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Economic Theory

Exposita Notes

Regulating a monopolist with limited funds*

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Summary. We consider the problem of regulating a monopolist with unknown costs when the regulator has limited funds. The optimal regulatory mechanism satisfies four properties. The first property is bunching at the top, that is the more efficient types produce the same quantity irrespective of their costs. The second property is separability of less efficient types. The third property is full bunching of types when the available fund is small enough. The fourth property of the mechanism is that it is a third best one, that is, the output under this regulatory mechanism is strictly lower than the second best output for any given type.

Keywords and Phrases: Regulation, Asymmetric information, Limited funds.

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1 Introduction

We analyze the problem of regulating a monopolist with unknown cost when the regulator has limited funds. Baron and Myerson (1982) and Laffont and Tirole

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(1993) developed a procedure to regulate a monopolist with unknown cost in the absence of any fund constraints. The main property of the optimal (or second best) mechanism is full separability of types, that is, if the monopolist is a high (low) cost type then she produces lower (higher) quantity and recieve a lower (higher) transfer. The cost of separation, induced by the optimal mechanism, is the information rent enjoyed by the lower cost or more efficient types. The regulator uses its fund to pay this information rent. However, if there are numerous projects, public funds are usually scarce. It is reasonable to imagine that the fund provider may be unable to finance the monopolist at the level prescribed by the optimal mechanism.

When funds are limited, the regulator has two instruments to limit the transfer: (a) bunching the more efficient types and (b) under-production. The optimal regulatory mechanism, which we call the constrained optimal mechanism, prescribes that the monopolist supplies a good of lower quantity (compared to the second best quantity) and that the more efficient types produce the same quantity. These two distortions reduce the information rent and the quantity produced by the more efficient types. However, if the fund crisis is "too" strong, the constrained optimal mechanism prescribes full bunching. We also highlight the difference between the optimal and the constrained optimal mechanism. Our comparative static result show that a reduction in available fund reduces the quantities produced by all types and increases the interval in which there is bunching.

There are several papers dealing with mechanism design problems under asymmetric information when there exists budget constraints. Laffont and Robert (1996) describe the optimal auction when all the bidders have a financial constraint which is common knowledge. Like in our constrained optimal mechanism, the financial constraint in Laffont and Roberts (1996) reduces the bids of all participants (even those with a low valuation for the good). Che and Gale (2000) extends the result in Laffont and Roberts (1996) by relaxing the assumption that financial constraints are common knowledge. Monteiro and Page Jr. (1998) describe the optimal selling mechanisms for multiproduct monopolists in the presence of budget constrained buyers. To construct the constrained optimal mechanism, we extend the methodology of Thomas (2002). Thomas (2002) considers the incentive problem of a monopolist who faces financially constrained buyers. Finally, Gautier (2002) considers the regulator's mechanism design problem under financial constraint when there are two types of firm. In Gautier (2002), bunching is an issue only if the financial constraint is sufficiently strong. We develop and analyze our model in Sections 2–4. All proofs are relegated in the Appendix.

2 The model

The utility of the monopolist is $U_m = t - \theta q$ where t is the transfer that she receives from the regulator and θ is her marginal cost and q is the quantity of the public good she produces. The utility function of the regulator is $U_r = S(q) - t$ where $S(q)$ is the consumer's surplus when a quantity q of public good is supplied and t is the transfer to the monopolist. $S(q)$ is assumed to be twice differentiable with $S'(q) > 0$, $S''(q) < 0$ and $S'(0) = \infty$. The regulator's main objective is to select the quantity q to maximize U_r. Since $S'(0) = \infty$, the good is always produced. If the regulator knows the marginal cost θ of the monopolist, then the optimal quantity is $q^f(\theta) = S'^{-1}(\theta)$ and the optimal transfer to the monopolist is $t(\theta) = \theta q^f(\theta)$. The pair $\langle q^f(\theta), t^f(\theta)\rangle$ is the first best outcome.[1]

We assume that the marginal cost of the monopolist is private information. In this context, we assume that the marginal cost of the monopolist θ belongs to the interval $[\underline{\theta}, \overline{\theta}]$ where $0 < \underline{\theta} < \overline{\theta}$. This interval is assumed to be common knowledge. It is also common knowledge that (i) the marginal cost has a differentiable density $f(\theta)$ and that (ii) $f(\theta) > 0$ for all $\theta \in [\underline{\theta}, \overline{\theta}]$. The regulator's objective is to maximize $\int_{\underline{\theta}}^{\overline{\theta}} \{S(q(\theta)) - t(\theta)\} f(\theta) d\theta$ subject to incentive compatibility constraint (or IC) and participation constraint (or PC). A direct mechanism $M = \langle q(.), t(.)\rangle$, in this context, simply specifies a type contingent quantity-transfer pair. Here $q : [\underline{\theta}, \overline{\theta}] \to \mathbf{R}_+$ and $t : [\underline{\theta}, \overline{\theta}] \to \mathbf{R}_+$. For simplicity we restrict attention to continuous mechanisms. Let $U_m(\theta; \theta') = t(\theta') - \theta q(\theta')$ be the utility of the monopolist under the mechanism M if her true type is θ and if she announces $\theta' \in [\underline{\theta}, \overline{\theta}]$. With slight abuse of notation, let us define $U_m(\theta) \equiv U_m(\theta; \theta)$, for all $\theta \in [\underline{\theta}, \overline{\theta}]$. Incentive compatibility requires that $U_m(\theta) \geq U_m(\theta; \theta')$, for all $\theta, \theta' \in [\underline{\theta}, \overline{\theta}]$ and participation constraint states that $U_m(\theta) \geq 0$, for all $\theta \in [\underline{\theta}, \overline{\theta}]$. It is well known in the literature that the optimal mechanism M satisfies both the incentive compatibility constraint and the participation constraint if and only if the utility of any type $\theta \in [\underline{\theta}, \overline{\theta}]$ is given by $U_m(\theta) = \int_{\theta}^{\overline{\theta}} q(\tau) d\tau$ and the optimal type-contingent quantity $q(\theta)$ is non-increasing in θ (see Baron and Myerson, 1982). Before stating our first Proposition, we provide two relevant definitions. For any $\theta \in [\underline{\theta}, \overline{\theta}]$, let $L(\theta) \equiv \frac{F(\theta)}{f(\theta)}$ be the hazard rate function where $F(.)$ is the distribution function associated with the density function $f(.)$. For any $\theta \in [\underline{\theta}, \overline{\theta}]$, let $z(\theta) \equiv \theta + L(\theta)$ be the virtual type function.

Proposition 2.1. *The optimal mechanism is* $M^b = \langle q^b(.), t^b(.)\rangle$ *where*

1. $q^b(\theta) = S'^{-1}(z(\theta))$ *and*
2. $t^b(\theta) = \theta q^b(\theta) + \int\limits_{\theta}^{\overline{\theta}} q^b(\tau) d\tau \ \forall \theta \in [\underline{\theta}, \overline{\theta}]$

We omit the proof of Proposition 2.1 since it is quite well known in the literature (see Baron and Myerson, 1982). It is important to observe that for the benchmark model it is necessary that $q^b(\theta)$ is non-increasing in $\theta \in [\underline{\theta}, \overline{\theta}]$. Non-increasingness of quantity is satisfied if and only if the virtual type $z(\theta)$ is non-decreasing in $\theta \in [\underline{\theta}, \overline{\theta}]$. Given that $z(\theta)$ is non-decreasing, we get $q^b(\theta)$ is non-increasing in $\theta \in [\underline{\theta}, \overline{\theta}]$. Moreover, since $L(\theta) > 0$ for all $\theta \in (\underline{\theta}, \overline{\theta}]$ and $z(\theta) > \theta$, we get $q^b(\theta) < q^f(\theta)$ for all $\theta \in (\underline{\theta}, \overline{\theta}]$ and $q^b(\underline{\theta}) = q^f(\underline{\theta})$. Thus, for all but the lowest cost firm, we have underproduction under the optimal mechanism compared to the first best outcome. For our main problem, to be analyzed in the next section, we take

[1] In many models of regulation, it is assumed that public subsidies are costly, that is transferring one dollar to the monopolist costs the authority $(1 + \lambda)$ dollars, where λ represents the shadow cost of public funds (see Laffont and Tirole, 1993). We assume that this shadow cost of public funds is zero in the relevant range.

the following assumption which is stronger than non-decreasingness of the virtual type function $z(.)$.

Assumption 1. *For all $\theta \in [\underline{\theta}, \overline{\theta}]$, $\theta \neq L(\theta)$ and for all $\theta \in (\underline{\theta}, \overline{\theta})$, $\theta(1 + L'(\theta)) \geq 2L(\theta)$.*

For Assumption 1, it is necessary that $z(\theta) = \theta + L(\theta)$ is non-decreasing for all $\theta \in [\underline{\theta}, \overline{\theta}]$. This follows from the the second part of Assumption 1 since for all $\theta \in [\underline{\theta}, \overline{\theta}]$, $\theta > 0$ and $L(\theta) \geq 0$. It is quite easy to verify that Uniform Distribution satisfy assumption 1. For all density functions with the property that $f'(\theta) \leq 0$ for all $\theta \in (\underline{\theta}, \overline{\theta})$, assumption 1 is satisfied provided $\overline{\theta} f(\overline{\theta}) > 1$. Therefore, under certain restrictions, Exponential Distribution, Beta Distributions, Gamma Distributions, Pareto Distribution and Weibull Distributions satisfy Assumption 1. Among the class of distributions with the property that there exists a non-empty interval (a, b) such that $f'(\theta) > 0$ for all $\theta \in (a, b)$, Normal Distribution with mean $\mu = \frac{(\underline{\theta}+\overline{\theta})}{2}$ and standard deviation σ satisfies Assumption 1 if and only if $\frac{2\sigma^2\theta}{(2\sigma^2+\theta(\mu-\theta))} \geq L(\theta)$ for all $\theta \in (\underline{\theta}, \mu)$ and $\overline{\theta} f(\overline{\theta}) > 1$. For a Logistic Distribution with $\mu = \frac{(\underline{\theta}+\overline{\theta})}{2}$ and standard deviation σ, the sufficient conditions for Assumption 1 are $\underline{\theta} \geq \frac{\sqrt{3}}{\pi}\sigma$ and $\overline{\theta} f(\overline{\theta}) > 1$. We obtained the condition for Normal Distribution by taking doubly-truncated Normal Distribution following Hald's (1952) convention. The same double truncation technique was applied to obtain the sufficiency conditions with Logistic Distribution. For both Normal and Logistic Distributions we assumed symmetry around the mean $\mu = \frac{\underline{\theta}+\overline{\theta}}{2}$.

3 The constrained optimal mechanism

Complete separation of types is feasible if $\bar{T} \geq t^b(\underline{\theta})$ where $\bar{T}$ is the fund available to the regulator and $t^b(\underline{\theta})$ is the transfer of the lowest type under the optimal mechanism. This is because the transfer under the optimal mechanism is strictly decreasing, $t^b(\underline{\theta}) > t^b(\theta)$ for all $\theta \in (\underline{\theta}, \overline{\theta}]$. Therefore, if $\bar{T} \geq t^b(\underline{\theta})$ then the optimal mechanism is always feasible. However, if $\bar{T} < t^b(\underline{\theta})$, then the regulator's optimization problem is to select $\langle q(\theta), t(\theta)\rangle$ to maximize $\int_{\underline{\theta}}^{\overline{\theta}} \{S(q(\theta)) - t(\theta)\} f(\theta) d\theta$ subject to (1) $U_m(\theta) \geq U_m(\theta; \theta'), \forall \theta, \theta' \in [\underline{\theta}, \overline{\theta}]$, (2) $U_m(\theta) \geq 0, \forall \theta \in [\underline{\theta}, \overline{\theta}]$ and (3) $t(\theta) \leq \bar{T}\ \forall \theta \in [\underline{\theta}, \overline{\theta}]$. We refer to this problem as $[P^*]$ and the optimal solution $M^* = \langle q^*(.), t^*(.)\rangle$ for $[P^*]$ as the constrained optimal mechanism.

Theorem 3.1. *Under Assumption 1, the constrained optimal mechanism $M^* = \langle q^*(.), t^*(.)\rangle$ for the constrained optimization problem $[P^*]$ specifies the following:*

1. *If $\bar{T} \in \left(\mathbf{T}, t^b(\underline{\theta})\right)$ where $\mathbf{T} = \overline{\theta} S'^{-1}\left(z(\overline{\theta}) + \frac{\Psi(\tilde{\theta})}{f(\overline{\theta})}\right)$, then*
   (a) *the optimal type contingent quantities are*

$$q^*(\theta) = \begin{cases} q^*(\tilde{\theta}) & \forall \theta \in [\underline{\theta}, \tilde{\theta}) \\ S'^{-1}\left(z(\theta) + \frac{\Psi(\tilde{\theta})}{f(\theta)}\right) & \forall \theta \in [\tilde{\theta}, \overline{\theta}] \end{cases}$$

where $\Psi(\tilde{\theta}) = \frac{F(\tilde{\theta})^2}{\tilde{\theta} f(\tilde{\theta}) - F(\tilde{\theta})} > 0 = \Psi(\underline{\theta})\ \forall\ \tilde{\theta} \in (\underline{\theta}, \overline{\theta}]$, *(b) the optimal type contingent transfers are*

$$t^*(\theta) = \begin{cases} \bar{T}(= t^*(\tilde{\theta})) & \forall \theta \in [\underline{\theta}, \tilde{\theta}) \\ \theta q^*(\theta) + \int\limits_{\theta}^{\overline{\theta}} q^*(\tau) d\tau & \forall \theta \in [\tilde{\theta}, \overline{\theta}] \end{cases}$$

(c) the optimal cut-off point $\tilde{\theta} \in [\underline{\theta}, \overline{\theta}]$ *is obtained from* $\bar{T} = \tilde{\theta} q^*(\tilde{\theta}) + \int\limits_{\tilde{\theta}}^{\overline{\theta}} q^*(\tau) d\tau$.

2. *If* $\bar{T} \leq \mathbf{T}$, *then* $q^*(\theta) = \frac{\bar{T}}{\overline{\theta}}$ *and* $t^*(\theta) = \bar{T}\ \forall\ \theta \in [\underline{\theta}, \overline{\theta}]$ *and the optimal cut-off point is* $\tilde{\theta} = \overline{\theta}$.

Remark 3.1. Assumption 1 is sufficient to guarantee that for any given cut-off point $\tilde{\theta} \in [\underline{\theta}, \overline{\theta}]$, $z(\theta) + \frac{\Psi(\tilde{\theta})}{f(\theta)}$ is non-decreasing in θ for all $\theta \in (\tilde{\theta}, \overline{\theta}]$. Monotonicity of $z(\theta) + \frac{\Psi(\tilde{\theta})}{f(\theta)}$ is necessary for the optimal output $q^*(\theta)$ to be non-increasing in $\theta \in [\tilde{\theta}, \overline{\theta}]$. Moreover, assumption 1 also guarantees that $\Psi(\theta)$ is well defined and non-decreasing in $\theta \in [\underline{\theta}, \overline{\theta}]$. The second part of assumption 1, is sufficient for the monotonicity of $z(\theta) + \frac{\Psi(\tilde{\theta})}{f(\theta)}$ since it implies and is implied by non-decreasingness of $z(\theta) + \frac{\Psi(\theta)}{f(\theta)}$ for all $\theta \in [\underline{\theta}, \overline{\theta}]$. If the monotonicity of $z(\theta) + \frac{\Psi(\tilde{\theta})}{f(\theta)}$ is violated in the non-bunching interval $(\tilde{\theta}, \overline{\theta}]$, then the analysis can be modified à la Guesnerie and Laffont (1984).

From Theorem 3.1 it is obvious that the optimal quantity $\frac{\bar{T}}{\overline{\theta}}$ for the full bunching case (that is for $\bar{T} \leq \mathbf{T}$) is strictly lower than any $q^*(\theta)$ for the partial bunching case (that is for $\bar{T} > \mathbf{T}$). Moreover, from Theorem 3.1 it also follows that if the fund limit is not binding, that is if $\bar{T} \geq t^b(\underline{\theta})$, then $q^*(\theta) = q^b(\theta)\ \forall \theta \in [\underline{\theta}, \overline{\theta}]$ since $\Psi(\underline{\theta}) = 0$. If, instead, $\bar{T} < t^b(\underline{\theta})$, then the following two Propositions summarize a comparative study between the constrained optimal mechanism M^* and the optimal mechanism M^b.

Proposition 3.2. *If* $\bar{T} < t^b(\underline{\theta})$, *then* $q^*(\theta) < q^b(\theta)\ \forall \theta \in [\underline{\theta}, \overline{\theta}]$.

Proposition 3.3. *Call* $\hat{\theta} = \{\theta \in (\underline{\theta}, \overline{\theta}) \mid t^b(\theta) = \bar{T}\}$. *If* $\bar{T} < t^b(\underline{\theta})$ *then* $\hat{\theta} \geq \tilde{\theta}$.

Theorem 3.1 and its two complementary Propositions (3.2 and 3.3) describe the constrained optimal mechanism and compare it with the optimal mechanism. While full separability of types is the main property of the optimal mechanism M^b, this property does not hold in the constrained optimal mechanism M^*, at least for the lower cost firms. Hence, with limited fund, the optimal quantity under the constrained optimal mechanism is strictly lower than that of the optimal mechanism (see Proposition 3.2). In the constrained optimal mechanism, there is a conflict between the necessity of separability (the IC constraints) and the fund constraint. Separability of types implies increasing information rents for the lower cost firms. With limited funds, it becomes impossible to finance the information rents of the

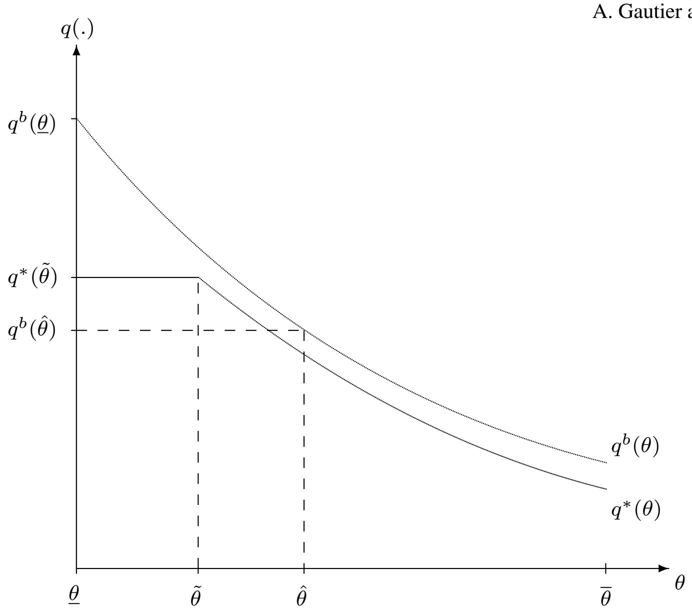

Figure 1. The optimal quantities for $\bar{T} = t^b(\underline{\theta})$ and $\bar{T} \in (\mathbf{T}, t^b(\underline{\theta}))$

more efficient firms. Hence, there is bunching for the lower cost firms. However, the regulator optimally limits the bunching zone (see Proposition 3.3). For that, the contract offered to the higher cost firms (that is firms for which the fund limit is non-binding) is distorted compared to the optimal mechanism M^b. Reducing the quantities of the less efficient firms (compared to the optimal mechanism M^b), reduces the information rent, and hence, it is possible to finance separability for a larger fringe of firms. *Without any distortions in quantity, the bunching zone would have been* $[\underline{\theta}, \hat{\theta}]$, *while by imposing optimal distortions in quantity, the bunching zone is reduced to* $[\underline{\theta}, \tilde{\theta}]$. This is shown in Figure 1. The constrained optimal mechanism, described in the Theorem 3.1, takes care of the trade off between the cost of abandoning separability for the more efficient firms and the cost of larger distortions to preserve it. However, the cost of keeping separability for high cost firm may be too high. In that case we have a full bunching solution.

Remark 3.2. In Baron and Myerson (1982), the decision to provide the public good is itself a regulatory instrument. In their optimal mechanism, the public good is provided whenever the associated surplus is larger than the cost and this decision does not interfere with the optimal mechanism. Likewise, if, in our problem, $S'(0)$ is finite, then exclusion of the higher cost firms from the mechanism is another instrument that can be used to tackle the problem of limited funds. In our problem, the decision of whether to provide the public good or not can be incorporated ex-post, together with the cut-off point. Given $S'(0) < \infty$, let θ^{**} denote the highest type for which $q^*(\theta^{**}) > 0$. Then θ^{**} and $\tilde{\theta}$ are determined by the following conditions:

$$S(q^*(\theta^{**})) = t^*(\theta^{**}),$$
$$\bar{T} = \tilde{\theta} q^*(\tilde{\theta}) + \int_{\tilde{\theta}}^{\theta^{**}} q^*(\tau) d\tau.$$

It is obvious that θ^{**} is lower than its corresponding value in the optimal mechanism M^b. Hence, the presence of limited funds also reduces the probability $F(\theta^{**})$ that the public good is provided. Thus, if $S'(0)$ is finite, then it is possible that the provision of public good is delayed due to fund crisis.

We conclude our analysis on constrained optimal mechanism with a comparative static result. Consider any two fund limits $\bar{T}_1$ and $\bar{T}_2$ such that $\bar{T}_1 < \bar{T}_2 \leq t^b(\underline{\theta})$. With slight abuse of notation, let $q_i^*(\theta)$ be the type contingent output and $\tilde{\theta}_i$ be the cut-off point, both associated with the fund limits $\bar{T}_i$ for $i = 1, 2$.

Proposition 3.4. *If* $\mathbf{T} \leq \bar{T}_1 < \bar{T}_2 \leq t^b(\underline{\theta})$*, then* $\Psi(\tilde{\theta}_1) \geq \Psi(\tilde{\theta}_2)$ *and* $\tilde{\theta}_1 > \tilde{\theta}_2$ *which together imply* $q_1^*(\theta) \leq q_2^*(\theta) \forall \theta \in [\underline{\theta}, \overline{\theta}]$.

A reduction in available funds reduces the optimal quantities and the cut-off point (provided $\mathbf{T} \leq \bar{T}_1$). This comparative static result is intuitive. Due to the scarcity of resources, the opportunity cost of paying information rents to the more efficient firms increases and hence the regulator prefers to save on these rents to finance the infrastructure with its available resources. This result also explains why for a sufficiently small fund $\bar{T}(< \mathbf{T})$, the constrained optimal mechanism prescribes full bunching.

3.1 Welfare implications

The constrained optimal mechanism leads to welfare loss. The welfare is reduced because each type produces a lower quantity of the public good and hence the consumer surplus is lower. Moreover, the welfare is also reduced because there is bunching for the more efficient firms. To satisfy the wealth constraint, the regulator gives up separability for the more efficient firms. From our comparative static result it is obvious that welfare loss is decreasing in available resources ($\bar{T}$).

Finally, what happens when the regulator, instead of maximizing only the expected gains to the consumers, maximizes a weighted sum of the expected gains to the consumers and the expected utility of the monopolist? To see what happens then, let $\alpha \in [0, 1]$ be the welfare weight attached to the expected utility of the monoplolist. Observe that $\alpha = 0$ corresponds to the case we have analyzed so far. If $\alpha \in (0, 1]$, the regulators objective is to maximize $\int_{\underline{\theta}}^{\overline{\theta}} \{S(q(\theta)) - t(\theta)\} f(\theta) d\theta + \alpha \int_{\underline{\theta}}^{\overline{\theta}} U_m(\theta) f(\theta) d\theta$. Here the optimal (or second best) mechanism specifies $q_\alpha^b(\theta) = S'^{-1}(z_\alpha(\theta))$ for all $\theta \in [\underline{\theta}, \overline{\theta}]$ where $z_\alpha(\theta) = \theta + (1 - \alpha) L(\theta)$. With limited funds, the constrained optimal mechanism is such that in the separating zone the optimal quantity is $q_\alpha^*(\theta) = S'^{-1}(z_\alpha(\theta) + \frac{\Psi_\alpha(\tilde{\theta})}{f(\theta)})$

where $\Psi_\alpha(\tilde{\theta}) = \Psi(\tilde{\theta})\{(1-\alpha) + \alpha(\frac{\tilde{\theta}-E(\theta|\theta\leq\tilde{\theta})}{L(\tilde{\theta})})\}$ and $E(\theta \mid \theta \leq \tilde{\theta})$ is the conditional expectation of θ given that $\theta \leq \tilde{\theta}$. Thus, with welfare weight, the results are qualitatively similar to the results obtained so far as long as the distribution function $F(.)$ is such that $q^*_\alpha(\theta)$ is non-increasing in θ.

4 Summary

The results and observations of Sections 2 and 3 are the following:

1. The constrained optimal mechanism satisfies four properties (a) bunching of the low cost types (b) separation of the high cost types (c) full bunching if the available fund is small enough and (d) lower output compared to the optimal mechanism for all types.
2. The constrained optimal mechanism adds distortion in order to optimally reduce the bunching zone of the low cost (or more efficient) types. This is achieved by reducing the optimal quantity of the high cost types in comparison to the optimal mechanism.
3. In our problem we have assumed that $S'(0) = \infty$. If instead $S'(0)$ is finite, then exclusion of the higher cost firms from the mechanism is another instrument that can be used to tackle the problem of limited funds. In that case we have the possibility that the provision of public good is delayed due to fund crisis.
4. If the available fund is not "too" small, then a reduction in fund reduces the optimal quantity for all types and it reduces the cut-off point.
5. The constrained optimal mechanism leads to welfare loss because, relative to the optimal mechanism, each type produces a lower quantity of the public good that leads to a lower consumer surplus.
6. If the regulator adds non-zero welfare weight to the monopolist's utility, then the results are qualitatively similar to the results obtained in Sections 2 and 3 provided the distribution function is such that the optimal type contingent quantity is non-increasing.

5 Appendix

Proof of Theorem 3.1. For a continuous mechanism, IC implies that truth-telling is a best response of the monopolist, that is $\{\frac{\partial U_m(\theta;\theta')}{\partial\theta'}\}_{\theta'=\theta} = 0$ almost everywhere. This condition implies that $t'(\theta) = \theta q'(\theta)$ almost everywhere. From IC we also know that $q(\theta)$ must be non-increasing in θ and hence $t(\theta)$ must be non-increasing in θ.[2] For the optimization problem $[P^*]$, let $\tilde{\theta}$ be the first type for which the fund limit is not binding. Therefore, for all $\theta \in [\underline{\theta}, \tilde{\theta})$, the fund limit is binding and for all $\theta \in [\tilde{\theta}, \overline{\theta}]$ it is not binding (or free).[3] This means that $t'(\theta) = 0$ for all $\theta \in (\underline{\theta}, \tilde{\theta})$ and

[2] Given that (a) the mechanism is continuous and (b) both $q(\theta)$ and $t(\theta)$ are non-increasing in θ, we get almost everywhere differentiability of the mechanism. We are thankful to the referee for pointing this out.

[3] Observe that we are assuming that it is possible to find type contingent quantity-transfer pairs which allows for partial bunching and partial separability. In otherwords, we are trying to find the

$t'(\theta) \leq 0$ for all $\theta \in (\tilde{\theta}, \overline{\theta})$. From IC and PC we also know that $U'_m(\theta) = -q(\theta) < 0$ almost everywhere and optimality of the mechanism guarantees that $U_m(\overline{\theta}) = 0$. Finally, non-increasingness of $q(\theta)$ implies that $q'(\theta) = 0$ for all $\theta \in (\underline{\theta}, \tilde{\theta})$, $q'(\theta) \leq 0$ for all $\theta \in (\tilde{\theta}, \overline{\theta})$ and since $t(.)$ is not differentiable at $\tilde{\theta}$,

$$q(\tilde{\theta}^-) \geq q(\tilde{\theta}^+) \tag{5.1}$$

The regulator's optimization problem $[P^*]$ can now be divided into two sub-problems $[P_1^*]$ and $[P_2^*]$ for the intervals $[\underline{\theta}, \tilde{\theta})$ and $[\tilde{\theta}, \overline{\theta}]$ respectively.

$[P_1^*]$ $\max \int_{\underline{\theta}}^{\tilde{\theta}} \{S(q_1(\theta)) - U_m(\theta) - \theta q_1(\theta)\} f(\theta) d\theta$ subject to

1. $U'_m(\theta) = -q_1(\theta)$,
2. $\bar{T} - U_m(\theta) - \theta q_1(\theta) = 0$,
3. $U_m(\underline{\theta})$ free, $\tilde{\theta}$ and $U_m(\tilde{\theta})$ given, and
4. $q_1(\theta) \equiv q(\theta)$.

$[P_2^*]$ $\max \int_{\tilde{\theta}}^{\overline{\theta}} \{S(q_2(\theta)) - U_m(\theta) - \theta q_2(\theta)\} f(\theta) d\theta$ subject to

1. $U'_m(\theta) = -q_2(\theta)$,
2. $U_m(\overline{\theta}) = 0$,
3. $\tilde{\theta}$ and $U_m(\tilde{\theta})$ given, and
4. $q_2(\theta) \equiv q(\theta)$.

$[P_1^*]$ and $[P_2^*]$ are two optimal control problems. In both these sub-problems $q(.)$ is the control variable and $U_m(.)$ is the state variable. Finally, $\tilde{\theta}$ is the optimal cut-off point that links the two problems.

The Hamiltonian function associated with $[P_i^*]$ is

$$H_i(\theta) = \{S(q_i(\theta)) - U_m(\theta) - \theta q_i(\theta)\} f(\theta) - \lambda_i(\theta) q_i(\theta)$$

for $i = 1, 2$. Here $\lambda_i(\theta)$ is the co-state (or auxiliary) variable associated with the Hamiltonian $H_i(\theta)$ for the type θ. The Lagrangian associated with the sub-problem $[P_1^*]$ is $L_1(\theta) = H_1(\theta) + \mu(\theta)[\bar{T} - U_m(\theta) - \theta q_1(\theta)]$ where $\mu(\theta)$ is the Lagrangian multiplier associated with the type θ.
The necessary conditions for $[P_1^*]$ are

$[P_1^*(1)]$ $\frac{\partial L_1(\theta)}{\partial q_1(\theta)} = \{S'(q_1(\theta)) - \theta\} f(\theta) - \lambda_1(\theta) - \theta\mu(\theta) = 0$,

$[P_1^*(2)]$ $\lambda'_1(\theta) = -\frac{\partial L_1(\theta)}{\partial U_m(\theta)} = f(\theta) + \mu(\theta)$,

$[P_1^*(3)]$ $\lambda_1(\tilde{\theta})$ is free,

$[P_1^*(4)]$ $\lambda_1(\underline{\theta}) = 0$,

optimal constrained mechanism for the case when the available fund $\bar{T}$ is above some critical level $\mathbf{T}$ which allows for partial bunching and partial separation. The solution to this program will provide the exact amount of this critical level $\mathbf{T}$.

$[P_1^*(5)]$ $\mu(\theta) \geq 0$ and

$[P_1^*(6)]$ $\bar{T} - U_m(\theta) - \theta q_1(\theta) = 0.$

Similarly, the necessary conditions for $[P_2^*]$ are

$[P_2^*(1)]$ $\frac{\partial H_2(\theta)}{\partial q_2(\theta)} = \{S'(q_2(\theta)) - \theta\}f(\theta) - \lambda_2(\theta) = 0,$

$[P_2^*(2)]$ $\lambda_2'(\theta) = -\frac{\partial H_2(\theta)}{\partial U_m(\theta)} = f(\theta),$

$[P_2^*(3)]$ $\lambda_2(\tilde{\theta})$ is free and

$[P_2^*(4)]$ $\lambda_2(\overline{\theta})$ is free.

From $[P_1^*(2)]$ we get

$$\lambda_1(\theta) = F(\theta) + \Psi(\theta) + k_1 \tag{5.2}$$

where $\Psi(\theta) = \int_{\underline{\theta}}^{\theta} \mu(\tau)d\tau$ and k_1 is the constant of integration.[4] Since $\Psi(\underline{\theta}) = F(\underline{\theta}) = 0$ and since $\lambda_1(\underline{\theta}) = 0$ from $[P_1^*(4)]$, we get $k_1 = 0$. Therefore, from (5.2) we get

$$\lambda_1(\theta) = F(\theta) + \Psi(\theta) \tag{5.3}$$

From $[P_2^*(2)]$ we get

$$\lambda_2(\theta) = F(\theta) + k_2 \tag{5.4}$$

where k_2 is the constant of integration. Since $\tilde{\theta}$ is the optimal cut-off point for the program $[P^*]$, we get $\lambda_1(\tilde{\theta}) = \lambda_2(\tilde{\theta})$. Then from conditions (5.3) and (5.4) we get $k_2 = \Psi(\tilde{\theta})$ and hence

$$\lambda_2(\theta) = F(\theta) + \Psi(\tilde{\theta}) \tag{5.5}$$

Substituting (5.3) in $[P_1^*(1)]$ and then simplifying it, using $q(\theta) = q(\tilde{\theta})$ for all $\theta \in [\underline{\theta}, \tilde{\theta})$, we get

$$S'(q_1(\tilde{\theta})) = \theta + \frac{F(\theta) + \Psi(\theta) + \theta\mu(\theta)}{f(\theta)} \tag{5.6}$$

for all $\theta \in [\underline{\theta}, \tilde{\theta})$.
Similarly, substituting (5.5) in $[P_2^*(1)]$ and then simplifying it we get for all $\theta \in [\tilde{\theta}, \overline{\theta}]$

$$S'(q_2(\theta)) = \theta + \frac{F(\theta) + \Psi(\tilde{\theta})}{f(\theta)} \tag{5.7}$$

[4] It is important to note that $\Psi'(\theta) = \mu(\theta)$. This fact will be used later to determine the functional form of $\Psi(\theta)$.

To show that $q(.)$ is continuous at the cut-off point $\tilde{\theta}$, we must show that the left hand side of $[P_1^*(1)]$ and $[P_2^*(1)]$ are the same at $\tilde{\theta}$, that is $\{S'(q_1(\tilde{\theta})) - \tilde{\theta}\}f(\tilde{\theta}) - \lambda_1(\tilde{\theta}) - \tilde{\theta}\mu(\tilde{\theta}) = \{S'(q_2(\tilde{\theta})) - \tilde{\theta}\}f(\tilde{\theta}) - \lambda_2(\tilde{\theta})$. Using $\lambda_1(\tilde{\theta}) = \lambda_2(\tilde{\theta})$ we get

$$S'(q_1(\tilde{\theta})) - S'(q_2(\tilde{\theta})) = \frac{\tilde{\theta}\mu(\tilde{\theta})}{f(\tilde{\theta})} \tag{5.8}$$

If $\mu(\tilde{\theta}) > 0$, then the right hand side of condition (5.8) is positive. This means that $S'(q_1(\tilde{\theta})) > S'(q_2(\tilde{\theta}))$ and hence by strict concavity of $S(.)$ we get $q_1(\tilde{\theta})) < q_2(\tilde{\theta})$. This violates condition (5.1). Therefore, it must be the case that $\mu(\tilde{\theta}) = 0$ and hence $q_1(\tilde{\theta})) = q_2(\tilde{\theta})$.

Therefore, the constrained optimal mechanism M^* for the partial bunching case satisfies the following three conditions:

(p1) $S'(q^*(\tilde{\theta})) = \theta + \frac{F(\theta)+\Psi(\theta)+\theta\mu(\theta)}{f(\theta)}$ for all $\theta \in [\underline{\theta}, \tilde{\theta})$, $\mu(\tilde{\theta}) = 0$,

(p2) $S'(q^*(\theta)) = \theta + \frac{F(\theta)+\Psi(\tilde{\theta})}{f(\theta)}$ for all $\theta \in [\tilde{\theta}, \overline{\theta}]$ and

(p3) $\bar{T} = U_m(\tilde{\theta}) + \tilde{\theta}q^*(\tilde{\theta})$

Here (p1) follows from condition (5.6), (p2) follows from condition (5.7) and (p3) is obtained from $[P_1^*(6)]$ which gives us the optimal cut-off point $\tilde{\theta}$.

We now determine $\Psi(\tilde{\theta})$. Integrating condition (5.6) after substituting $\frac{d}{d\theta}[\theta F(\theta)] = \theta f(\theta) + F(\theta)$, $\frac{d}{d\theta}[\theta\Psi(\theta)] = \theta\mu(\theta) + \Psi(\theta)$ and $S'(q(\tilde{\theta})) \equiv c(\tilde{\theta})$ we get

$$\theta F(\theta) + \theta\Psi(\theta) = c(\tilde{\theta})F(\theta) + k_3 \tag{5.9}$$

for all $\theta \in [\underline{\theta}, \tilde{\theta})$. Here k_3 is the constant of integration. Using $F(\underline{\theta}) = \Psi(\underline{\theta}) = 0$ in condition (5.9) we get $k_3 = 0$. Substituting $k_3 = 0$ in condition (5.9) and then simplifying it we get

$$\Psi(\theta) = \left(\frac{c(\tilde{\theta}) - \theta}{\theta}\right) F(\theta) \tag{5.10}$$

Differentiating (5.10) with respect to θ and then using $\mu(\tilde{\theta}) = 0$ we get $S'(q(\tilde{\theta})) \equiv c(\tilde{\theta}) = (\frac{\tilde{\theta}^2 f(\tilde{\theta})}{\tilde{\theta}f(\tilde{\theta})-F(\tilde{\theta})})$. Substituting $\theta = \tilde{\theta}$ and $c(\tilde{\theta})$ in condition (5.10) we get $\Psi(\tilde{\theta}) = (\frac{F^2(\tilde{\theta})}{\tilde{\theta}f(\tilde{\theta})-F(\tilde{\theta})})$. Observe that from the first part of Assumption 1 it follows that $\Psi(\tilde{\theta})$ is well defined. Moreover, since $\Psi(\tilde{\theta}) = \int_{\underline{\theta}}^{\tilde{\theta}} \mu(\tau)d\tau$ and the Lagrangian multiplier $\mu(\theta) \geq 0$ for all $\theta \in [\underline{\theta}, \tilde{\theta})$, it is necessary that $\Psi(\tilde{\theta}) \geq 0$. Observe that $\Psi(\underline{\theta}) = 0$. Therefore, to show that $\Psi(\theta) \geq 0$ for all $\theta \in (\underline{\theta}, \overline{\theta})$ it is now more than enough to show that $\Psi'(\theta) \geq 0$ for all $\theta \in (\underline{\theta}, \overline{\theta})$. By differentiating $\Psi(\theta)$ with respect to $\theta \in (\underline{\theta}, \overline{\theta})$ and then setting it to be non-negative we get $\theta(1 + L'(\theta)) \geq 2L(\theta)$ for all $\theta \in (\underline{\theta}, \overline{\theta})$ which is the second part of assumption 1. Hence, $\Psi'(\theta) \geq 0$ for all $\theta \in (\underline{\theta}, \overline{\theta})$ which means that $\Psi(\theta) \geq 0$ for all $\theta \in (\underline{\theta}, \overline{\theta})$. This implies that $\Psi(\theta) = (\frac{F^2(\theta)}{\theta f(\theta)-F(\theta)}) > 0$ for all $\theta \in (\underline{\theta}, \overline{\theta})$ since our assumption that $f(\theta) > 0$ for all $\theta \in [\underline{\theta}, \overline{\theta}]$ implies that $F(\theta) > 0$ for all $\theta \in (\underline{\theta}, \overline{\theta})$ and since for $\Psi(\theta)$ to be non-negative it is always necessary that $\theta f(\theta) - F(\theta) > 0$. Thus, conditions (p1), (p2) and (p3) together with $\Psi(\tilde{\theta}) = (\frac{F^2(\tilde{\theta})}{\tilde{\theta} f(\tilde{\theta}) - F(\tilde{\theta})}) > 0$ gives us the conditions in Theorem 3.1 when partial bunching is optimal.

We will show that given $\Psi(\tilde{\theta}) = (\frac{F^2(\tilde{\theta})}{\tilde{\theta} f(\tilde{\theta}) - F(\tilde{\theta})})$, $q^*(\theta)$ is non-increasing in $\theta \in [\underline{\theta}, \overline{\theta}]$. Observe first that from condition (p1) it follows that $q^*(\theta) = q^*(\tilde{\theta})$ for all $\theta \in [\underline{\theta}, \tilde{\theta})$. To show that $q^*(\theta)$ is non-increasing in $\theta \in [\tilde{\theta}, \overline{\theta}]$ we have to show that $\bar{z}(\theta) \equiv z(\theta) + \frac{\Psi(\tilde{\theta})}{f(\theta)}$ is non-decreasing in $\theta \in [\tilde{\theta}, \overline{\theta}]$. Differentiating $\bar{z}(\theta)$ with respect to $\theta \in (\tilde{\theta}, \overline{\theta})$ and then setting it to be non-negative we get (c) $\frac{f'(\theta)(F(\theta) + \Psi(\tilde{\theta}))}{f^2(\theta)} \leq 2$. To show that condition (c) is true, it is more than enough to show that $\frac{f'(\theta)(F(\theta) + \Psi(\theta))}{f^2(\theta)} \leq 2$ since $\Psi'(\theta) \geq 0$ for all $\theta \in (\underline{\theta}, \overline{\theta})$ implies that $\frac{f'(\theta)(F(\theta) + \Psi(\tilde{\theta}))}{f^2(\theta)} \leq \frac{f'(\theta)(F(\theta) + \Psi(\theta))}{f^2(\theta)}$ for all $\theta \in (\tilde{\theta}, \overline{\theta})$. From the second part of assumption 1 we know that for all $\theta \in (\underline{\theta}, \overline{\theta})$,

$$\theta(1 + L'(\theta)) \geq 2L(\theta)$$

$$\text{or } \theta \left(2 - \frac{f'(\theta)F(\theta)}{f^2(\theta)}\right) \geq 2L(\theta)$$

$$\text{or } \frac{\theta}{L(\theta)} \left(2 - \frac{f'(\theta)F(\theta)}{f^2(\theta)}\right) \geq 2$$

$$\text{or } 2\left(\frac{\theta - L(\theta)}{L(\theta)}\right) \geq \left(\frac{\theta}{L(\theta)}\right)\left(\frac{f'(\theta)F(\theta)}{f^2(\theta)}\right)$$

$$\text{or } 2\left(\frac{F(\theta)}{\Psi(\theta)}\right) \geq \left(\frac{(F(\theta) + \Psi(\theta)}{\Psi(\theta)}\right)\left(\frac{f'(\theta)F(\theta)}{f^2(\theta)}\right) \text{ (since } \Psi(\theta) = \left(\frac{L(\theta)F(\theta)}{\theta - L(\theta)}\right) \text{ for all } \theta)$$

$$\text{or } 2 \geq \left(\frac{f'(\theta)(F(\theta) + \Psi(\theta))}{f^2(\theta)}\right)$$

Thus from Assumption 1, we get $\frac{f'(\theta)(F(\theta) + \Psi(\theta))}{f^2(\theta)} \leq 2$ for all $\theta \in (\underline{\theta}, \overline{\theta})$. Therefore, condition (c) holds. This proves that $\bar{z}(\theta) \equiv z(\theta) + \frac{\Psi(\tilde{\theta})}{f(\theta)}$ is non-decreasing in $\theta \in [\tilde{\theta}, \overline{\theta}]$ and hence $q^*(\theta)$ is non-increasing in $\theta \in [\tilde{\theta}, \overline{\theta}]$. Observe that $\tilde{\theta} = \overline{\theta}$, corresponds to the transfer $\mathbf{T} = \overline{\theta} S'^{-1}(\overline{\theta} + \frac{F(\overline{\theta}) + \Psi(\overline{\theta})}{f(\overline{\theta})}) > 0$. Therefore, for all $\bar{T} > \mathbf{T}$ the optimal mechanism is a partial bunching one. Finally, since the Hamiltonian H_2 is concave in $q(.)$ and linear in $U_m(.)$, the necessary conditions are also sufficient for $[P_2^*]$. The necessary conditions are also sufficient for $[P_1^*]$ since the Lagrangian $L_1(\theta)$ is concave in (q, U_m) for all $\theta \in [\underline{\theta}, \tilde{\theta})$ (see Chiang, 1992).

If $\bar{T} \leq \mathbf{T}$, then a partial bunching contract is not feasible. Hence, for $\bar{T} \leq \mathbf{T}$, the optimal solution is a full-bunching one implying $q^*(\theta) = \bar{q}$ for all $\theta \in [\underline{\theta}, \overline{\theta}]$. Given that the mechanism is optimal, from IC and PC it follows that $U_m(\theta) = \int_\theta^{\overline{\theta}} q^*(\tau) d\tau$ and hence we get $\bar{T} - \theta\bar{q} = (\overline{\theta} - \theta)\bar{q}$. Therefore, $\bar{T} = \overline{\theta}\bar{q}$. Thus, in the full bunching case $\tilde{\theta} = \overline{\theta}$ and $q^*(\theta) = \frac{\bar{T}}{\overline{\theta}}$ for all $\theta \in [\underline{\theta}, \overline{\theta}]$. □

Proof of 3.2. Consider first the partial bunching case, that is consider $\mathbf{T} < \bar{T} < t^b(\underline{\theta})$. Observe first that the number $\Psi(\tilde{\theta})$ is strictly positive. This implies that for all $\theta \in [\tilde{\theta}, \overline{\theta}]$, $S'^{-1}(z(\theta) + \frac{\Psi(\tilde{\theta})}{f(\theta)}) < S'^{-1}(z(\theta))$. Therefore, $q^*(\theta) < q^b(\theta)$ for all $\theta \in [\tilde{\theta}, \overline{\theta}]$. Moreover, for all $\theta \in [\underline{\theta}, \tilde{\theta})$, $q^*(\theta) = q^*(\tilde{\theta}) < q^b(\theta)$ since $q^b(\theta) \geq q^b(\tilde{\theta})$ for all $\theta \in [\underline{\theta}, \tilde{\theta})$. For the full bunching case, that is for $\bar{T} \leq \mathbf{T} < t^b(\underline{\theta})$, it is obvious that the optimal fixed quantity $\frac{\bar{T}}{\overline{\theta}}$ is strictly smaller than any $q^*(\theta)$ for the partial bunching case. Hence the result follows. □

Proof of 3.3. Observe first that by definition $\hat{\theta}q^b(\hat{\theta}) + \int_{\hat{\theta}}^{\overline{\theta}} q^b(\tau)d\tau = \bar{T} = \tilde{\theta}q^*(\tilde{\theta}) + \int_{\tilde{\theta}}^{\overline{\theta}} q^*(\tau)d\tau$. Using this observation, we prove the proposition by contradiction. We first assume that $\hat{\theta} < \tilde{\theta}$. Then define $h(\theta) = q^b(\theta) - q^*(\theta)$ for all $\theta \in (\underline{\theta}, \overline{\theta}]$. Given Proposition 3.2, $h(\theta) > 0$ for all $\theta \in (\underline{\theta}, \overline{\theta}]$. Using the observation we get

$$\hat{\theta}(h(\hat{\theta}) + q^*(\hat{\theta})) - \tilde{\theta}q^*(\tilde{\theta}) + \int_{\tilde{\theta}}^{\overline{\theta}} h(\tau)d\tau + \int_{\hat{\theta}}^{\tilde{\theta}} q^b(\tau)d\tau = 0 \tag{5.11}$$

Since by assumption $\hat{\theta} < \tilde{\theta}$, from the constrained optimal mechanism M^* we get $q^*(\hat{\theta}) = q^*(\tilde{\theta})$. Substituting $q^*(\hat{\theta}) = q^*(\tilde{\theta})$ in (5.11) we get

$$\hat{\theta}h(\hat{\theta}) - (\tilde{\theta} - \hat{\theta})q^*(\tilde{\theta}) + \int_{\tilde{\theta}}^{\overline{\theta}} h(\tau)d\tau + \int_{\hat{\theta}}^{\tilde{\theta}} q^b(\tau)d\tau = 0 \tag{5.12}$$

Since $\int_{\hat{\theta}}^{\tilde{\theta}} q^b(\tau)d\tau \geq (\tilde{\theta} - \hat{\theta})q^*(\tilde{\theta})$, the left hand side of (5.12) is strictly positive. Hence we have a contradiction. □

Proof of 3.4. From the constrained optimal mechanism, we know that $\bar{T}_i = \tilde{\theta}_i q_i^*(\tilde{\theta}_i) + \int_{\tilde{\theta}_i}^{\overline{\theta}} q_i^*(\tau)d\tau$ for $i = 1, 2$. Given that $\Psi'(\theta) \geq 0$, we have the following possibilities:

1. $\tilde{\theta}_1 > \tilde{\theta}_2$ and $\Psi(\tilde{\theta}_1) \geq \Psi(\tilde{\theta}_2)$ and
2. $\tilde{\theta}_1 \leq \tilde{\theta}_2$ and $\Psi(\tilde{\theta}_1) \leq \Psi(\tilde{\theta}_2)$.

We now show that condition (2) is incompatible with $\bar{T}_1 < \bar{T}_2$. Observe that $\bar{T}_1 < \bar{T}_2$ implies that

$$\tilde{\theta}_1 q_1^*(\tilde{\theta}_1) - \tilde{\theta}_2 q_2^*(\tilde{\theta}_2) + \int_{\tilde{\theta}_2}^{\overline{\theta}} (q_1^*(\tau) - q_2^*(\tau))d\tau + \int_{\tilde{\theta}_1}^{\tilde{\theta}_2} q_1^*(\tau)d\tau < 0 \tag{5.13}$$

If $\tilde{\theta}_1 q_1^*(\tilde{\theta}_1) \geq \tilde{\theta}_2 q_2^*(\tilde{\theta}_2)$, then we already have a contradiction to condition (5.13) since $\int_{\tilde{\theta}_2}^{\overline{\theta}} (q_1^*(\tau) - q_2^*(\tau))d\tau \geq 0$ and $\int_{\tilde{\theta}_1}^{\tilde{\theta}_2} q_1^*(\tau)d\tau > 0$. Therefore, for condition (5.13) to be true it is necessary that $\tilde{\theta}_1 q_1^*(\tilde{\theta}_1) < \tilde{\theta}_2 q_2^*(\tilde{\theta}_2)$. Moreover, for condition (5.13) to hold it is also necessary that $\tilde{\theta}_2 q_2^*(\tilde{\theta}_2) - \tilde{\theta}_1 q_1^*(\tilde{\theta}_1) > \int_{\tilde{\theta}_1}^{\tilde{\theta}_2} q_1^*(\tau)d\tau$. We now show that this condition is not true. Observe first that $\int_{\tilde{\theta}_1}^{\tilde{\theta}_2} q_1^*(\tau)d\tau > (\tilde{\theta}_2 - \tilde{\theta}_1)q_1^*(\tilde{\theta}_2)$ since $q_1^*(\theta) > q_1^*(\tilde{\theta}_2)$ for all $\theta \in [\tilde{\theta}_1, \tilde{\theta}_2)$. Observe next that $(\tilde{\theta}_2 - \tilde{\theta}_1)q_1^*(\tilde{\theta}_2) \geq \tilde{\theta}_2 q_2^*(\tilde{\theta}_2) - \tilde{\theta}_1 q_1^*(\tilde{\theta}_1)$. These two observations together imply that

$\tilde{\theta}_2 q_2^*(\tilde{\theta}_2) - \tilde{\theta}_1 q_1^*(\tilde{\theta}_1) < \int_{\tilde{\theta}_1}^{\tilde{\theta}_2} q_1^*(\tau)d\tau$. Thus, for this case we get $\bar{T}_1 > \bar{T}_2$ which is a contradiction.

Thus, we have proved that only condition (1) is compatible with $\bar{T}_1 < \bar{T}_2$. Hence for all $\theta \in [\underline{\theta}, \overline{\theta}]$, $q_1^*(\theta) < q_2^*(\theta)$. □

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