Large pion pole in Z_S^{MOM}/Z_P^{MOM} from Wilson action data

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Abstract

We show that, contrarily to recent claims, data from the Wilson (unimproved) fermionic action at three different β values demonstrate the presence of a large Goldstone boson contribution in the quark pseudoscalar vertex, quantitatively close to our previous estimate based on the SW action with $c_{SW} = 1.769$. We show that discretisation errors on Z_S^{MOM}/Z_P^{MOM} seem to be much smaller than the Goldstone pole contribution over a very large range of momenta. The subtraction of this non perturbative contribution leads to numbers close to one-loop BPT.

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1 Introduction

In a recent paper [1], using data[2] from the QCDSF collaboration at $\beta = 6.0$ and $c_{SW} = 1.769$ for three different values of κ [2], we have shown that the quark pseudoscalar vertex contains an unexpectedly large contribution from the Goldstone boson pole, and that this contribution accounts for a third of Z_P at 2 GeV. Qualitative indication of this large Goldstone contribution had been noted previously by the QCDSF group [3], and by the Rome group [4]. A large quantitative estimate has been found independently by JLQCD [5] with staggered fermions.

Giusti and Vladikas [6] have recently presented a criticism of our paper; they have reexamined this problem using Wilson data at two values of $\beta = 6.2,\ 6.4$, and have come to the conclusion that the Goldstone boson term would be below the level of discretisation errors "around $p = 1/a$", and therefore not significant. However, they do not compare their data with our result − given at 2 GeV ($ap \simeq 1$ at $\beta = 6.0$)− but rather consider higher momenta $p = 3.3$ or 4.6 GeV ($sin^2(ap) = 0.8$ in their Fig. 4 and Table 1), where of course the Goldstone is much smaller[3]. We present here an analysis based on a set of previously published data of the same origin [7], which shows that, contrary to their objections, the Goldstone boson contribution in Wilson data is in fact completely compatible with our previous estimate, and much above discretisation errors at 2 GeV and probably at notably higher momenta[4].

The present study in fact improves our determination of the Goldstone pole. First of all, the set of Wilson data analysed here is obtained at larger values of β, up to $\beta = 6.4$, where discretisation errors should become really small at moderate $p \simeq 2$ GeV ($a^2p^2 \simeq 0.25$ at $\beta = 6.4$), and perhaps better than with the previous $\beta = 6.0$ data [2] with ALPHA SW improved action. Secondly, we are now in a position to give a reliable estimate of the discretisation errors by considering the evolution of the parameters with

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[2] To calculate Z_ψ, we have also used the propagator data at $\beta = 6.0$ kindly communicated by the Rome group.

[3]One should not speak of the magnitude of the Goldstone at $p = 1/a$ independently of the value of β, since this magnitude depends on p, and $p = 1/a$ depends of course on β. We have never spoken ourselves of such a thing, as stated in the abstract of [6], but of its magnitude at 2 GeV ($\beta = 6.0$).

[4] The present study could presumably be improved using the raw data of [6], which were unfortunately not available to us up to now. We now hope to improve our results in the future with the help of the authors of [6].

β. Thirdly, thanks to the $\beta = 6.2, 6.4$ data, we can improve the large momentum tail of our analysis of power corrections. Finally, we shall also improve our previous work by the inclusion of statistical errors.

Having shown that the Wilson data confirm rather beautifully our first estimate, exhibiting a remarkable stability of the effect with increasing β, we shall conclude by a critical analysis of the procedures of [6] which lead to erroneous conclusions.

2 Previous results on the Goldstone pole in the pseudoscalar vertex

2.1 Our previous results

The theoretical expectation from the continuum is that a pole in $1/m_q$ must be present in the pseudoscalar quark vertex at $q = 0$, as a consequence of the existence of the Goldstone boson. In [1], we analysed the lattice data kindly communicated by QCDSF collaboration [2] for the PS vertex at $\beta = 6.0$ with SW action at $c_{SW} = 1.769$, at several κ, combined with propagator data from the Rome group, at the same β with the same action. We have obtained, in the MOM renormalisation scheme, a Goldstone-like fit of $\left(Z_P^{-1}\right)_{MOM} = \Gamma_P/Z_\psi$ as function of κ. Namely, we have shown that, at $ap = 1$ and $\beta = 6.0$, i.e. around $1.9 - 2$ GeV:

$$Z_P^{-1}(2\ \text{GeV}) = 1.88 + \frac{0.023}{am_q}. \tag{1}$$

The first term on the r.h.s. is the (β-dependent) short-distance contribution. The Goldstone pole corresponds to the second term[5], i.e. a pole in m_q at $m_q = 0$. We have also checked that, as function of p^2, one has the expected behavior: the short distance term is compatible with a logarithmic dependence $\sim \left[\alpha_s(p^2)\right]^{4/11}$ and the Goldstone term has a $1/p^2$ decrease. Converting to physical units, with $a^{-1} = 1.9$ GeV, one obtains:

$$Z_P^{-1}(p^2) = Z_P^{-1}(short\ distance) + \frac{0.158\ \text{GeV}^3}{m_q\, p^2} \tag{2}$$

Of course, the effect of the uncertainty due to the error on a^{-1} could be relevant for the Goldstone contribution since it is $\propto a^{-3}$. Despite this fact, and despite the presence of other uncertainties, it seems difficult to escape the conclusion that the magnitude of the Goldstone term is large at the smallest quark mass (around $m_q = 50$ MeV) and at 2 GeV: 30% of the total $Z_P^{-1} = 2.7$ at 1.9 GeV, although it is decreasing rapidly with increasing p^2. The result can be translated into an estimate of the Georgi-Politzer mass at 1.9 GeV in the chiral limit: $m_R = 34$ MeV.

2.2 Related findings of JLQCD and ALPHA

Our evaluation (1,2) is quantitatively supported by the remarkable JLQCD results on the pseudoscalar vertex and the mass operator with staggered fermions [5]. These results are important as they benefit from two advantages: they go down to very small quark masses (about 20 MeV), and they have, in principle, small discretisation errors. The phenomenon appears very stable with respect to β as they considered $\beta = 6.0$, 6.2 and 6.4.

The above estimate of the Goldstone term is also supported by the estimate of the ALPHA group for the short distance Z_P [9], which must be considered as very solid, since they work at ultra-short distances, and since their discretisation errors are very well controlled. When their short-distance result is converted into the MOM scheme[6] and evolved perturbatively (at 3 loops) down to 2 GeV, one obtains $Z_P^{-1}(2\ \text{GeV}) = 1.8$. This result is close to the first term of Eq.(1), and quite different from the total $Z_P^{-1}(2\ \text{GeV}) \simeq 2.5 - 2.7$: the difference must be filled by the Goldstone boson pole, unless there be incredibly large discretisation errors in the total Z_P^{-1} . The latter is very unlikely in view of the following discussion of Wilson data.

[5] Recently, on investigating the quark propagator and the Ward identity relating the PS and the propagator [8], we have improved the precision of the determination of Z_P,and obtained similar numbers. However, we shall stick here to our first determination, to which [6] is referring.

[6] The initial idea of this conversion is due to Vittorio Lubicz.

3 The Goldstone pole in Wilson data

The most valuable part of [6] is the introduction of the ratio $(Z_P/Z_S)^{RI/MOM} = Z_P^{MOM}/Z_S^{MOM}$ and, on the other hand, of some interesting Ward-Takahashi identities. Let us emphasize indeed that Z_P^{MOM}/Z_S^{MOM} is a scale-dependent quantity, in contrast to $(Z_P/Z_S)^{WI}$, but with a p^2 dependence due only to power corrections . This gives it an important advantage over Z_P which necessarily contains a purely perturbative contribution with logarithmic behaviour, complicating the determination of the power term.

We therefore discuss Z_P^{MOM}/Z_S^{MOM} as *the best probe of the Goldstone pole*, which should be seen as a $1/m_q\, p^2$ term in the *inverse*, Z_S^{MOM}/Z_P^{MOM}. Moreover, we shall show later that, according to equation (19) of [6], a p^2 change in Z_P^{MOM}/Z_S^{MOM} can only be due to the presence of a Goldstone $1/m_q$ pole. The physical source of any departure from the $p^2 \to \infty$ asymptotic value must then be a Goldstone contribution. That it is present is recognized in [6]; we differ on the estimate of its magnitude.

The first question to be answered is whether the effect of the Goldstone pole has the large magnitude that we have estimated, or whether it is sub-dominant with respect to discretisation errors already at $p =$ 2 GeV, as claimed in [6]. This can be answered only by considering the behaviour of Z_S^{MOM}/Z_P^{MOM} around 2 GeV, or, in a scale independent manner, by comparing the *coefficient* of the power corrections to the one we have given in Eq. (2).

The second point concerns the estimate of the discretisation error itself: its magnitude can be estimated by examining the stability of the result as a function of β with fixed physical parameters.

3.1 Methodological considerations

3.1.1 Use of physical units and comparison of different actions

We have to compare different sets of data, with different actions and different β's. It is a delicate task, especially for scale-dependent quantities. Since the Goldstone pole residue is a physical effect − although perhaps gauge dependent − seen in the renormalised pseudoscalar (PS) vertex, a minimum requirement is to compare the results at *identical momenta for the same quark mass*, not at identical ap if β varies, as done in ref. [6], e.g. when making statements about the magnitude of the Goldstone as compared to ours, at "$p = 1/a$". Our Fig. 1 below illustrates the effect of comparing data in terms of ap instead of p: the data at various β's, which show large discrepancies in terms of ap (Fig. 1 a), almost superpose in terms of physical p (Fig. 1 b)[7].

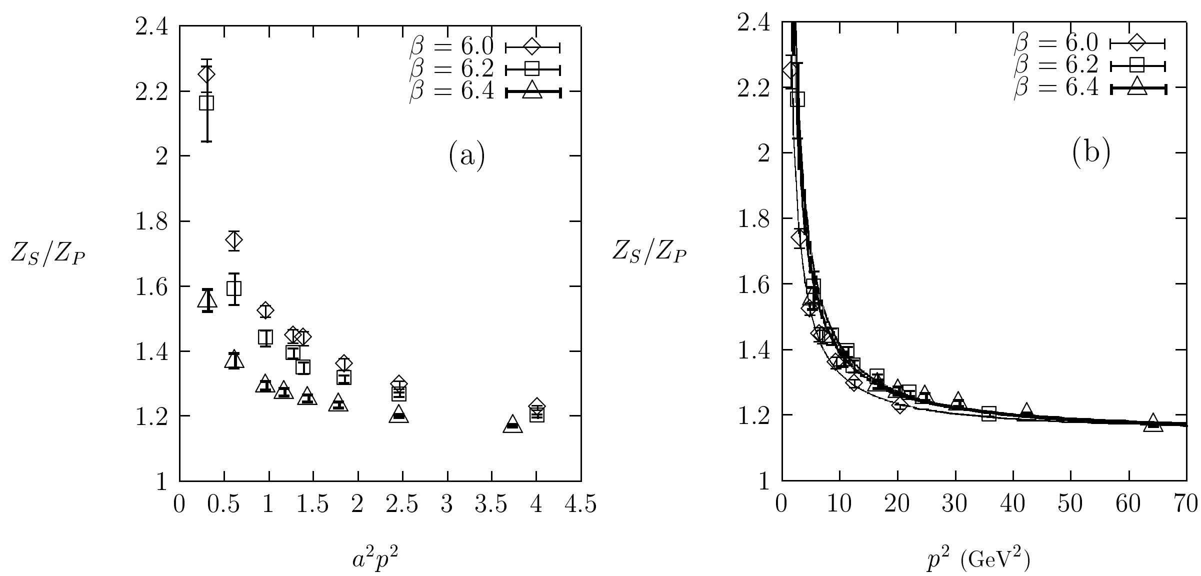

Figure 1: (a) the data from [7] for Z_S^{MOM}/Z_P^{MOM} for three values of β, as function of the lattice a^2p^2 and (b) our fits to these data represented in *physical units*.

[7]This is also quite visible in Figs. 3 to 5 of [7].

The comparison of data from the same action at different β requires only the ratio of the lattice units a, which are rather well determined. When we consider the absolute magnitude of the Goldstone term, the uncertainty becomes larger since the strength (i.e. the coefficient of $1/(m_q p^2)$) is proportional to a^{-3}.

Furthermore, a dependence on the action, and therefrom an additional cutoff dependence, is to be expected even on the the finite Z's as shown by lattice perturbation theory. In particular, the Wilson term and the clover term induce contributions to finite Z's of the form $Cg^2 = C6.0/\beta$, with a coefficient C dependent on the action. These contributions are quite sizable − even for large β, close to the continuum. On the other hand, one may expect that the non-perturbative Goldstone part is independent of the action, since it is a long distance effect. Both these expectations are confirmed in the present analysis.

3.1.2 Discretisation errors

At a given β, one can appreciate discretisation errors, as done in [6], by observing the discrepancies between various quantities which should have been equal, for instance between various estimates of $(Z_P/Z_S)^{WI}$, from various Ward identities (WI), or else from the asymptotic value of Z_P^{MOM}/Z_S^{MOM}. We shall return to their conclusions at the end of the paper.

However, since one has a series of values for β, it is possible to do better; by observing the variation of a quantity when one increases β, one can estimate its discretisation error *separately*. Indeed, the discretisation error will then correspond to the deviation by powers of a from what is expected close to the continuum : namely, as we said, one expects a cutoff independent Goldstone pole, and a very slow dependence of the perturbative part of Z_P/Z_S itself through g^2. Note that these expectations amount, on the whole, to saying that Z_P/Z_S should be rather stable with respect to β, on the limited range of available β values, and we will refer to this, from now on as "stability"; the discretisation errors can then be estimated as the deviation from this stability.

Such a study of the β dependence is indeed possible for the Goldstone contribution to Z_P^{MOM}/Z_S^{MOM} since one has three β's, and to some extent for Z_P^{MOM}/Z_S^{MOM} itself, although one must then take into account the dependence expected from $\mathcal{O}(g^2)$ corrections, which is very slow, and one must make sure that the comparisons are performed for the same parameter values in physical units.

3.1.3 Extrapolating to $m_q = 0$

In the presence of a Goldstone pole, and because of its $1/m_q$ behavior, there is no $m_q \to 0$ limit at all for $1/Z_P^{MOM}$, and there is the trivial one 0 for Z_P^{MOM}, or Z_P^{MOM}/Z_S^{MOM}. One can define a chiral limit only after subtracting the Goldstone pole.

On the other hand, if, as in [6, 7], one considers Z_P^{MOM} as it is, without subtraction of the pole, and if one then makes as usual a linear fit in m_q, *the extrapolation to $m_q = 0$ is not the chiral limit.* Two questions then arise :

1) Is such a linear fit possible, given that the real behaviour includes a $1/m_q$ term?

2) What is the meaning of the quantity obtained by this linear extrapolation?

As to question 1), a fit to $1/m_q$ linear in m_q seems possible for the values of m_q reached in standard numerical simulations, at least for p^2 not too small, but is not with smaller masses, as reached by JLQCD. As to question 2), we show below that in fact the linear fit used in [6] gives to a good approximation Z_P^{MOM}/Z_S^{MOM} at an effective mass $m_{eff}(\beta)$ which can be determined by the various masses used in the extrapolation and which is close to the lowest mass.

3.2 Results

First of all, the extrapolated results of [7] can then still be used to observe the $1/p^2$ power behavior of the Goldstone pole, and a fit in $1/p^2$ will first allow us to quantify this contribution. Secondly, we shall determine $m_{eff}(\beta)$ and this will enable us to observe the typical Goldstone sensitivity to the mass: the apparent discretisation errors are in fact due to the hidden $m_{eff}(\beta)$ dependence. Finally, we shall be able to confirm the quantitative estimate of the Goldstone coefficient we have made previously, and to give an estimate of the true discretisation errors in Wilson data.

3.2.1 Large power corrections

To display the power corrections, we perform a fit on the Wilson Z_P^{MOM}/Z_S^{MOM} ratio, given with its statistical errors in [7], and shown in Fig. 1. Note that the momentum variable is the true p, not $\bar{p} = sin(ap)/a$ as in [6]. The values of the parameters corresponding to the data are given in Table 1.

β	6.0	6.2	6.4
1/a (GeV)	2.258±0.050	2.993±0.094	4.149±0.161
κ_1	0.1530	0.1510	0.1488
κ_2	0.1540	0.1515	0.1492
κ_3	0.1550	0.1520	0.1496
κ_4		0.1526	0.1500
κ_{crit}	0.15683	0.15337	0.15058

Table 1: the parameters corresponding to the data analysed here, from [7].

One can clearly see in these data the presence of a pole contribution at small p^2. To quantify it, we fit the points at each β separately, to the form:

$$Z_S^{MOM}/Z_P^{MOM} = a_{S/P}(\beta) + \frac{b(\beta)}{p^2} \tag{3}$$

with b in GeV^2 and p in GeV. The results of this fit are shown in Fig. 1(b) and in Table 2.

The χ^2/dof is rather high, but this is due to the points at high p^2 (this can be verified: cutting out high values of p^2 reduces the χ^2/dof substantially), where the discretisation errors and/or the logarithmic corrections should be the largest. As expected from BPT, the p^2-independent term is remarkably stable with β (even at 6.0), $a_{S/P} \simeq 1.14$. The coefficient of the power correction is less stable, changing by about 15%, but is consistently very large. Hence b is clearly incompatible with zero, and gives a very large effect of around 30% on the total Z_S^{MOM}/Z_P^{MOM} at 2 GeV.

β	6.0	6.2	6.4
$a_{S/P}$	1.1414±0.0072	1.1266 ± 0.0082	1.1364±0.0049
b	1.8710±0.063	2.8996±0.15	2.5844±0.15
χ^2/dof	1.49	1.22	1.83

Table 2: the values of the coefficients of Eq. (3) fit to the data of [7].

The consistency of the values of b at $\beta = 6.2$ and $\beta = 6.4$ shows that this contribution is much beyond the discretisation error on Z_S^{MOM}/Z_P^{MOM}. In fact, these can be estimated to about 2% from the difference between Z_S^{MOM}/Z_P^{MOM} at 6.2 and 6.4 at $p \simeq 2$ GeV.

The difference of b at $\beta = 6.2$ and $\beta = 6.4$ might be taken as indicating the discretisation artefact on the coefficient itself. However, we show in the next section that even this difference is most probably a physical effect, and that the real discretisation error on the power correction is still smaller.

3.2.2 Power corrections are of Goldstone origin

Having proven the existence of large power corrections, stable with β, and therefore probably not artefacts of discretisation, we must now prove that these come from a Goldstone boson. This is in agreement with the dominance of the divergence of axial current (the pseudoscalar density) at small pion mass, but in fact we can show that the data itself favours this interpretation.

However, we would like first to comment on the dominance of the Goldstone, and, for that purpose, to establish the connection with the quantity WIq discussed in [6]: power corrections to Z_S^{MOM}/Z_P^{MOM} can originate only from the Goldstone boson pole $\sim 1/m_q$, if one is close enough to the chiral limit.

We start with the Ward identity given in equation (19) of [6]:

$$(\frac{Z_S}{Z_P})^{WI} = \frac{m_1\Gamma_P(ap; am_1, am_1) - m_2\Gamma_P(ap; am_2, am_2)}{(m_1 - m_2)\Gamma_S(ap; am_1, am_2)} \tag{4}$$

where Γ_P and Γ_S are the bare vertex functions. If we assume that Γ_P has a Goldstone contribution, whereas Γ_S doesn't, we get:

$$\Gamma_P = A_P(p^2) + \frac{B_P(p^2)}{m_q} + \mathcal{O}(m_q)$$

$$\Gamma_S = A_S(p^2)(1 + \lambda_S(p^2)m_q)$$

We see that the r.h.s. of Eq. (4) must then be $A_P(p^2)/A_S(p^2) + \mathcal{O}(m_q)$. As $(Z_S/Z_P)^{WI}$ is a constant, we have:

$$\frac{A_P(p^2)}{A_S(p^2)} = C + O(m_q)$$

with C a constant, independent of p^2.

But in the MOM scheme, the ratio is given by:

$$\frac{Z_S^{MOM}}{Z_P^{MOM}} = \frac{\Gamma_P}{\Gamma_S}$$

Hence :

$$\frac{Z_S^{MOM}}{Z_P^{MOM}} = C + (m_q)^{-1}\frac{B_P(p^2)}{A_S(p^2)} - (m_q)^0\frac{\lambda_S(p^2)B_P(p^2)}{A_S(p^2)} + O(m_q)$$

We see that the power corrections are dominated by the Goldstone pole $1/m_q$. There seems to be possible additional $O\left((m_q)^0\right)$ power corrections, although $\lambda_S(p^2)$ is probably small[8]. These are themselves roughly proportional to the residue of the Goldstone term, since $\lambda_S(p^2)$ is not expected to have quick variation with p^2. At least, they are connected with the presence of the Goldstone pole, and we can say that all the power corrections, to this order $O(m_q^0)$ included, originate in the Goldstone pole. In our fits, we shall neglect the $\lambda_S(p^2)$ term in Eq. (5), as well as the smaller $O(m_q)$ terms.

We see also that the the difference :

$$(\frac{Z_S}{Z_P})^{WI} - \frac{Z_S^{MOM}}{Z_P^{MOM}} = \frac{B_P(p^2)}{m_q A_S(p^2)} - \frac{\lambda_S(p^2)B_P(p^2)}{A_S(p^2)} + O(m_q) \qquad (5)$$

is entirely due to the Goldstone boson.

3.2.3 The effective quark mass; Goldstone fit

If the $O(m_q)$ corrections are not large, the linear extrapolation to κ_{crit} which is usually performed amounts to making a linear fit in m_q to $1/m_q$, for the 3 or 4 values of m_q with $2am_q = 1/\kappa - 1/\kappa_{crit}$. The extrapolation of the resulting straight line to $m_q = 0$ (or to κ_{crit}) then defines the inverse of an effective mass $1/m_{eff}(\beta)$. The extrapolated Z_S^{MOM}/Z_P^{MOM} is thus really calculated, not at the chiral limit, but at $m_{eff}(\beta)$, and has the form:

$$\frac{Z_S^{MOM}}{Z_P^{MOM}} = a_{S/P}(\beta) + \frac{b'}{m_{eff}(\beta)\, p^2}, \qquad (6)$$

in other words:

$$b = \frac{b'}{m_{eff}(\beta)} \quad , \qquad (7)$$

where b' is a constant, i.e. a number independent of p, m_q and β, which we call the Goldstone strength.

[8] This $(m_q)^0$ term was ommitted in the initial version of our paper. We thank D. Becirevic for having helped us realise this. That this λ_S is indeed small is implied by the observations of the QCDSF group for Wilson action, hep-lat/9807044, p.16 ; for Kogut-Susskind action, it is striking in the Fig 1 of the JLQCD paper, hep-lat/9901019 ; for ALPHA action, we thank D. Becirevic for confirming that Z_S is incredibly stable with respect to variations of m_q over a very large range of light masses.

As long as the Goldstone term is not too large, this result is maintained to a good approximation when the linear extrapolation is made on the inverse, Z_P^{MOM}/Z_S^{MOM}, which is what is actually done in [7].

We show in Fig. 2(a) and in Table 3 the result of this extrapolation for the values of κ given in Table 1. As we see, $m_{eff}(\beta)$ is close to the lowest mass used in the extrapolation.

β	6.0	6.2	6.4
$m_{eff}(\beta)$ (GeV)	0.0591	0.0400	0.0434
b' (GeV^3)	0.1106±0.004	0.116±0.006	0.112±0.006

Table 3: the values of the effective mass defined by the linear extrapolation, and the resulting values of the coefficient of the Goldstone term.

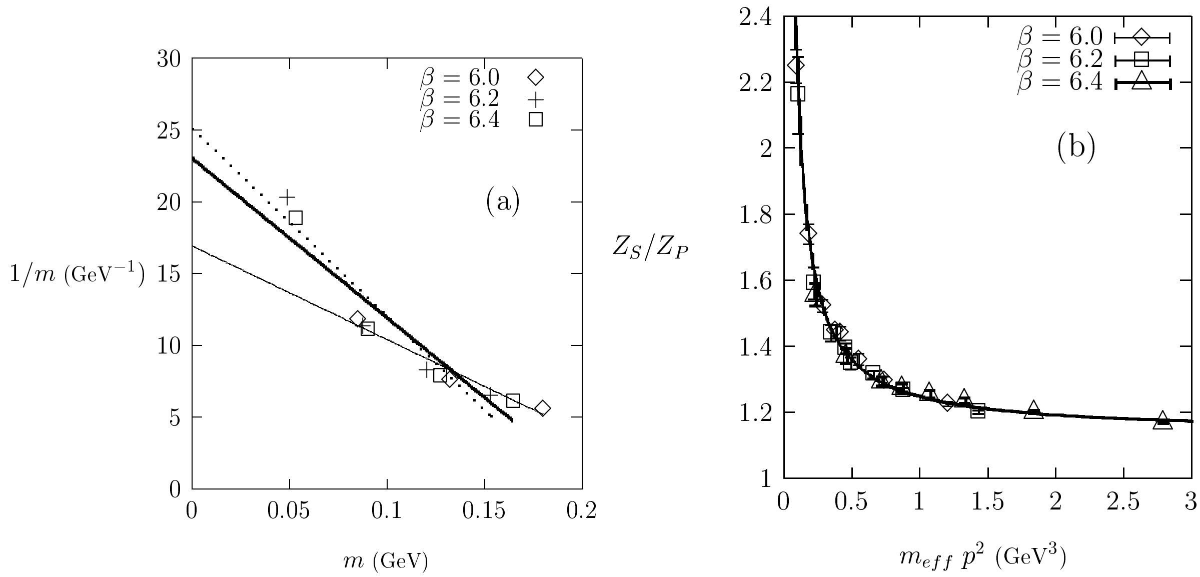

Figure 2: (a) The three straight lines extrapolate the values of $1/m_q$ at each β. Their intersection with the $m = 0$ ordinate defines $1/m_{eff}(\beta)$ and (b) the result of a joint fit to all data, after the $1/m_{eff}(\beta)$ dependence of the Goldstone term has been taken into account.

We can now deduce b' from the fit to b calculated previously: $b' = bm_{eff}(\beta)$. The striking result, shown in Table 3, is that the rather different values of b obtained previously in Table 2 correspond to very good approximation to the same b', i.e., the Goldstone strengths extracted from the data are almost the *same*, and the difference of the b's is mainly due to the different values of $m_{eff}(\beta)$ for each β, which are due to the choice of values of κ.

This effect is also evident in Fig. 5 of [7] and in our Fig. 1(b) , where Z_P^{MOM}/Z_S^{MOM} at $\beta = 6.0$, with p in physical units, deviates significantly from its value at the higher β's, at small momenta. Such a deviation cannot be explained by discretisation effects, which are not confined at small momenta. The natural explanation is that the effective mass is notably higher than at higher β, and that the apparent deviation of Z_P^{MOM}/Z_S^{MOM} at $\beta = 6.0$ is almost entirely due to the quark mass dependence of the Goldstone effect, and disappears when results are compared not only at same physical p, but also at same physical quark mass. The typical quark mass dependence of the Goldstone had been hidden by the extrapolation procedure, but has reappeared as a completely spurious discretisation effect. The true Goldstone origin of this fictitious discretisation effect is revealed by its $1/(p^2 m_q)$ behaviour.

We are now in a position to perform a joint fit to the data at the three values of β, with the variable $1/(m_{eff}(\beta)\,p^2)$ instead of $1/p^2$. We find, with a $\chi^2/dof = 1.39$:

$$Z_S^{MOM}/Z_P^{MOM} = a_{S/P}(\beta) + \frac{(0.112 \pm 0.025)\ \mathrm{GeV}^3}{m_{eff}(\beta)\,p^2} \tag{8}$$

The dependence of $a_{S/P}$ on β is very weak: we find $1/a_{S/P}(\beta) = 0.88\pm0.05, 0.88\pm0.04$ and 0.88 ± 0.03 respectively at $\beta = 6.0$, 6.2 and 6.4.

The Goldstone contribution is stable, compatible at the 3 β's within statistical errors, and very large when $p \simeq 2$ GeV and $m_q \simeq 50$ MeV. Note that these results, shown in Fig. 2(b), are in direct contradiction with the conclusions of [6].

The mildness of cutoff dependence is manifest in the possibility of making such a good common fit to the data for the three different β's. This possibility also gives strong support to the Goldstone interpretation, since it would not be possible without accounting for the $1/m_q$ dependence.

3.2.4 a^2p^2 discretisation errors

From the fit (8), we can deduce the asymptotic value of $Z_P^{MOM}/Z_S^{MOM} = 1/a_{S/P} \approx 0.88$, close to the BPT result. This should be equal to the value given by the Ward identity (4), from which however one gets a lower result $0.79 - 0.80$ [6].

In fact, there seems to be a p-dependent effect, which is signaled by the fact that the strength of the Goldstone term becomes slightly lower and that the χ^2 improves when cutting off the large a^2p^2 points.

Also, correspondingly, the right-hand side of Eq. (4), which equates to Z_S^{MOM}/Z_P^{MOM} minus the Goldstone, is not perfectly constant as it should, although it is much more so than Z_P^{MOM}/Z_S^{MOM} itself, illustrating the Goldstone interpretation (see Fig. 3 of [6]). It is also somewhat different from $1/a_{S/P}$, although the comparison between a constant and a varying quantity is difficult.

The residual cutoff dependence (leaving aside the $\mathcal{O}(g^2)$ effect) can be fitted by a small *negative* a^2p^2 term in Z_S^{MOM}/Z_P^{MOM}. We can then obtain very good fits at the three β's with *a universal Goldstone strength* $b' = 0.098$ GeV^3 and a constant term $a_{S/P}$ which decreases with β just as expected from BPT, and we obtain our final result:

$$Z_S^{MOM}/Z_P^{MOM} = a_{S/P}(\beta) + \frac{(0.098 \pm 0.004)GeV^3}{m_{eff}(\beta)\, p^2} - (0.013 \pm 0.003)a^2p^2 \tag{9}$$

with $\chi^2/dof = 0.45$ and $dof = 19$. The corresponding values of $a_{S/P}$, together with the expectations from BPT and the Ward identities, are shown in Table 3. From this table, it is visible that, once more, after due subtraction of the essentially non perturbative Goldstone pole effect, one has a result close to BPT. Note however that the BPT estimate quoted here is the ratio of the one-loop BPT estimates of Z_P and Z_S at $ap = 1$; one would obtain a somewhat different result, and one exactly scale independent, by applying one loop BPT directly to the ratio.

β	6.0	6.2	6.4
$1/a_{S/P}$ (our fit)	0.835±0.010	0.845±0.009	0.845±0.007
$1/a_{S/P}$ (BPT) [7]	0.83	0.84	0.85
$1/a_{S/P}$ (WIq) [6]	-	0.79 ± 0.02	0.80 ± 0.02

Table 3: our determination of $1/a_{S/P}$ compared with other determinations from [7],[6].

We show in Fig. 3 that the Goldstone contribution dominates the discretisation artefact described by our a^2p^2 term, up to the highest considered a^2p^2 for the two higher values of β.

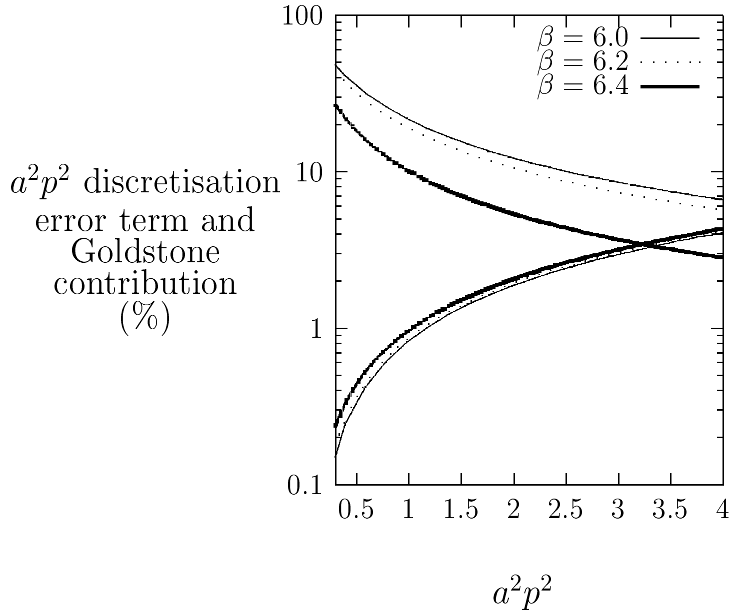

Figure 3: the contribution of the a^2p^2 term (rising curves) and of the Goldstone contribution (decreasing curves) relative to the total $\frac{Z_S}{Z_P}$.

This fit, as shown in Table 3, also gives an estimate of the asymptotic contribution to the Ward Identity, $1/a_{S/P}$, lower and closer to (4). We also give our estimate of the relative sizes of the three terms at $a^2p^2 = 1$ in Table 4.

β	6.0	6.2	6.4
Goldstone term	21.5%	19.0%	10.1%
a^2p^2 discretisation errors	-0.83%	-0.87%	-0.96%

Table 4: the relative values of the Goldstone boson and of the a^2p^2 discretisation errors at $a^2p^2 = 1$.

We must of course keep in mind that the procedure used here is crude, and that a proper analysis of the data must take the Goldstone contribution into account for each κ, before extrapolating to the chiral limit. Hence the agreement with BPT and the difference with the WI determination may reflect the crudeness of our method, due to the unavailability of better data. It is also possible to get fits still closer to $1/a_{S/P} = 0.8$ by allowing for a^4p^4 terms, at the cost of some variation of b' with β. On the other hand it is clear that a similar analysis of discretisation artefacts of the type a^np^n could be usefully applied to WIq, which does not appear to be perfectly constant. At any rate, we observe that even with our latter fits with a^4p^4 terms, the main conclusion is still the same: the Goldstone strength is large, close to our previous estimate, and well above any estimate of discretisation errors over a large range of momenta above 2 GeV.

3.2.5 Comparison with Goldstone residue extracted from QCDSF improved data

We can compare the Goldstone residue with our result of Eq. (2), extracted from the QCDSF data. Since we do not know the value of Z_S from the ALPHA action, the simplest thing to do, disregarding slow logarithmic evolutions, is to compare the magnitudes of the ratio of the power correction to the perturbative term at p and m_q similar in physical units, or of the Goldstone strength b' to the perturbative term in the same unit GeV^3. We immediately notice that the latter ratio is in perfect agreement: 0.82 GeV^3 for the improved action, 0.818 ± 0.003 GeV^3 for the Wilson action.

Hence there is full compatibility between the various determinations of the Goldstone strength, both at various β's and for various actions, converging towards very large values, some 30% of the total around 2 GeV and for a mass around 50 MeV.

4 Discretisation errors; an overall discussion

We now come to the paper [6] of Giusti and Vladikas (G&V), and to a discussion of the origin of its conclusions, opposite to ours. Admittedly, many of the procedures used there are common in the literature, but they turn out to be inappropriate for the present discussion of the Goldstone contribution. Hence we think that beyond answering the criticisms of [6], commenting upon them is of general interest.

Although they intended to discuss specifically our estimate of the magnitude of the Goldstone contribution, the core of the argument of G&V is the comparison of various determinations of $(Z_P/Z_S)^{WI}$. The spread of the values directly extracted from Ward identities, and the deviation with the values obtained for Z_P^{MOM}/Z_S^{MOM}, are supposed to be a measure of the discretisation error on Z_P^{MOM}/Z_S^{MOM} itself. This way of estimating the errors is one of our main disagreements, for reasons expressed below in points 2 and 3. The other main difference is that G&V do not take into account the strong scale dependence of Z_P^{MOM}/Z_S^{MOM}, as explained in points 1 and 4.

1. Let us first emphasize that, even admitting the $10-15$ % estimate of discretisation errors made by [6], these errors cannot dominate the 30 % estimate of the Goldstone pole that we have given at 2 GeV. The reason why G&V have missed this point is clear. The argument, illustrated in their Fig. 4 and Table 1, relies on a point with large momentum for each β. The numbers of ref. [6] are given at $\sin^2(ap) = 0.8$, i.e. at $p = 3.3$ GeV ($\beta = 6.2$) or $p = 4.6$ GeV ($\beta = 6.4$). This choice of a large physical momentum is not appropriate when the manifest goal is to discuss the Goldstone pole overall strength. Indeed, if there is a Goldstone pole, Z_P^{MOM}/Z_S^{MOM} strongly depends on p^2, and its difference with the asymptotic value decreases rapidly with increasing p^2, rendering difficult or eventually impossible the determination of the power correction. Had G&V taken $p = 2$ GeV, they would have had to quote a central value for Z_P^{MOM}/Z_S^{MOM} around 0.6 or less (as can be seen from their Fig. 3), much below $WIq = 0.79$ and also below $WIh = 0.68$ – making manifest the large magnitude of the Goldstone −. In the introduction to their new version of the paper [6], they state however that Z_P^{MOM}/Z_S^{MOM} is compatible with the WI's even at 2 GeV within discretisation errors; this, from their own numbers, amounts to admitting still larger discretisation errors of the order of 30 % or more at $\beta = 6.2$ and $a^2p^2 = 0.45$ (difference between 0.79 and 0.6). One the other hand, the only known way to explain the data with a reasonable error estimate is through the Goldstone interpretation.

2. Furthermore, let us emphasize that their estimated discretisation errors contradict the evolution of data with β as far as Z_P^{MOM}/Z_S^{MOM} is concerned. Indeed in [7], the statistical errors on Z_P^{MOM}/Z_S^{MOM} are small, and the discretisation errors seem also small, since the values of Z_P^{MOM}/Z_S^{MOM} taken at the same physical momenta differ only by a few percent between $\beta = 6.2$ and $\beta = 6.4$ (see Fig. 1(a)).

   A careful reading of the text reveals that the much larger error introduced in [6] has actually nothing to do with the error on Z_P^{MOM}/Z_S^{MOM} itself, but really concerns the estimated error made on the indirect estimate of the Ward identity result through the asymptotic value of Z_P^{MOM}/Z_S^{MOM}. This error was already given in [7], and in fact, in [7], the *same* numbers were quoted as "RGI" (i.e., estimate of the asymptotic, renormalisation group invariant quantity). Indeed, in [7], the lack of the expected plateau was interpreted as an error of 10 to 15% on $(Z_P/Z_S)^{RGI}$. In [6], the same number is re-expressed arbitrarily as an error on the value of Z_P^{MOM}/Z_S^{MOM} at the lower end of the range, $a^2\mu^2 = 0.8$, though it is not an actual error on Z_P^{MOM}/Z_S^{MOM}.

   In our opinion, the lack of plateau signals power corrections and the need to subtract them. The procedure of [6], which amounts to including them automatically into the errors on Z_P^{MOM}/Z_S^{MOM}, makes it of course impossible to discuss the power corrections.

3. Moreover, G&V further substantiate their estimate of the discretisation effect by observing a $10-15$% discrepancy between determinations from two Ward identities, called *WIq* and *WIh*, at $\beta = 6.2$. This procedure has the advantage that the Ward identities are scale independent. It is also a natural approach, if one is working at only one β, to look for the difference between quantities which should be equal. However, in this approach, one does not know which is the best estimate, or whether both equally fail: the same discretisation error is attributed to both, and to any other quantity, such as $(Z_P/Z_S)^{RI/MOM}$, which may be over-pessimistic.

As already emphasized, a better approach is to examine the variation of the specific quantity which one wants to study, when one increases β. Admittedly, small variations can present from the variation of α_S, but they should be $O(g^2)$ in BPT and vary slowly with β, and hence the quantity should be very stable when β changes. If it is stable up to logarithms, this particular quantity has probably a small discretisation error. It seems that the *WIq* determination changes very slightly between 6.2 and 6.4, from 0.79 to 0.8. In fact, the values are compatible within statistical errors, and the small increase is expected from BPT. This is not so for *WIh*, which shows a strong variation from 0.68 to 0.73. In fact, *WIh* corresponds to Z_A times the ratio ρ/m_q of the axial to the subtracted mass, and the latter ratio is known to exhibit rather large variations. The natural conclusion would be then that *WIq* deserves more trust than *WIh*. The same can be said probably of Z_P^{MOM}/Z_S^{MOM}, which is remarkably stable, with a small variation in agreement with BPT. . Thus the large discretisation error should be probably attributed to *WIh* only, not to the three quantities at the same time. It is then rewarding that *WIq* and Z_P^{MOM}/Z_S^{MOM} give compatible results for the estimate of WI, after due subtraction of the Goldstone pole, as we have shown above.

4. We note that in their Fig. 4 and Table 1, G&V consider the spread of values of various determinations of Z_P/Z_S with increasing β, including Z_P^{MOM}/Z_S^{MOM}, at the same $a^2\mu^2 = 0.8$, and not at at the same physical p, as one should do when discussing the error on a scale-dependent quantity. G&V are in fact not comparing the same quantity at two different β's, but two different quantities: the values of Z_P^{MOM}/Z_S^{MOM} respectively at $p = 3.3$ GeV and $p = 4.6$ GeV.

   The natural explanation of the increase of Z_P^{MOM}/Z_S^{MOM} with β in their figure and table is the decrease of the power correction with increasing p, which is a physical effect, not the discretisation errors, except at very large p. If we duly compare at identical p, we see once again that Z_P^{MOM}/Z_S^{MOM} is very stable with β, and that the discrepancy with WIq does not decrease, unlike suggested by the Fig. 4 of [6]: it is a physical effect, the sign of Goldstone contribution as shown above.

5 Conclusion

The quantity Z_P^{MOM}/Z_S^{MOM} appears to be the best indicator of the Goldstone pole. Contrarily to [6], we find a large Goldstone contribution to the Wilson data, of the same magnitude as found previously with data for Z_P^{MOM} from the QCDSF improved action. The results are consistent for 3 values of β and for momenta ranging from about 1 GeV to 8 GeV. Of course, the determination of the Goldstone strength comes mainly from moderate momenta, where the contribution is the largest. The discretisation uncertainty, as estimated from the variation of Z_P^{MOM}/Z_S^{MOM} with β, appears in fact to be rather small, and the Goldstone contribution at $p = 2 - 4$ GeV is far above it. Even admitting the larger discretisation error advocated by [6], which is not relevant in our opinion, our claimed Goldstone contribution at 2 GeV is so large, as already found previously, that it is clearly dominating. Evidently, it is smaller at the higher momenta considered by G&V in their Fig. 4 and Table 1, but this is as it should be: it must be $\propto 1/p^2$!

This large Goldstone contribution explains in a natural manner the discrepancy of the MOM Z_P with the ALPHA group determination of Z_P at large distance (around 30% at 2 GeV). It also explains for the most part the absence of plateau in Z_P^{MOM}/Z_S^{MOM}, even at the highest acceptable momenta. True, we find some contribution from a^2p^2 artefacts, but certainly not a dominant one. Given this absence of a plateau, one should not insist on extracting an estimate of $(Z_P/Z_S)^{WI}$ directly from Z_P^{MOM}/Z_S^{MOM}, even if one assumes large errors. The only way to proceed, which we have illustrated here, is to subtract the Goldstone contribution. The result of the subtraction is, once more, a number close to the BPT expectation, which is quite encouraging. Another formulation of this is to use Eq. (4), which automatically subtracts the Goldstone, and which is found to be rather stable with β.

Of course, some slight changes in the conclusions must be expected from a more thorough analysis of the complete data, where it may be possible, in particular, to explain the small discrepancy between WIq and $1/a_{S/P}$ given in Table 3.

It remains to explain the apparently different conclusion from [10], at $\beta = 6.2$, which find smaller power corrections; this may be related to off-shell improvement.

The interesting and intriguing physical question is now to find the reason why the Goldstone residue is so large in Z_P^{MOM} or Z_P^{MOM}/Z_S^{MOM}. This is connected with the behavior of the pion wave function (BS amplitude) at short distance. The apparent contradiction with the standard OPE is puzzling. A naïve interpretation of our results[9] would be to claim that the quark condensate is 10 times the standard value, but this is certainly not probable, and the ultimate physical reason of this disagreement must surely be more subtle.

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Note added in proof:

While we were writing this letter, C. Dawson [12] and Y. Zhestkov [13] stressed again the necessity of subtracting the Goldstone pole to obtain a chiral limit.

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[9] Note that, contrarily to what was written in the first version of [6], we have never ourselves proposed such an interpretation. Note however that in the mechanism of spontaneous color symmetry breaking proposed recently by C. Wetterich [11], the propagator could get large fluctuations in the octet and this could be connected with our observation (private communication).