j}(L) \geq \alpha_j^* : L \in \bar{\mathcal{L}}\}$ for $(i,j) \in V^{\mathrm{in}} \times E$, and the sets $\{\beta_{i,k}(L) \geq \beta_k^* : L \in \bar{\mathcal{L}}\}$ for $(i,k) \in V^{\mathrm{in}} \times (V \setminus V^{\mathrm{in}})$, are finite. An equivalence relation on $\bar{\mathcal{L}}$ can now be defined based on the following sets. Given $L \in \bar{\mathcal{L}}$, let

$$\mathrm{IP}^e(L) := \{(i,j,\alpha_{i,j}(L)) \in V^c \times E \times \mathbb{R}_+ : \alpha_{i,j}(L) \geq \alpha_j^*\}, \text{ and}$$
$$\mathrm{IP}^v(L) := \{(i,k,\beta_{i,k}(L)) \in V^{\mathrm{in}} \times (V \setminus V^{\mathrm{in}}) \times \mathbb{R}_+ : \beta_{i,k}(L) \geq \beta_k^*\}$$

denote the set of intersection points between L and the halflines $\{v^i + \alpha r^j : \alpha \geq \alpha_j^*\}$ for $(i,j) \in V^{\mathrm{in}} \times E$, and the set of intersection points between L and the line segments $\{v^i + \beta(v^k - v^i) : \beta_k^* \leq \beta \leq 1\}$ for

$(i,k) \in V^{\text{in}} \times (V \setminus V^{\text{in}})$ respectively. The equivalence relation on the mixed integer split polyhedra in $\bar{\mathcal{L}}$ is defined from the sets $\text{IP}^e(L)$ and $\text{IP}^v(L)$ for $L \in \bar{\mathcal{L}}$ as follows.

$$\text{For all } L^1, L^2 \in \bar{\mathcal{L}} : \ L^1 \equiv L^2 \iff \text{IP}^v(L^1) = \text{IP}^v(L^2) \text{ and } \text{IP}^e(L^1) = \text{IP}^e(L2).$$

Properties (i) and (ii) of Theorem 4.2 imply that the number of equivalence classes corresponding to the above equivalence relation is finite. Let $S^1, S^2, \ldots, S^{\text{ns}} \subseteq \bar{\mathcal{L}}$ denote these equivalence classes. For every equivalence class $m \in \{1, 2, \ldots, \text{ns}\}$ we have the following.

(a) For every $(i,j) \in V^{\text{in}} \times E$, if there exists $L \in S^m$ s.t. $\alpha_{i,j}(L) \geq \alpha_j^*$, then $(i,j) \in \text{SIP}^e(S^m)$.

(b) For every $(i,k) \in V^{\text{in}} \times (V \setminus V^{\text{in}})$, if there exists $L \in S^m$ s.t. $\beta_{i,k}(L) \geq \beta_k^*$, then $(i,k) \in \text{SIP}^v(S^m)$.

(c) If $S^m \cap \mathcal{L}^f \neq \emptyset$, then $\text{SIP}^e(S^m) = V^{\text{in}} \times E$ and $\text{SIP}^v(S^m) = V^{\text{in}} \times (V \setminus V^{\text{in}})$.

As in the proof of Theorem 3.1, and with the same argument, one can use (a)-(c) to show that if $s(S^m) = s(\bar{\mathcal{L}})$, then by choosing $\bar{L} \in \mathcal{L}^f$ arbitrarily, the mixed integer split polyhedron $\bar{L}$ dominates all mixed integer split polyhedra $L \in \bar{\mathcal{L}}$. This then completes the proof of Theorem 4.2.

We next present one application of Theorem 4.2. Specifically, we will show that if $\bar{\mathcal{L}}$ is a family of mixed integer split polyhedra, where each mixed integer split polyhedron $L \in \bar{\mathcal{L}}$ has max-facet-width at most a constant $w > 0$, then $\text{Cl}(P, \bar{\mathcal{L}})$ is a polyhedron.

Theorem 4.3 *Let $\bar{\mathcal{L}} \subseteq \mathcal{L}^w$ be any family of mixed integer split polyhedra that have max-facet-width at most $w > 0$. The set $Cl(P, \bar{\mathcal{L}}) = \cap_{L \in \bar{\mathcal{L}}} R(L, P)$ is a polyhedron.*

We can assume $V^{\text{in}} := V^{\text{in}}(L)$ for all $L \in \bar{\mathcal{L}}$. To apply Theorem 4.2, we need to argue that the sets $\{\alpha_{i,j}(L) \geq \alpha^* : L \in \bar{\mathcal{L}}\}$ and $\{\beta_{i,k}(L) \geq \beta^* : L \in \bar{\mathcal{L}}\}$ are finite, where $i \in V^{\text{in}}$, $j \in E$, $k \in V \setminus V^{\text{in}}$, $\alpha^* > 0$ and $\beta^* > 0$ are arbitrary. Our argument does not depend on the particular halfline or line segment, so we only consider the set $\{\alpha_{1,1}(L) \geq \alpha^* : L \in \bar{\mathcal{L}}\}$. We prove Theorem 4.3 in two steps. We first give a representation of $\alpha_{1,1}(L)$ for an arbitrary mixed integer split polyhedron $L \in \bar{\mathcal{L}}$. This is then used to show that the set $\{\alpha_{1,1}(L) \geq \alpha^* : L \in \bar{\mathcal{L}}\}$ is finite for any $\alpha^* > 0$. We will use that v^1 is rational, and that r^1 is integer.

Lemma 4.3 *(A generalization of [**?**, Lemma 5]). Let $L \in \mathcal{L}^w$ satisfy $v^1 \in \text{int}(L)$ and $\alpha_{1,1}(L) < +\infty$.*

(i) $0 < \alpha_{1,1}(L) \leq w$, and

(ii) $\alpha_{1,1}(L) = \frac{s(L)}{gt(L)}$, where g, $s(L)$,$t(L) > 0$ are integers satisfying $s(L) \leq gw$.

(Note that the integer g is independent of L).

Proof. We may write $L = \{x \in \mathbb{R}^n : (\pi^k)^T x \leq \pi_0^k \text{ for all } k \in N(L)\}$, where $N(L)$ is an index set for the facets of L, $(\pi^k, \pi_0^k) \in \mathbb{Z}^{n+1}$ and $\gcd(\pi^k, \pi_0^k) = 1$ for all $k \in N(L)$. From $v^1 \in \text{int}(L)$, it follows that $(\pi^k)^T v^1 < \pi_0^k$ for all $k \in N(L)$, and therefore $\alpha_{1,1}(L) = \frac{\pi_0^{\bar{k}} - (\pi^{\bar{k}})^T v^1}{(\pi^{\bar{k}})^T r^1}$ for some $\bar{k} \in N(L)$. Since L has max-facet-width at most w and $v^1 \in \text{int}(L)$, we have $0 < \pi_0^{\bar{k}} - (\pi^{\bar{k}})^T v^1 \leq w$. Furthermore, since $(\pi^{\bar{k}})^T r^1$ is integer, we have $(\pi^{\bar{k}})^T r^1 \geq 1$, and therefore $\alpha_{1,1}(L) \leq w$.

Recall that we assumed $v^1 \in \mathbb{Q}^n$ and $r^1 \in \mathbb{Z}^n$. We can therefore write $v^1 = (\frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots, \frac{p_n}{q_n})$, where $p_k \in \mathbb{Z}$ and $q_k \in \mathbb{N}$ for $k = 1, 2, \ldots, n$. Define integers $g := \Pi_{k=1}^n q_k$, $d_m := \Pi_{k=1, k \neq m}^n q_k$ for $m \in \{1, 2, \ldots, n\}$, $s(L) := g\pi_0^{\bar{k}} - \sum_{m=1}^n d_m p_m \pi_m^{\bar{k}}$ and $t(L) := (\pi^{\bar{k}})^T r^1$. Observe that $\frac{s(L)}{g} = \pi_0^{\bar{k}} - (\pi^{\bar{k}})^T v^1 \leq w$. With these choices, (ii) is satisfied. □

By using the above lemma, we can now bound the number of possible intersection points on the halfline $\{v^1 + \alpha r^1 : \alpha \geq \alpha^*\}$ for a given $\alpha^* > 0$.

Lemma 4.4 *([**?**, Lemma 6]). Let $\alpha^* > 0$, and let $\bar{\mathcal{L}} \subseteq \mathcal{L}^w$ be a family of mixed integer split polyhedra that have max-facet-width at most $w > 0$. The set $\{\alpha_{1,1}(L) : L \in \bar{\mathcal{L}} \text{ and } \alpha_{1,1}(L) \geq \alpha^*\}$ is finite.*

Proof. Let $L \in \bar{\mathcal{L}}$ satisfying $\alpha^* \leq \alpha_{1,1}(L) < +\infty$ be arbitrary. We may assume α^* is of the form $\alpha^* = \frac{s^*}{gt^*}$ for some integers $s^*, t^* > 0$ satisfying $0 < s^* < gw$.

Let $s(L)$ and $t(L)$ be as in Lemma 4.3. Hence we have $\alpha_{1,1}(L) = \frac{s(L)}{gt(L)}$, where $s(L) \in \{1, 2, \ldots, gw\}$. Hence there is only a finite number of possible values for $s(L)$. Finally, Lemma 4.3.(i) and $\alpha_{1,1}(L) \geq \alpha^*$ gives $\frac{s^*}{gt^*} \leq \frac{s(L)}{gt(L)} \leq w$, and therefore $\frac{s(L)}{gw} \leq t(L) \leq \frac{s(L)t^*}{s^*}$. Hence, for a fixed value $s(L) \in \{1, 2, \ldots, gw\}$, there is only a finite number of possible values for $t(L)$. □

5. Finite split polyhedron proofs Mixed integer split polyhedra can be used to design finite cutting plane proofs for the validity of an inequality for P_I as follows. Let $\delta^T x \geq \delta_0$ be a valid inequality for P_I. Observe that, if $\delta^T x \geq \delta_0$ is valid for $R(L, P)$ for some mixed integer split polyhedron L, then L provides a finite cutting plane proof for the validity of $\delta^T x \geq \delta_0$ for P_I. More generally, a finite family $\mathcal{S}$ of mixed integer split polyhedra gives an approximation of P_I of the form

$$\mathrm{Cl}(\mathcal{S}, P) := \bigcap_{L \in \mathcal{S}} R(L, P).$$

Improved approximations of P_I can be obtained by iteratively computing closures $P^1(\mathcal{S}, P), P^2(\mathcal{S}, P), \ldots$, where $P^0(\mathcal{S}, P) := P$, $P^1(\mathcal{S}, P) := \mathrm{Cl}(\mathcal{S}, P^0(\mathcal{S}, P))$, $P^2(\mathcal{S}, P) := \mathrm{Cl}(\mathcal{S}, P^1(\mathcal{S}, P))$ etc. A *finite split polyhedron proof* of validity of $\delta^T x \geq \delta_0$ for P_I is a finite family $\mathcal{S}$ of mixed integer split polyhedra such that $\delta^T x \geq \delta_0$ is valid for $P^k(\mathcal{S}, P)$ for some $k < \infty$, and a finite cutting plane proof is given from a finite split polyhedron proof by the valid inequalities for the polyhedron $P^k(\mathcal{S}, P)$.

Given a measure of "size" or "complexity" of a mixed integer split polyhedron, and a finite family $\mathcal{S}$ of mixed integer split polyhedra, a natural question is the following : How large a "size" of the mixed integer split polyhedra $L \in \mathcal{S}$ is necessary to prove validity of an inequality $\delta^T x \geq \delta_0$ for P_I with a finite split polyhedron proof? This is the main question in this section.

Possible measures for the size of a mixed integer split polyhedron L are the max-facet-width of L and the lattice width of L [**?**]. In addition, given that every mixed integer split polyhedron L can be written in the form $L = \mathcal{P} + \mathcal{L}$, where $\mathcal{P}$ is a polytope and $\mathcal{L}$ is a linear space, an alternative measure of the size of L could be the dimension of the polytope $\mathcal{P}$. However, in this section, we do not choose a specific measure. We simply use a generic function size(L) on the mixed integer split polyhedra L, and we assume that small values of size(L) are desirable in a finite split polyhedron proof.

The simplest mixed integer split polyhedra seem to be the *split sets*. A split set is a mixed integer split polyhedron of the form $\{x \in \mathbb{R}^n : \pi_0 \leq \pi^T x \leq \pi_0 + 1\}$, where $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$ and $\pi_j = 0$ whenever x_j is not integer constrained. A split set has the smallest possible lattice width and the smallest possible max-facet-width among all mixed integer split polyhedra. An example of an inequality for which split sets are *not* sufficient for providing a finite split polyhedron proof was given in [**?**].

Example 5.1 *Consider the mixed integer linear program (MILP)*

$$\max y$$

$$s.t.$$

$$-x_i + y \leq 0, \qquad \textit{for } i = 1, 2, \ldots, p, \tag{17}$$

$$\sum_{i=1}^{p} x_i + y \leq p, \tag{18}$$

$$y \geq 0, \tag{19}$$

$$x_i \textit{ integer} \quad \textit{for } i = 1, 2, \ldots, p. \tag{20}$$

The optimal solutions to (MILP) are of the form $(x^*, y^*) = (x^*, 0)$ with $x^* \in S^p \cap \mathbb{Z}^p$, where S^p is the set $S^p := \{x \in \mathbb{R}^p : x \geq 0$ and $\sum_{i=1}^{p} x_i \leq p\}$. The set S^2 is the split polyhedron L shown in Figure 1(a). The unique optimal solution to the LP relaxation of (MILP) is given by $x_i^{lp} = \frac{p}{p+1}$ for $i = 1, 2, \ldots, p$ and $y^{lp} = \frac{p}{p+1}$. Hence the only missing inequality to describe conv(P_I) *is the inequality $y \leq 0$.*

It is well-known that split sets do not suffice to give a finite split polyhedron proof for the validity of $y \leq 0$ [?]. On the other hand, the polyhedron $\tilde{L} := \{(x, y) \in \mathbb{R}^{p+1} : x \in S^p\}$ is a mixed integer split polyhedron, and $y \leq 0$ is valid for $R(\tilde{L}, P)$. Using $\tilde{L}$ in a split polyhedron proof therefore proves validity of $y \leq 0$. However, $\tilde{L}$ has both lattice width and max-lattice-width equal to p, and is therefore "larger" in the sense of lattice width and max-lattice-width than split sets, since split sets have lattice width and max-lattice-width equal to one.

We will also use the function size(L) to measure the size of lattice point free polyhedra that are not necessarily maximal wrt. inclusion. Specifically, if T is a lattice point free polyhedron, then size(T) is defined to be the smallest size of a mixed integer split polyhedron that includes T. In other words, $\text{size}(T) := \inf\{\text{size}(L) : T \subseteq L$ and L is a mixed integer split polyhedron$\}$, and we assume the infimum is achieved, *i.e.*, we assume there exists a mixed integer split polyhedron L such that $\text{size}(T) = \text{size}(L)$.

A measure of the complexity of a finite split polyhedron proof $\mathcal{S}$ is the size of the mixed integer split polyhedron $L \in \mathcal{S}$ of the largest size. We call this number the size of a split polyhedron proof. A measure of the complexity of a valid inequality $\delta^T x \geq \delta_0$ for P_I is then the smallest number $s_{(\delta,\delta_0)}$ for which there exists a finite split polyhedron proof of validity of $\delta^T x \geq \delta_0$ for P_I of size $s_{(\delta,\delta_0)}$. This number is called the size of $\delta^T x \geq \delta_0$, and it is denoted $\text{size}(\delta, \delta_0)$. Finally, since validity of every facet defining inequality for $\text{conv}(P_I)$ must be proved to generate $\text{conv}(P_I)$, the largest of the numbers $\text{size}(\delta, \delta_0)$ over all facet defining inequalities $\delta^T x \geq \delta_0$ for $\text{conv}(P_I)$ gives a measure of the complexity of P_I. We call this number the size of P_I, and it is denoted $\text{size}(P_I)$.

We now characterize exactly which size is necessary to prove validity of an inequality $\delta^T x \geq \delta_0$ for P_I with a finite split polyhedron proof, *i.e.*, we characterize the number $\text{size}(\delta, \delta_0)$. We will partition the inequality $\delta^T x \geq \delta_0$ into its integer part and its continuous part. Throughout the remainder of this section, $(\delta^x)^T x + (\delta^y)^T y \geq \delta_0$ denotes an arbitrary valid inequality for P_I, where $x \in \mathbb{R}^p$ is integer constrained, $y \in \mathbb{R}^q$ is continuous, $\delta^x \in \mathbb{Q}^p$, $\delta^y \in \mathbb{Q}^q$ and $\delta_0 \in \mathbb{Q}$. We assume $(\delta^x)^T x + (\delta^y)^T y \geq \delta_0$ is tight at a mixed integer point of P_I.

It is possible to prove validity of $(\delta^x)^T x + (\delta^y)^T y \geq \delta_0$ for $\text{conv}(P_I)$ by solving the mixed integer linear problem (MIP)

$$\begin{aligned} &\min\, (\delta^x)^T x + (\delta^y)^T y \\ &s.t. \\ &\qquad (x, y) \in\ P_I. \end{aligned}$$

The following notation is used. The point $(x^*, y^*) \in P_I$ denotes an optimal solution to (MIP), and $(x^{\text{lp}}, y^{\text{lp}}) \in P$ denotes an optimal solution to the linear relaxation of (MIP). We assume $\delta_0 = (\delta^x)^T x^* + (\delta^y)^T y^*$ and $(\delta^x)^T x^{\text{lp}} + (\delta^y)^T y^{\text{lp}} < \delta_0$. From the inequality $(\delta^x)^T x + (\delta^y)^T y \geq \delta_0$, we can create the following subsets of P and P_I

$$\begin{aligned} P(\delta, \delta_0) &:= \{(x, y) \in P : (\delta^x)^T x + (\delta^y)^T y \leq \delta_0\} \text{ and} \\ P_I(\delta, \delta_0) &:= \{(x, y) \in P(\delta, \delta_0) : x \in \mathbb{Z}^p\}. \end{aligned}$$

To prove validity of $(\delta^x)^T x + (\delta^y)^T y \geq \delta_0$ for $\text{conv}(P_I)$, we consider the following projections of $P(\delta, \delta_0)$ and $P_I(\delta, \delta_0)$ onto the space of the integer constrained x variables

$$\begin{aligned} P^x(\delta, \delta_0) &:= \{x \in \mathbb{R}^p : \exists y \in \mathbb{R}^q \text{ such that } (x, y) \in P(\delta, \delta_0)\} \text{ and} \\ P^x_I(\delta, \delta_0) &:= P^x(\delta, \delta_0) \cap \mathbb{Z}^p. \end{aligned}$$

The validity proofs we derive for $(\delta^x)^T x + (\delta^y)^T y \geq \delta_0$ are based on the following important property.

Lemma 5.1 *The polyhedron $P^x(\delta, \delta_0)$ is lattice point free.*

Proof. The relative interior of $P^x(\delta, \delta_0)$ is given by

$$\text{ri}(P^x(\delta, \delta_0)) = \{x \in \mathbb{R}^p : \exists y \in \mathbb{R}^q \text{ such that } (x, y) \in \text{ri}(P) \text{ and } (\delta^x)^T x + (\delta^y)^T y < \delta_0\}.$$

Since δ_0 is the optimal objective value of (MIP), $\text{ri}(P^x(\delta, \delta_0))$ does not contain lattice points. □

We assume that split sets are those mixed integer split polyhedra that have the smallest size. This is the case, for instance, when size is measured in terms of lattice width or max-facet-width. Furthermore, it is well known that split sets are sufficient to generate the integer hull of a pure integer set. It follows from this that there exists a finite number of split polyhedra of the smallest possible size such that a polyhedron $\overline{P}$ can be obtained in a finite number of iterations that satisfies $\overline{P}^x(\delta, \delta_0) = \text{conv}(\overline{P}^x_I(\delta, \delta_0))$.

Hence, since the purpose in this section is to provide finite split polyhedron proofs, we can assume $P^x(\delta,\delta_0) = \mathrm{conv}(P^x_I(\delta,\delta_0))$ in the remainder of this section.

The split polyhedra that are needed to prove validity of $(\delta^x)^T x + (\delta^y)^T y \geq \delta_0$ for P_I depend on the facial structure of $P^x(\delta,\delta_0)$. To obtain a description of the faces of $P^x(\delta,\delta_0)$, we need the following reformulation of $P^x(\delta,\delta_0)$.

Lemma 5.2 *Assume $P^x(\delta,\delta_0) = \mathrm{conv}(P^x_I(\delta,\delta_0))$. For every $x \in P^x(\delta,\delta_0)$, there exists $y \in \mathbb{R}^q$ such that $(x,y) \in P$ and $(\delta^x)^T x + (\delta^y)^T y = \delta_0$. Hence*

$$P^x(\delta,\delta_0) = \{x \in \mathbb{R}^p : \text{ there exists } y \in \mathbb{R}^q \text{ s.t. } (x,y) \in P \text{ and } (\delta^x)^T x + (\delta^y)^T y = \delta_0\}.$$

Proof. First suppose $\bar{x} \in P^x(\delta,\delta_0)$ is integer. By definition of $P^x(\delta,\delta_0)$, there exists $\bar{y} \in \mathbb{R}^q$ such that $(\bar{x},\bar{y}) \in P$ and $(\delta^x)^T \bar{x} + (\delta^y)^T \bar{y} \leq \delta_0$. We can not have $(\delta^x)^T \bar{x} + (\delta^y)^T \bar{y} < \delta_0$, since δ_0 is the optimal objective of MIP. Hence $(\bar{x},\bar{y}) \in P$ and $(\delta^x)^T \bar{x} + (\delta^y)^T \bar{y} = \delta_0$.

Now suppose $x^r \in \mathbb{Q}^p$ is a ray of $P^x(\delta,\delta_0)$. We claim that for every $\mu \geq 0$ and $\bar{x} \in P^x_I(\delta,\delta_0)$, there exists $\bar{y} \in \mathbb{R}^q$ such that $(\bar{x}+\mu x^r,\bar{y}) \in P$ and $(\delta^x)^T(\bar{x}+\mu x^r) + (\delta^y)^T \bar{y} = \delta_0$. Indeed, let $\mu \geq 0$ and $\bar{x} \in P^x_I(\delta,\delta_0)$ be arbitrary. We can choose a non-negative integer $\mu^I \geq \mu$ such that $\bar{x}+\mu^I x^r$ is integer. We therefore have that there exists $y^1 \in \mathbb{R}^q$ such that $(\bar{x}+\mu^I x^r, y^1) \in P$ and $(\delta^x)^T(\bar{x}+\mu^I x^r) + (\delta^y)^T y^1 = \delta_0$. Since $\bar{x} \in P^x_I(\delta,\delta_0)$, we also have that there exists $y^2 \in \mathbb{R}^q$ such that $(\bar{x},y^2) \in P$ and $(\delta^x)^T \bar{x} + (\delta^y)^T y^2 = \delta_0$. By choosing $\lambda := \frac{\mu}{\mu_I}$ and $\bar{y} := \lambda y^1 + (1-\lambda) y^2$, we have $(\bar{x}+\mu x^r,\bar{y}) = \lambda(\bar{x}+\mu^I x^r, y^1) + (1-\lambda)(\bar{x},y^2)$, and therefore $(\bar{x}+\mu x^r,\bar{y}) \in P$. In addition we have that $(\delta^x)^T(\bar{x}+\mu x^r) + (\delta^y)^T \bar{y} = \delta_0$.

Finally let $\bar{x} \in P^x(\delta,\delta_0)$ be arbitrary. We may write $\bar{x} = \sum_{i=1}^k \lambda_i x^i + d = \sum_{i=1}^k \lambda_i (x^i + d)$, where $\{x^i\}_{i=1}^k$ are the vertices of $P^x(\delta,\delta_0)$, $d \in \mathbb{Q}^p$ is a non-negative combination of the extreme rays of $P^x(\delta,\delta_0)$, $\lambda_1,\lambda_2,\ldots,\lambda_k \geq 0$ and $\sum_{i=1}^k \lambda_i = 1$. From what was shown above, we have that for every $i \in \{1,2,\ldots,k\}$, there exists $y^i \in \mathbb{R}^q$ such that $(x^i+d,y^i) \in P$ and $(\delta^x)^T(x^i+d) + (\delta^y)^T y^i = \delta_0$. By letting $\bar{y} := \sum_{i=1}^k \lambda_i y^i$, we have that $(\bar{x},\bar{y}) \in P$ and $(\delta^x)^T \bar{x} + (\delta^y)^T \bar{y} = \delta_0$. □

The faces of $P^x(\alpha,\beta)$ can now be characterized. Let $P = \{(x,y) \in \mathbb{R}^p \times \mathbb{R}^q : Ax + Dy \leq b\}$ be an outer description of P, where $A \in \mathbb{Q}^{m\times p}$, $D \in \mathbb{Q}^{m\times q}$ and $b \in \mathbb{Q}^m$, and let $M := \{1,2,\ldots,m\}$. Lemma 5.2 shows that $P^x(\delta,\delta_0)$ can be written in the form

$$\begin{aligned} P^x(\delta,\delta_0) = \{x \in \mathbb{R}^p : \quad & a_{i.}^T x + d_{i.}^T y = b_i, i \in M^=, \\ & a_{i.}^T x + d_{i.}^T y \leq b_i, i \in M \setminus M^=, \\ & (\delta^x)^T x + (\delta^y)^T y = \delta_0\}, \end{aligned}$$

where $M^= \subseteq M$ denotes those constrains $i \in M$ for which $a_{i.}^T x + d_{i.} y = b_i$ for all $(x,y) \in P(\delta,\delta_0)$ that satisfy $(\delta^x)^T x + (\delta^y)^T y = \delta_0$. Also, for every $i \in M \setminus M^=$, there exists $(x,y) \in P(\delta,\delta_0)$ that satisfies $(\delta^x)^T x + (\delta^y)^T y = \delta_0$ and $a_{i.}^T x + d_{i.} y < b_i$.

A non-empty face F of $P^x(\delta,\delta_0)$ can be characterized by a set $M^F \subseteq M$ of inequalities that satisfies $M^= \subseteq M^F$. Every face F of $P^x(\delta,\delta_0)$ can be written in the form

$$\begin{aligned} F = \{x \in \mathbb{R}^p : \quad & a_{i.}^T x + d_{i.}^T y = b_i, i \in M^F, \\ & a_{i.}^T x + d_{i.}^T y \leq b_i, i \in M \setminus M^F, \\ & (\delta^x)^T x + (\delta^y)^T y = \delta_0\}. \end{aligned}$$

Consider an arbitrary proper face F of $P^x(\delta,\delta_0)$. In order for $(\delta^x)^T x + (\delta^y)^T y \geq \delta_0$ to be valid for P_I, $(\delta^x)^T x + (\delta^y)^T y \geq \delta_0$ must be valid for all $(x,y) \in P$ such that $x \in F$. The following lemma shows that F is of exactly one of two types depending on the coefficient vectors on the continuous variables in the tight constraints.

Lemma 5.3 *(A characterization of the faces of $P^x(\delta,\delta_0)$)*
Assume $P^x(\delta,\delta_0) = \mathrm{conv}(P^x_I(\delta,\delta_0))$. Let F be a face of $P^x(\delta,\delta_0)$.

(i) If $\delta^y \notin \mathrm{span}(\{d_{i.}\}_{i \in M^F})$:

(a) F is lattice point free.

(b) For every $x \in \mathrm{ri}(F)$, there exists $y \in \mathbb{R}^q$ s.t. $(x, y) \in P$ and $(\delta^x)^T x + (\delta^y)^T y < \delta_0$.

(ii) If $\delta^y \in \mathrm{span}(\{d_{i.}\}_{i\in M^F})$:
The inequality $(\delta^x)^T x + (\delta^y)^T y \geq \delta_0$ holds for all $(x, y) \in P$ satisfying $x \in \mathrm{ri}(F)$.

Proof. (i) Suppose $\delta^y \notin \mathrm{span}(\{d_{i.}\}_{i\in M^F})$, and let $\bar{x} \in \mathrm{ri}(F)$ be arbitrary. This implies there exists $\bar{y} \in \mathbb{R}^q$ such that $a_{i.}^T\bar{x} + d_{i.}^T\bar{y} < b_i$ for all $i \in M \setminus M^F$. Since $\delta^y \notin \mathrm{span}(\{d_{i.}\}_{i\in M^F})$, the linear program $\min\{(\delta^y)^T r : d_{i.}^T r = 0, \forall i \in M^F\}$ is unbounded. Choose $\bar{r} \in \mathbb{R}^q$ such that $(\delta^y)^T\bar{r} < 0$ and $d_{i.}^T\bar{r} = 0$ for all $i \in M^F$. We have that $(\delta^x)^T\bar{x} + (\delta^y)^T(\bar{y} + \mu\bar{r}) < (\delta^x)^T\bar{x} + (\delta^y)^T\bar{y} = \delta_0$ for every $\mu > 0$. Furthermore, since $(\bar{x}, \bar{y})$ satisfies $a_{i.}^T\bar{x} + d_{i.}^T\bar{y} < b_i$ for all $i \in M \setminus M^F$, there exists $\bar{\mu} > 0$ such that $(\bar{x}, \bar{y} + \bar{\mu}\bar{r}) \in P$ and $(\delta^x)^T\bar{x} + (\delta^y)^T(\bar{y} + \bar{\mu}\bar{r}) < \delta_0$. We can not have $\bar{x}$ integer, since this would contradict that δ_0 is the optimal objective of MIP.

(ii) Let $(\bar{x}, \bar{y}) \in P$ satisfy $\bar{x} \in \mathrm{ri}(F)$, and suppose $\delta^y \in \mathrm{span}(\{d_{i.}\}_{i\in M^F})$. If $(\delta^x)^T\bar{x} + (\delta^y)^T\bar{y} \geq \delta_0$, we are done, so suppose for a contradiction that $(\delta^x)^T\bar{x} + (\delta^y)^T\bar{y} < \delta_0$. Since $\bar{x} \in \mathrm{ri}(F)$, there exists $\tilde{y} \in \mathbb{R}^q$ such that $(\bar{x}, \tilde{y}) \in P$, $(\delta^x)^T\bar{x} + (\delta^y)^T\tilde{y} = \delta_0$ and $a_{i.}^T\bar{x} + d_{i.}^T\tilde{y} < b_i$ for all $i \in M \setminus M^F$. Consider the vector $\bar{r} := \bar{y} - \tilde{y}$. We have $d_{i.}^T\bar{r} = 0$ for all $i \in M^F$ and $(\delta^y)^T\bar{r} < 0$. However, this contradicts $\delta^y \in \mathrm{span}(\{d_{i.}\}_{i\in M^F})$. □

We can now identify the mixed integer split polyhedra that are needed to provide a finite split polyhedron proof of validity of $(\delta^x)^T x + (\delta^y)^T y \geq \delta_0$ for P_I. Let $\mathcal{F}$ denote the finite set of all faces of $P^x(\delta, \delta_0)$, and let $\mathcal{F}^V := \{F \in \mathcal{F} : \exists (x, y) \in P \text{ s.t. } x \in F \text{ and } (\delta^x)^T x + (\delta^y)^T y < \delta_0\}$ denote those faces $F \in \mathcal{F}$ for which there exists $(x, y) \in P$ such that $x \in F$ and (x, y) violates the inequality $(\delta^x)^T x + (\delta^y)^T y \geq \delta_0$. A face $F \in \mathcal{F}^V$ is called a *violated face*. Lemma 5.3.(i) shows that every violated face is lattice point free. A mixed integer split polyhedron $L \subseteq \mathbb{R}^n$ that satisfies $(\delta^x)^T x + (\delta^y)^T y \geq \delta_0$ for every $(x, y) \in R(L, P)$ such that $x \in F$ is said to *prove validity of* $(\delta^x)^T x + (\delta^y)^T y \geq \delta_0$ *on* F. Given a violated face $F \in \mathcal{F}^V$, the following lemma gives a class of split polyhedra that can prove validity of $(\delta^x)^T x + (\delta^y)^T y \geq \delta_0$ on F.

Lemma 5.4 *(Split polyhedra for proving validity of $(\delta^x)^T x + (\delta^y)^T y \geq \delta_0$ on a face of $P^x(\delta, \delta_0)$)*
Assume $P^x(\delta, \delta_0) = \mathrm{conv}(P_I^x(\delta, \delta_0))$. Let $F \in \mathcal{F}^V$ be a violated face of $P^x(\delta, \delta_0)$, and suppose $G \notin \mathcal{F}^V$ for every proper face G of F. Every mixed integer split polyhedron $L \subseteq \mathbb{R}^n$ that satisfies $\mathrm{ri}(F) \subseteq \mathrm{int}(L)$ proves validity of $(\delta^x)^T x + (\delta^y)^T y \geq \delta_0$ on F.

Proof. Let L be a mixed integer split polyhedron that satisfies $\mathrm{ri}(F) \subseteq \mathrm{int}(L)$, and let $(\bar{x}, \bar{y}) \in P$ satisfy $\bar{x} \in F$ and $\bar{x} \notin \mathrm{int}(L)$. Since $\mathrm{ri}(F) \subseteq \mathrm{int}(L)$, it follows that $\bar{x} \notin \mathrm{ri}(F)$. Since $\bar{x} \in F \setminus \mathrm{ri}(F)$, $\bar{x}$ must be on some proper face G of F. Since $G \notin \mathcal{F}^V$, we have $(\delta^x)^T\bar{x} + (\delta^y)^T\bar{y} \geq \delta_0$. Since $R(L, F) = \mathrm{conv}(\{(x, y) \in F : x \notin \mathrm{int}(L)\})$, the result follows. □

By iteratively considering the finite number $|\mathcal{F}^V|$ of violated faces of $P^x(\delta, \delta_0)$, we obtain a finite split polyhedron proof for the validity of the inequality $(\delta^x)^T x + (\delta^y)^T y \geq \delta_0$ for P_I.

Corollary 5.1 *(Upper bound on the size of the inequality $(\delta^x)^T x + (\delta^y)^T y \geq \delta_0$)*
There exists a split polyhedron proof for the validity of $(\delta^x)^T x + (\delta^y)^T y \geq \delta_0$ for P_I of size

$$\max\{\mathit{size}(F) : F \in \mathcal{F}^V\}.$$

We can now prove the main theorem of this section.

Theorem 5.1 *(A formula for the size of the inequality $(\delta^x)^T x + (\delta^y)^T y \geq \delta_0$)*
Let $\mathit{size}(\delta, \delta_0)$ denote the smallest number $s_{(\delta,\delta_0)}$ for which there exists a finite split polyhedron proof of validity of $(\delta^x)^T x + (\delta^y)^T y \geq \delta_0$ for P_I of size $s_{(\delta,\delta_0)}$. Then

$$\mathit{size}(\delta, \delta_0) = \max\{\mathit{size}(F) : F \in \mathcal{F}^V\}.$$

Proof. Let L be a mixed integer split polyhedron of smaller size than $\max\{\mathrm{size}(F) : F \in \mathcal{F}^V\}$. This implies there exists $F \in \mathcal{F}^V$ and $x' \in \mathrm{ri}(F)$ such that $x' \notin \mathrm{int}(L)$. Furthermore, since $x' \in \mathrm{ri}(F)$, it follows from Lemma 5.3.(i) that there exists $y' \in \mathbb{R}^q$ such that $(x', y') \in P$ and $(\delta^x)^T x' + (\delta^y)^T y' < \delta_0$. We now have $(x', y') \in R(L, P)$ and $(\delta^x)^T x' + (\delta^y)^T y' < \delta_0$. □

Example 5.2 *Consider the mixed integer linear program (MILP), and the inequality* $y \leq 0$ *considered earlier. We have* $\delta^x = 0$*,* $\delta^y = -1$ *and* $\delta_0 = 0$*.*

Observe that any proper face G *of* $P^x(\delta, \delta_0)$ *contains mixed integer points in their relative interior. It follows that the inequality* $y \leq 0$ *is valid for every* $(x, y) \in P$ *such that* x *belongs to a proper face of* $P^x(\delta, \delta_0)$*. Hence the only interesting face of* $P^x(\delta, \delta_0)$ *to consider is the improper face* $F := P^x(\delta, \delta_0) = S^p$*. The only mixed integer split polyhedron* L *that satisfies* $\mathrm{ri}(F) \subseteq \mathrm{int}(L)$ *is the split polyhedron* $L = S^p$*. The polyhedron* S^p *has both max-facet-width and lattice width equal to* p*, and is therefore relatively "large" compared to split sets. However, it follows from Theorem 5.1 that the mixed integer split polyhedron* $L = S^p$ *is necessary in order to obtain a finite split polyhedron proof of* $y \leq 0$ *in a finite number of steps.*