$\mathcal{S}$-ADIC CHARACTERIZATION OF MINIMAL DENDRIC SHIFTS

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Abstract. Dendric shifts are defined by combinatorial restrictions of the extensions of the words in their languages. This family generalizes well-known families of shifts such as Sturmian shifts, Arnoux-Rauzy shifts and codings of interval exchange transformations. It is known that any minimal dendric shift has a primitive $\mathcal{S}$-adic representation where the morphisms in $\mathcal{S}$ are positive tame automorphisms of the free group generated by the alphabet. In this paper we give an $\mathcal{S}$-adic characterization of this family by means of two finite graphs. As an application, we are able to decide whether a shift space generated by a uniformly recurrent morphic word is (eventually) dendric.

1. Introduction

Dendric shifts are defined in terms of extension graphs that describe the left and right extensions of their factors. Extension graphs are bipartite graphs that can roughly be described as follows: if u is a word in the language $\mathcal{L}(X)$ of the shift space X, one puts an edge between the left and right copies of letters a and b such that aub is in $\mathcal{L}(X)$. A shift space is then said to be dendric if the extension graph of every word of its language is a tree. These shift spaces were initially defined through their languages under the name of tree sets [6] and were studied in a series of papers. They generalize classical families of shift spaces such as Sturmian shifts [29], Arnoux-Rauzy shifts [2], codings of regular interval exchange transformations [30, 1] (IET) or else shift spaces arising from the application of the Cassaigne multidimensional continued fraction algorithm [13] (MCF).

Minimal dendric shifts exhibit striking combinatorial [9, 4], algebraic [6, 8] and ergodic properties [5]. They for instance have factor complexity $\#(\mathcal{L}(X)\cap\mathcal{A}^n) = (\#\mathcal{A}-1)n+1$ [6] and topological rank $\#\mathcal{A}$ [5], where $\mathcal{A}$ is the alphabet of the shift space. They also fall into the class of eventually dendric shift spaces, which are exactly those satisfying the regular bispecial condition [17]. This implies that the number of their ergodic measures is at most $\#\mathcal{A}/2$ [16]. An important property for our work is that the derived shift of a minimal dendric shift is again a minimal dendric shift on the same alphabet, where derivation is here understood as derivation by return words. This allows to give $\mathcal{S}$-adic representations of such shift spaces [25], i.e., to define a set $\mathcal{S}$ of endomorphisms of the free monoid $\mathcal{A}^*$ and a sequence $\boldsymbol{\sigma} = (\sigma_n)_{n\geq 0} \in \mathcal{S}^{\mathbb{N}}$, called an $\mathcal{S}$-adic representation, such that

$$X = \{x \in \mathcal{A}^{\mathbb{Z}} \mid u \in \mathcal{L}(x) \Rightarrow \exists n \in \mathbb{N}, a \in \mathcal{A} : u \in \mathcal{L}(\sigma_0\sigma_2 \cdots \sigma_n(a))\}.$$

$\mathcal{S}$-adic representations are a classical tool that allows to study several properties of shift spaces such as the factor complexity [23, 18], the number of ergodic measures [10, 3, 12], the dimension group and topological rank [5] or yet the automorphism group [24]. In the case of minimal dendric shifts, the involved endomorphisms are particular tame automorphisms of the free group generated by the alphabet [9, 4]. This in particular allows to prove that minimal dendric shifts have topological rank equal to the cardinality of the alphabet and that ergodic measures are completely determined by the measures of the letter cylinders [5, 12].

In the case of the ternary alphabet $\{1, 2, 3\}$, we were able to give an $\mathcal{S}$-adic characterization of minimal dendric shifts [28]. We explicitly define an infinite set $\mathcal{S}_3$ of positive automorphisms of the free group F_3 and a labeled directed graph $\mathcal{G}$ with two vertices and labels in $\Sigma_3\mathcal{S}_3\Sigma_3$ (where Σ_3 is the symmetric group of $\{1, 2, 3\}$) such that a shift space over $\{1, 2, 3\}$ is a minimal dendric

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shift if and only if it has a primitive $\Sigma_3\mathcal{S}_3\Sigma_3$-adic representation labelling an infinite path in $\mathcal{G}$. We were then able to localize in $\mathcal{G}$ classical families of dendric shifts such as codings of regular interval exchanges or Arnoux-Rauzy shifts.

In this paper we extend this work to any alphabet as follows.

Theorem 1. *Let $\mathcal{S}$ be a family of dendric return morphisms from $\mathcal{A}^*$ to $\mathcal{A}^*$ and let X be a shift space having an $\mathcal{S}$-adic representation $\boldsymbol{\sigma} = (\sigma_n)_{n\geq 0}$. Then X is minimal dendric if and only if $\boldsymbol{\sigma}$ is primitive and labels infinite paths in the graphs $\mathcal{G}^L(\mathcal{S})$ and $\mathcal{G}^R(\mathcal{S})$.*

Observe that, contrary to the ternary case, we are not able to explicitely define the set $\mathcal{S}$. We also consider two finite graphs instead of one. A key argument in the ternary case is that for all N, the $\mathcal{S}$-adic shift $X^{(N)}$ generated by $(\sigma_n)_{n\geq N}$ is minimal and dendric, so that σ_N preserves dendricity from $X^{(N+1)}$. The finiteness of $\mathcal{G}$ was obtained by defining an equivalence relation with finite index on the set of dendric shifts such that two shifts X, Y are equivalent if and only if the same morphisms of $\Sigma_3\mathcal{S}_3\Sigma_3$ preserve dendricity from X and from Y. The vertices of $\mathcal{G}$ are the two equivalent classes of this relation.

In the general case, while we are not able to explicitly define $\mathcal{S}$, we have enough information on the morphisms in $\mathcal{S}$ to similarly define an equivalence relation characterizing how dendricity is preserved. It is defined by means of two finite graphs describing the left and right extensions of infinite special factors of X (see Section 4).

The paper is organized as follows. We start by giving, in Section 2, the basic definitions for the study of shift spaces. We introduce the notion of extension graph of a word and of a dendric shift. In Section 3, we recall and extend some results of [28] about the stability of dendricity when taking the image under a return morphism.

In Section 4, we introduce new graphs associated with a shift space. They provide another characterization of dendric shift spaces. We also study the link with eventual dendricity in Section 5.

After that, in Section 6, we reformulate the results of Section 3 and study the image under a return morphism using these graphs. We then use these results to obtain an S-adic characterization of dendric shift spaces (Theorem 1) in Section 7. We illustrate our results by giving an explicit graph characterizing the minimal dendric shift spaces over four letters having exactly one right special factor of each length. We also show that dendricity is decidable for substitutive shift spaces.

Finally, in Section 8, we focus on the sub-family of interval exchanges and provide an $\mathcal{S}$-adic characterization (Theorem 71) using a subgraph of the graph obtained in the dendric case.

2. Definitions

2.1. **Words, languages and shift spaces.** Let $\mathcal{A}$ be a finite alphabet of cardinality $d \geq 2$. Let us denote by ε the empty word of the free monoid $\mathcal{A}^*$ (endowed with concatenation), by $\mathcal{A}^+$ the set of non-empty finite words on $\mathcal{A}$ and by $\mathcal{A}^{\mathbb{Z}}$ the set of bi-infinite words over $\mathcal{A}$. For a word $w = w_1 \cdots w_\ell \in \mathcal{A}^\ell$, its *length* is denoted $|w|$ and equals ℓ. We say that a word u is a *factor* of a word w if there exist words p, s such that $w = pus$. If $p = \varepsilon$ (resp., $s = \varepsilon$) we say that u is a *prefix* (resp., *suffix*) of w. For a word $u \in \mathcal{A}^*$ and a word $w \in \mathcal{A}^* \cup \mathcal{A}^{\mathbb{Z}}$, an index j such that $w_j \cdots w_{j+|u|-1} = u$ is called an *occurrence* of u in w. The number of occurrences of a word $u \in \mathcal{A}^*$ in a finite word w is denoted by $|w|_u$.

The set $\mathcal{A}^{\mathbb{Z}}$ endowed with the product topology of the discrete topology on each copy of $\mathcal{A}$ is topologically a Cantor set. The *shift map* S defined by $S\left((x_n)_{n\in\mathbb{Z}}\right) = (x_{n+1})_{n\in\mathbb{Z}}$ is a homeomorphism of $\mathcal{A}^{\mathbb{Z}}$. A *shift space* is a pair (X, S) where X is a closed shift-invariant subset of some $\mathcal{A}^{\mathbb{Z}}$. It is thus a *topological dynamical system*. It is *minimal* if the only closed shift-invariant subset $Y \subset X$ are $\emptyset$ and X. Equivalently, (X, S) is minimal if and only if the orbit of every $x \in X$ is dense in X. Usually we say that the set X is itself a shift space.

The *language* of a sequence $x \in \mathcal{A}^{\mathbb{Z}}$ is its set of factors and is denoted $\mathcal{L}(x)$. For a shift space X, its *language* $\mathcal{L}(X)$ is $\bigcup_{x\in X} \mathcal{L}(x)$ and we set $\mathcal{L}_n(X) = \mathcal{L}(X) \cap \mathcal{A}^n$, $n \in \mathbb{N}$, and $\mathcal{L}_{\leq N}(X) = \bigcup_{n\leq N} \mathcal{L}_n(X)$. Its *factor complexity* is the function $p_X : \mathbb{N} \to \mathbb{N}$ defined by $p_X(n) = \operatorname{Card} \mathcal{L}_n(X)$. We say that a shift space X is *over* $\mathcal{A}$ if $\mathcal{L}_1(X) = \mathcal{A}$.

2.2. **Extension graphs and dendric shifts.** Dendric shifts are defined with respect to combinatorial properties of their language expressed in terms of extension graphs. Let F be a set of finite words on the alphabet $\mathcal{A}$ which is factorial, i.e., if $u \in F$ and v is a factor of u, then $v \in F$. For $w \in F$, we define the sets of left, right and bi-extensions of w by

$$\begin{aligned} E_F^L(w) &= \{a \in \mathcal{A} \mid aw \in F\}; \\ E_F^R(w) &= \{b \in \mathcal{A} \mid wb \in F\}; \\ E_F(w) &= \{(a,b) \in \mathcal{A} \times \mathcal{A} \mid awb \in F\}. \end{aligned}$$

The elements of $E_F^L(w)$, $E_F^R(w)$ and $E_F^L(w)$ are respectively called the *left extensions*, the *right extensions* and the *bi-extensions* of w in F. If X is a shift space over $\mathcal{A}$, we will use the terminology extensions in X instead of extensions in $\mathcal{L}(X)$ and the index $\mathcal{L}(X)$ will be replaced by X or even omitted if the context is clear. Observe that as $X \subset \mathcal{A}^{\mathbb{Z}}$, the set $E_X(w)$ completely determines $E_X^L(w)$ and $E_X^R(w)$. A word w is said to be *right special* (resp., *left special*) if $\mathrm{Card}(E^R(w)) \geq 2$ (resp., $\mathrm{Card}(E^L(w)) \geq 2$). It is *bispecial* if it is both left and right special. The factor complexity of a shift space is completely governed by the extensions of its special factors. In particular, we have the following result.

Proposition 2 (Cassaigne and Nicolas [14]). *Let X be a shift space. For all n, we have*

$$\begin{aligned} p_X(n+1) - p_X(n) &= \sum_{w \in \mathcal{L}_n(X)} (\mathrm{Card}(E^R(w)) - 1) \\ &= \sum_{w \in \mathcal{L}_n(X)} (\mathrm{Card}(E^L(w)) - 1) \end{aligned}$$

In addition, if for every bispecial factor $w \in \mathcal{L}(X)$, one has

$$\mathrm{Card}(E(w)) - \mathrm{Card}(E^L(w)) - \mathrm{Card}(E^R(w)) + 1 = 0, \tag{1}$$

then $p_X(n) = (p_X(1) - 1)n + 1$ for every n.

A classical family of bispecial factors satisfying Equation (1) is made of the *ordinary* bispecial factors that are defined by $E(w) \subset (\{a\} \times \mathcal{A}) \cup (\mathcal{A} \times \{b\})$ for some $(a,b) \in E(w)$. A larger family of bispecial factors also satisfying Equation (1) are the dendric bispecial factors defined below.

For a word $w \in F$, we consider the undirected bipartite graph $\mathcal{E}_F(w)$ called its *extension graph* with respect to F and defined as follows: its set of vertices is the disjoint union of $E_F^L(w)$ and $E_F^R(w)$ and its edges are the pairs $(a,b) \in E_F^L(w) \times E_F^R(w)$ such that $awb \in F$. For an illustration, see Example 3 below. Note that a letter a can label a vertex both on the left and on the right side. To distinguish the two, we will denote the left vertex a^L and the right vertex a^R. We say that w is *dendric* if $\mathcal{E}(w)$ is a tree. We then say that a shift space X is a *dendric shift* if all its factors are dendric in $\mathcal{L}(X)$ and it is an *eventually dendric shift (with threshold N)* if all factors of length at least N are dendric (where N is chosen minimal).

Note that every non-bispecial word and every ordinary bispecial word is trivially dendric. In particular, the Arnoux-Rauzy shift spaces are dendric (recall that Arnoux-Rauzy shift spaces are the minimal shift spaces having exactly one left special factor u_n and one right special factor v_n of each length n and such that $E^L(u_n) = \mathcal{A} = E^R(v_n)$; all bispecial factors of an Arnoux-Rauzy shift are ordinary). By Proposition 2, we deduce that any dendric shift has factor complexity $p_X(n) = (p_X(1) - 1)n + 1$ for every n.

Example 3. If X is the *Fibonacci shift* generated by the Fibonacci word $0100101001001010\cdots$. We have $\mathcal{L}_3(X) = \{001, 010, 100, 101\}$. The extension graphs of the empty word and of the two letters 0 and 1 are represented in Figure 1.

3. Extension graphs in morphic images

Let $\mathcal{A}, \mathcal{B}$ be finite alphabets with cardinality at least 2. By a *morphism* $\sigma : \mathcal{A}^* \to \mathcal{B}^*$, we mean a monoid homomorphism (also called a *substitution* when $\mathcal{A} = \mathcal{B}$). We will always assume that $\mathcal{B}$ is minimal, i.e. each letter appear in the image of some word. A morphism is said to be *non-erasing*

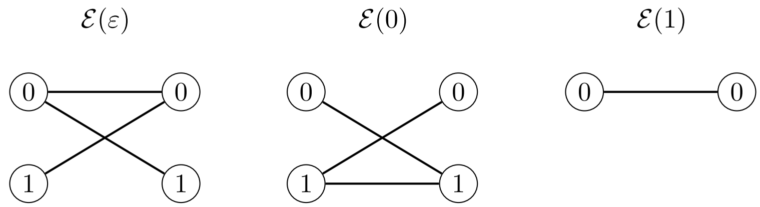


FIGURE 1. The extension graphs of ε (on the left), 0 (in the center) and 1 (on the right) are trees.

if the image of any letter is a non-empty word. We stress the fact that all morphisms are assumed to be non-erasing in the following. Using concatenation, we extend σ to $\mathcal{A}^{\mathbb{Z}}$. In particular, if X is a shift space over $\mathcal{A}$, the *image of X under σ* is the shift space

$$\sigma \cdot X := \{S^k\sigma(x) \mid x \in X, 0 \leq k < |\sigma(x_0)|\}.$$

In general, it can be difficult to deduce extension graphs of words in $\mathcal{L}(\sigma \cdot X)$ from those of $\mathcal{L}(X)$. In this paper, we will restrict ourselves to particular morphisms which offer some nice recognizability properties. They are defined using the notion of return words. Let X be a shift space. If $w \in \mathcal{L}(X)$ is non-empty, a *return word* to w in X is a non-empty word r such that $rw \in \mathcal{L}(X)$ and rw contains exactly two occurrences of w, one as a prefix and one as a suffix.

Definition 4. A *return morphism for a word $w \in \mathcal{B}^+$* is an injective morphism $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ such that, for all $a \in \mathcal{A}$, $\sigma(a)w$ contains exactly two occurrences of w, one as a (proper) prefix and one as a (proper) suffix.

In particular, the set $\sigma(\mathcal{A})$ is then a suffix code, i.e. $\sigma(a)$ is not a suffix of $\sigma(b)$ if $b \neq a$. Hence, injectivity on the letters is equivalent to injectivity on the words.

Example 5. The morphism $\sigma : 0 \mapsto 01, 1 \mapsto 010, 2 \mapsto 0102$ is a return morphism for 01 and it is also a return morphism for 010.

As we have seen in the previous example, the word w for which σ is a return morphism is not always unique. However, we have the following result.

Proposition 6. *Let σ be a return morphism for two distinct words w and w' and let X be a shift space. If $|w| \leq |w'|$, then*

(1) *w is a proper prefix of w';*
(2) *σ is a return morphism for all prefixes of w' of length at least $|w|$;*
(3) *w is not right special in $\sigma \cdot X$;*
(4) *if w' is of maximal length, it is right special in $\sigma \cdot X$.*

Proof.

(1) Let $a \in \mathcal{A}$. If $|w| \leq |\sigma(a)|$, then w is a prefix of $\sigma(a)$. As w' is also prefix of $\sigma(a)w'$ and $|w| \leq |w'|$, w is a proper prefix of w'. Otherwise, $|w'| \geq |w| > |\sigma(a)|$ and both w and w' are of the form $(\sigma(a))^n p$, where p is a prefix of $\sigma(a)$. Thus w is a proper prefix of w'.
(2) Let wu be a prefix of w'. By definition, it is a prefix and a suffix of $\sigma(a)wu$ for all $a \in \mathcal{A}$. Moreover, since w only has two occurrences in $\sigma(a)w$, wu only has two occurrences in $\sigma(a)wu$ thus σ is a return morphism for wu.
(3) Since σ is a return morphism for w, any occurrence of w in $\sigma \cdot X$ is as a proper prefix of some $\sigma(a)w$ for $a \in \mathcal{A}$. However, for all $a \in \mathcal{A}$, w' is a prefix of $\sigma(a)w'$ thus $ww'_{|w|+1}$ is a prefix of $\sigma(a)w$. The only right extension of w is then $w'_{|w|+1}$.
(4) Assume that w' only has one right extension b. Thus $w'b$ is a prefix (and a suffix) of each $\sigma(a)w'b$, $a \in \mathcal{A}$. As w' only has two occurrences in $\sigma(a)w'$, the word $w'b$ only has two occurrences in $\sigma(a)w'b$, which proves that σ is a return morphism for $w'b$, a contradiction.

□

We will sometimes talk about a *return morphism* without specifying the word w for which it is a return morphism. By default, this w will be chosen of maximal length. Because of the previous proposition, most results are independent of this choice (or only depend in a minor way).

We now give a description of the extension graphs in $\sigma \cdot X$ when σ is a return morphism. We first introduce some notations.

Definition 7. Let $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ be a return morphism for the word $w \in \mathcal{B}^+$. For all $u \in \mathcal{B}^*$, we define the sets

$$\mathcal{A}^L_{\sigma,u} = \{a \in \mathcal{A} \mid \sigma(a) \in \mathcal{B}^+ u\}$$

and

$$\mathcal{A}^R_{\sigma,u} = \{a \in \mathcal{A} \mid \sigma(a)w \in u\mathcal{B}^+\}.$$

For a shift X over $\mathcal{A}$ and a word $v \in \mathcal{L}(X)$, we then note $E_{X,s,p}(v) = E_X(v) \cap (\mathcal{A}^L_{\sigma,s} \times \mathcal{A}^R_{\sigma,p})$.

Proposition 8. *Let X be a shift space over $\mathcal{A}$, $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ a return morphism for the word w and $Y = \sigma \cdot X$.*

- *If $u \in \mathcal{L}(Y)$ is such that $|u|_w = 0$, then*
$$\mathcal{E}_Y(u) = \mathcal{E}_{\mathcal{L}(\sigma)}(u),$$
where we define
$$\mathcal{L}(\sigma) = \bigcup_{a \in \mathcal{A}} \mathcal{L}(\sigma(a)w).$$
- *If $u \in \mathcal{L}(Y)$ is such that $|u|_w > 0$, then there exists a unique triplet $(s, v, p) \in \mathcal{B}^* \times \mathcal{L}(X) \times w\mathcal{B}^*$ such that $u = s\sigma(v)p$ and for which the set $E_{X,s,p}(v)$ is not empty. The bi-extensions of u are then governed by those of v through the equation*

$$E_Y(u) = \{(a', b') \in \mathcal{B} \times \mathcal{B} \mid \exists (a, b) \in E_{X,s,p}(v) : \sigma(a) \in \mathcal{B}^* a' s \wedge \sigma(b)w \in pb'\mathcal{B}^*\}. \tag{2}$$

Proof. Let us prove the first case. For any $(b, c) \in E_Y(u)$, there exist a letter a and a word $v \in \mathcal{L}(X)$ such that buc is a factor of $\sigma(a)\sigma(v)$. We can moreover assume that the first occurrence of buc begins in $\sigma(a)$. Since w is not a factor of u and $\sigma(v)$ is prefix comparable with w, i.e. $\sigma(v)$ is either a prefix of w or has w as a prefix, we deduce that buc is a factor of $\sigma(a)w$. This proves that $(b, c) \in E_{\mathcal{L}(\sigma)}(u)$. By definition, $\mathcal{L}(\sigma)$ is included in $\mathcal{L}(Y)$ and we conclude that $\mathcal{E}_Y(u) = \mathcal{E}_{\mathcal{L}(\sigma)}(u)$.

For the second case, the proof is similar to the proof of [28, Proposition 4.1]. □

Under the assumptions of the previous result, the words u that do not contain any occurrence of w are called *initial factors* of Y. As the initial factors of Y and their extension graphs only depend on σ, we will sometimes talk about the *initial factors of* σ. Note that a different convention for the choice of w leads to a smaller set of initial factors. However, by Proposition 6, the set of right special initial factors is the same.

Whenever u and v are as in the second case of the previous proposition, v is called the *antecedent* of u under σ and u is said to be an *extended image* of v.

Thus, any factor of Y is either initial or an extended image of some factor in X. Using Proposition 8, we directly know when the initial factors of Y are dendric. It only depends on σ and not on X. We introduce the following definition.

Definition 9. A return morphism σ is *dendric* if every initial factor is dendric in $\mathcal{L}(\sigma)$.

Note that, by Proposition 6 and because the proof of Proposition 8 works for any choice of w, the fact that a return morphism for w is dendric is independent of the choice of w.

Example 10. The morphism $\beta : 0 \mapsto 0, 1 \mapsto 01, 2 \mapsto 02, 3 \mapsto 032$ is a return morphism for 0. Its initial factors are ε, 1, 2, 3 and 32. Note that the only one which is right special is ε and we can easily check that it is dendric thus β is a dendric return morphism.

We will now use the second item of Proposition 8 to characterize the fact that all the extended images are dendric in Y. Equation 2 can also be seen in terms of extension graphs. To do that, we introduce two types of (partial) maps.

Definition 11. Let $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ be a return morphism for the word w. For $s, p \in \mathcal{B}^*$, we define the partial maps $\varphi^L_{\sigma,s}, \varphi^R_{\sigma,p} : \mathcal{A} \to \mathcal{B}$ as follows:

(1) for $a \in \mathcal{A}^L_{\sigma,s}$, $\varphi^L_{\sigma,s}(a)$ is the letter a' such that $\sigma(a) \in \mathcal{B}^* a' s$;
(2) for $b \in \mathcal{A}^R_{\sigma,p}$, $\varphi^R_{\sigma,p}(b)$ is the letter b' such that $\sigma(b)w \in pb'\mathcal{B}^*$.

We denote by $\mathcal{E}_{X,s,p}(v)$ the subgraph of $\mathcal{E}_X(v)$ generated by the edges in $E_{X,s,p}(v)$. Remark that, with this definition, no vertex is isolated. In particular, the set of vertices might be strictly included in the disjoint union of $E^L_X(v) \cap \mathcal{A}^L_{\sigma,s}$ and $E^R_X(v) \cap \mathcal{A}^R_{\sigma,p}$.

The following proposition then directly follows from Equation 2.

Proposition 12. *Let X be a shift space over $\mathcal{A}$, $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ a return morphism for the word w, $Y = \sigma \cdot X$ and $u \in \mathcal{L}(Y)$ such that $|u|_w > 0$. If (s, v, p) is the triplet given by Proposition 8, then the extension graph of u in Y is the image of the graph $\mathcal{E}_{X,s,p}(v)$ by the morphisms $\varphi^L_{\sigma,s}$ acting on the left vertices and $\varphi^R_{\sigma,p}$ acting on the right vertices.*

Remark that, as a consequence, $E^L_Y(u) \subset \varphi^L_{\sigma,s}(E^L_X(v))$ and $E^R_Y(u) \subset \varphi^R_{\sigma,p}(E^R_X(v))$. In particular, if u is left (resp., right) special, v is also left (resp., right) special and there exist two distinct letters $a, b \in \mathcal{A}$ and two distinct letters $a', b' \in \mathcal{B}$ such that

$$\sigma(a) \in \mathcal{B}^* a's, \quad \sigma(b) \in \mathcal{B}^* b's$$
$$(\text{resp., } \sigma(a)w \in pa'\mathcal{B}^*, \quad \sigma(b)w \in pb'\mathcal{B}^*).$$

This observation motivates the following definition.

Definition 13. Let $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ be a return morphism for the word w. For two distinct letters $a, b \in \mathcal{A}$, we denote by $s_\sigma(a, b)$ the longest common suffix of $\sigma(a)$ and $\sigma(b)$ and by $p_\sigma(a, b)$ the longest common prefix of $\sigma(a)w$ and $\sigma(b)w$. Finally, we define the two sets

$$\mathcal{T}^L(\sigma) = \{s_\sigma(a, b) \mid a, b \in \mathcal{A}, a \neq b\}$$

and

$$\mathcal{T}^R(\sigma) = \{p_\sigma(a, b) \mid a, b \in \mathcal{A}, a \neq b\}.$$

The following result is [28, Proposition 4.8].

Proposition 14. *Let X be a shift space over $\mathcal{A}$, $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ a return morphism, $Y = \sigma \cdot X$ and $v \in \mathcal{L}(X)$ a dendric factor. All the extended images of v under σ are dendric in Y if and only if the following conditions are satisfied*

(1) *for every $s \in \mathcal{T}^L(\sigma)$, $\mathcal{E}_{X,s,\varepsilon}(v)$ is connected;*
(2) *for every $p \in \mathcal{T}^R(\sigma)$, $\mathcal{E}_{X,\varepsilon,p}(v)$ is connected.*

Remark 15. Recall that a word $v \in \mathcal{L}(X)$ is ordinary if there exist $a, b \in \mathcal{A}$ such that $E_X(v) = (E^L_X(v) \times \{b\}) \cup (\{a\} \times E^R_X(v))$. In this case, for all $s \in \mathcal{T}^L(\sigma)$ (resp., $p \in \mathcal{T}^R(\sigma)$), the graph $\mathcal{E}_{X,s,\varepsilon}(v)$ (resp., $\mathcal{E}_{X,\varepsilon,p}(v)$) is connected (or empty). Thus v only has dendric extended images.

As a direct consequence of Propositions 8 and 14, we can determine when the image of a dendric shift under a return morphism is dendric.

Proposition 16. *Let X be a dendric shift over $\mathcal{A}$ and $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ be a return morphism. The shift space $Y = \sigma \cdot X$ is dendric if and only if σ is dendric and, for all $v \in \mathcal{L}(X)$, for all $s \in \mathcal{T}^L(\sigma)$ and for all $p \in \mathcal{T}^R(\sigma)$, the graphs $\mathcal{E}_{X,s,\varepsilon}(v)$ and $\mathcal{E}_{X,\varepsilon,p}(v)$ are connected.*

In particular, by Remark 15, the image of an Arnoux-Rauzy shift under a return morphism is dendric if and only if the morphism is dendric.

In the case of an eventually dendric shift, we recall the following characterization.

Proposition 17 (Dolce and Perrin [17])**.** *Let X be a shift space over $\mathcal{A}$. The following properties are equivalent.*

(1) *The shift X is eventually dendric.*
(2) *There exists $N \in \mathbb{N}$ such that for any left special word $v \in \mathcal{L}_{\geq N}(X)$, there exists a unique letter $a \in \mathcal{A}$ such that va is left special. Moreover, $E^L_X(va) = E^L_X(v)$.*

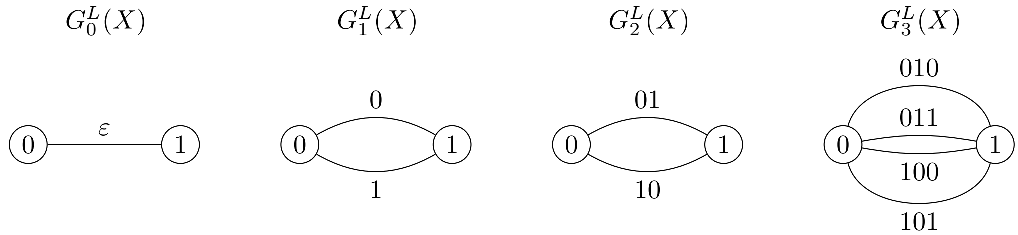


FIGURE 2. Graphs $G_n^L(X)$ for $n \in \{0, \ldots, 3\}$.

(3) *There exists $N \in \mathbb{N}$ such that for any right special word $v \in \mathcal{L}_{\geq N}(X)$, there exists a unique letter $a \in \mathcal{A}$ such that av is right special. Moreover, $E_X^R(av) = E_X^R(v)$.*

Corollary 18. *A shift space X is eventually dendric if and only if any long enough bispecial factor is ordinary.*

Thus, the image of an eventually dendric shift under a return morphism is also an eventually dendric shift, as stated in the following proposition.

Proposition 19. *Let X be an eventually dendric shift over $\mathcal{A}$ and $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ a return morphism. The shift space $Y = \sigma \cdot X$ is an eventually dendric shift.*

Proof. Let N be an integer satisfying conditions 2 and 3 in Proposition 17. Let w be a word such that σ is a return morphism for w and let $K = \max_{a \in \mathcal{A}} |\sigma(a)|$. Every word u in $\mathcal{L}(Y)$ of length at least $2K + |w|$ contains an occurrence of w and thus, using Proposition 8, has an antecedent $v \in \mathcal{L}(X)$. Furthermore, since u is of the form $s\sigma(v)p$, one has $|v| \geq \frac{|u|-2K-|w|}{K}$. Therefore, there exists $M \in \mathbb{N}$ such that, for each $u \in \mathcal{L}_{\geq M}(Y)$, the antecedent v of u is of length at least N. By Remark 15 and Proposition 17, v then only has dendric extended images thus u is dendric. □

4. OTHER GRAPHS CHARACTERIZING DENDRICITY

For any shift space, we introduce graphs which describe all the sets of left or right extensions for factors of length n in the shift space.

Definition 20. Let X be a shift space over $\mathcal{A}$ and $n \geq 0$. The graph $G_n^L(X)$ (resp., $G_n^R(X)$) is the multi-graph with labeled edges such that

- its vertices are the elements of $\mathcal{A}$,
- for any $v \in \mathcal{L}_n(X)$ and any distinct $a, b \in E_X^L(v)$ (resp., $a, b \in E_X^R(v)$) there is an (undirected) edge labeled by v between the vertices a and b.

Example 21. Let X be the Thue-Morse shift space over $\{0, 1\}$. The first few graphs $G_n^L(X)$ are given in Figure 2.

Remark 22. By definition, the edges are only labeled by left (resp., right) special factors and the edges with a given label v form a complete subgraph of $G_n^L(X)$ (resp., $G_n^R(X)$). In addition, if $v = v_1 \cdots v_n$ labels an edge between a and b in $G_n^L(X)$ (resp., $G_n^R(X)$), then $v_1 \cdots v_k$ (resp., $v_{n-k+1} \cdots v_n$) labels an edge between a and b in $G_k^L(X)$ (resp., $G_k^R(X)$) for all $k \leq n$.

We now explain the link between the properties of the extension graphs in X and of the graphs $G_n^L(X)$, $G_n^R(X)$ regarding both acyclicity and connectedness. We first need to define a notion of acyclicity for multi-graph with labeled edges.

Definition 23. A multi-graph with labeled edges G is *acyclic for the labeling* if any simple cycle in G only uses edges with the same label.

The following lemma is a direct consequence of the definition of the graphs $G_{n+1}^L(X)$ and $G_{n+1}^R(X)$.

Lemma 24. *Let X be a shift space and $v \in \mathcal{L}_n(X)$. The graph $\mathcal{E}_X(v)$ contains the path $(a_1^L, b_1^R, a_2^L, \dots, b_k^R, a_{k+1}^L)$ if and only if $G_{n+1}^L(X)$ contains the path*

$$a_1 \xrightarrow{vb_1} a_2 \dots \xrightarrow{vb_k} a_{k+1}.$$

Symmetrically, the graph $\mathcal{E}_X(v)$ contains the path $(a_1^R, b_1^L, a_2^R, \dots, b_k^L, a_{k+1}^R)$ if and only if $G_{n+1}^R(X)$ contains the path

$$a_1 \xrightarrow{b_1 v} a_2 \dots \xrightarrow{b_k v} a_{k+1}.$$

Proposition 25. *Let X be a shift space and $N \geq 0$. The following properties are equivalent.*

1. *The graph $\mathcal{E}_X(v)$ is acyclic for all $v \in \mathcal{L}_{<N}(X)$.*
2. *The graph $G_n^L(X)$ (resp. $G_n^R(X)$) is acyclic for the labeling for all $n \leq N$.*

Proof. Let us show the equivalence with the acyclicity of the graphs $G_n^L(X)$ by contraposition. The result with the graphs $G_n^R(X)$ is symmetric. If $v \in \mathcal{L}_n(X)$, $n < N$, is such that $\mathcal{E}_X(v)$ contains a (non-trivial) cycle, then $G_{n+1}^L(X)$ contains a cycle by Lemma 24 and the edges of this cycle do not have the same label. For the converse, assume that $G_n^L(X)$, $n \leq N$, contains a simple cycle

$$a_1 \xrightarrow{u^{(1)}} a_2 \dots \xrightarrow{u^{(k)}} a_1,$$

with at least two distinct labels. Let v be the longest common prefix to all the $u^{(i)}$, $i \leq k$, and, for all $i \leq k$, let b_i be the letter such that vb_i is a prefix of $u^{(i)}$. By Remark 22, the graph $G_{|v|+1}^L(X)$ thus contains the cycle

$$a_1 \xrightarrow{vb_1} a_2 \dots \xrightarrow{vb_k} a_1.$$

By definition of v, there exist $i < j$ such that $b_i \neq b_j$. We may assume that $b_1 \neq b_k$ because, if $b_1 = b_k$, since the subgraph of $G_{|v|+1}^L(X)$ generated by the edges labeled by vb_k is a complete graph, we can consider the path

$$a_2 \xrightarrow{vb_2} a_3 \dots \xrightarrow{vb_k} a_2$$

instead.

By Lemma 24, the graph $\mathcal{E}_X(v)$ contains the cycle $(a_1^L, b_1^R, a_2^L, \dots, b_k^R, a_1^L)$ which is non-trivial as $b_1 \neq b_k$. □

We now look at the connectedness properties.

Proposition 26. *Let X be a shift, $N \geq 0$ and $C \subset \mathcal{A}$. If the graph $\mathcal{E}_X(v)$ is acyclic for all $v \in \mathcal{L}_{<N}(X)$, then the following properties are equivalent.*

1. *For all $v \in \mathcal{L}_{<N}(X)$ and all $a, b \in E_X^L(v) \cap C$, a^L and b^L are connected in $\mathcal{E}_X(v)$ by a path avoiding vertices c^L, $c \notin C$.*
2. *For all $n \leq N$, the subgraph of $G_n^L(X)$ generated by the vertices in C is connected.*
3. *The subgraph of $G_N^L(X)$ generated by the vertices in C is connected.*

Similarly, the following are equivalent.

1. *For all $v \in \mathcal{L}_{<N}(X)$ and all $a, b \in E_X^R(v) \cap C$, a^R and b^R are connected in $\mathcal{E}_X(v)$ by a path avoiding vertices c^R, $c \notin C$.*
2. *For all $n \leq N$, the subgraph of $G_n^R(X)$ generated by the vertices in C is connected.*
3. *The subgraph of $G_N^R(X)$ generated by the vertices in C is connected.*

Proof. Let us show the first set of equivalences. We denote by H_n the subgraph of $G_n^L(X)$ generated by the vertices in C. Assume that the first property is satisfied. We prove that H_n is connected by induction on n. The graph $G_0^L(X)$ is a complete graph thus any subgraph is connected. If H_n is connected, then, to prove that H_{n+1} is also connected, it suffices to show that any two vertices $a, b \in C$ that were connected by an edge in H_n are connected by a path in H_{n+1}. Let $a, b \in C$ be two such vertices and v be the label of an edge between them. As a and b are left extensions of v, a^L and b^L are connected in $\mathcal{E}_X(v)$ by a path avoiding vertices c^L, $c \notin C$. The vertices a and b are thus connected by a path in H_{n+1} by Lemma 24. Notice that the acyclic hypothesis is not needed for this implication.

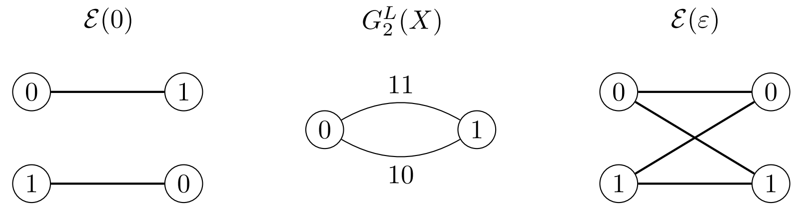


FIGURE 3. The acyclicity of $\mathcal{E}_X(v)$ is necessary in Proposition 27.

For the converse, let us proceed by contraposition. Let $v \in \mathcal{L}_n(X)$, $n < N$, be such that there exist two letters $a, b \in E_X^L(v) \cap C$ such that the vertices a^L and b^L are not connected in $\mathcal{E}_X(v)$ by a path avoiding vertices c^L, $c \notin C$. By definition, there is an edge labeled by v between a and b in $G_n^L(X)$ thus, as this graph is acyclic for the labeling by Proposition 25, any simple path between a and b in $G_n^L(X)$ only uses edges labeled by v. By Remark 22, any simple path between a and b in $G_{n+1}^L(X)$ uses edges labeled by elements of $v\mathcal{A}$. This is also true for the paths in H_{n+1}. However, using Lemma 24, the existence of such path in H_{n+1} would imply that a^L and b^L are connected in $\mathcal{E}_X(v)$ by a path avoiding vertices c^L, $c \notin C$. Thus H_{n+1} is not connected.

The equivalence between the second and the third properties follows from Remark 22 as, with any path in H_N, we can associate a path in H_n for $n \leq N$. □

In particular, when $C = \mathcal{A}$, Proposition 26 can be rewritten as follows.

Proposition 27. *Let X be a shift space and $N \geq 0$. If the graph $\mathcal{E}_X(v)$ is acyclic for all $v \in \mathcal{L}_{<N}(X)$, then the following properties are equivalent.*

1. *The graph $\mathcal{E}_X(v)$ is connected for all $v \in \mathcal{L}_{<N}(X)$.*
2. *The graph $G_n^L(X)$ (resp., $G_n^R(X)$) is connected for all $n \leq N$.*
3. *The graph $G_N^L(X)$ (resp., $G_N^R(X)$) is connected.*

Remark 28. The result is false if we remove the acyclic hypothesis. Indeed, if X is such that $\mathcal{L}_3(X) = \{001, 010, 011, 100, 110, 111\}$, then the graph $\mathcal{E}_X(0)$ is not connected. However, $G_2^L(X)$ is and, indeed, the graph $\mathcal{E}_X(\varepsilon)$ contains a cycle (see Figure 3) thus this does not contradict the previous result.

As a direct consequence of Proposition 25 and Proposition 27, we obtain the following characterization of dendric shifts.

Corollary 29. *A shift space X is dendric if and only if, for all $n \in \mathbb{N}$, the graph $G_n^L(X)$ (resp., $G_n^R(X)$) is acyclic for the labeling and connected.*

5. STABILIZATION AND EVENTUALLY DENDRIC SHIFTS

In the rest of the paper, we will be interested in whether two edges have the same label or not more than in the actual label of the edges. We will thus identify two multi-graphs G and G' with edges labeled by elements of C and C' respectively if they both have the same set of vertices and if there exists a bijection $\varphi : C \to C'$ such that there are k edges labeled by $c \in C$ between a and b in G if and only if there are k edges labeled by $\varphi(c)$ between a and b in G'.

To make the distinction clearer, we will talk about multi-graphs with *colored* edges instead of labeled edges. We can naturally extend the notion of acyclicity for the labeling to acyclicity for the coloring.

Among the multi-graphs with colored edges, we will mainly be interested in graphs that can correspond to $G_n^L(X)$ (or $G_n^R(X)$) for some shift space X. These graphs are exactly the ones that can be constructed as follows.

Definition 30. A multi-graph G with colored edges and with set of vertices V is *multi-clique* if there exist subsets $C_1, \dots, C_k$ of V such that the set of edges is the union of the sets of colored edges $((C_i \times C_i) \setminus \operatorname{diag}(C_i)) \times \{c_i\}$, $i \leq k$, where c_i is the color of the edges and $c_1, \dots, c_k$ are distinct colors. The multi-clique G is then denoted $G(\{C_1, \dots, C_k\})$.

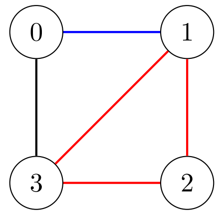


Figure 4. Multi-clique $G(\{\{0,1\},\{1,2,3\},\{0,3\}\})$

Example 31. If $C = \{0,1,2,3\}$, $C_1 = \{0,1\}$, $C_2 = \{1,2,3\}$, $C_3 = \{0,3\}$, the multi-clique $G(\{C_1, C_2, C_3\})$ is represented in Figure 4. It is not acyclic from the coloring because of the cycle $0, 1, 3, 0$.

With this definition, the colored version of $G^L_n(X)$ is the graph $G(\{E^L_X(v) : v \in \mathcal{L}_n(X)\})$. Remark that if $\mathrm{Card}(C_i) \leq 1$, then $G(\{C_1,\dots,C_k\}) = G(\{C_1,\dots,C_k\} \setminus \{C_i\})$ and, in particular, this shows once again that it suffices to consider the left-special factors to build $G^L_n(X)$.

In the case where the shift space X is eventually dendric, the graphs $G^L_n(X)$ (resp., $G^R_n(X)$) are eventually constant, when seen as multi-graphs with colored edges. This is stated in the following result.

Proposition 32. *Let X be an eventually dendric shift. There exists $N \in \mathbb{N}$ such that, for all $n \geq N$,*

$$G(\{E^L_X(v) \mid v \in \mathcal{L}_n(X)\}) = G(\{E^L_X(v) \mid v \in \mathcal{L}_N(X)\})$$
$$\textit{(resp., } G(\{E^R_X(v) \mid v \in \mathcal{L}_n(X)\}) = G(\{E^R_X(v) \mid v \in \mathcal{L}_N(X)\})\textit{).}$$

Proof. It suffices to take N satisfying condition 2 (resp., condition 3) of Proposition 17. □

The previous result is not an equivalence as there exist non eventually dendric shift spaces X whose graphs $G^L_n(X)$ and $G^R_n(X)$ are eventually constant, when seen as multi-graphs with colored edges. This is the case of the following example.

Example 33. The Chacon ternary shift space over $\{0,1,2\}$ is generated by the morphism $\sigma\colon 0 \mapsto 0012, 1 \mapsto 12, 2 \mapsto 012$. It is of complexity $2n+1$ [27] thus, by Proposition 2, for each length n, there is either a unique left special word with three extensions or two left special words with two extensions each. The graph $G^L_n(X)$ then contains three or two edges respectively. Since the graph $G^L_1(X)$ only contains two edges between 0 and 2 (labeled by 0 and 1), using Remark 22, it will be the case of $G^L_n(X)$ for all $n \geq 1$. We can similarly show that the graphs $G^R_n(X)$ correspond to the same colored graph for all $n \geq 1$. However, the Chacon shift space is not eventually dendric [17].

We can however show that, if X is eventually connected, i.e. the extension graph of any long enough factor is connected, and if the graphs $G^L_n(X)$ or $G^R_n(X)$ are eventually constant, then X is eventually dendric.

Definition 34. Let X be an eventually dendric shift. The graph $G^L(X)$ (resp., $G^R(X)$) is the graph $G(\{E^L_X(v) \mid v \in \mathcal{L}_N(X)\})$ (resp., $G(\{E^R_X(v) \mid v \in \mathcal{L}_N(X)\})$) where N is as in Proposition 32.

Remark 35. Another way to see the graph $G^L(X)$ is to consider the right infinite words and their left extensions. More precisely, for a right infinite word x, we denote by $E^L_X(x)$ the set of letters a such that $\mathcal{L}(ax) \subset \mathcal{L}(X)$. We then have

$$G^L(X) = G(\{E^R_X(x) \mid x \in \mathcal{A}^{\mathbb{N}}\}).$$

This is also related to the notion of asymptotic equivalence. Indeed, each edge of $G^L(X)$ corresponds to a (right) asymptotic pair, i.e. two elements $x, y \in X$ such that $x_n = y_n$ for each $n \geq 0$ and $x_{-1} \neq y_{-1}$.

By Corollary 29, if X is a dendric shift, then $G^L(X)$ and $G^R(X)$ are acyclic for the coloring and connected. However, the converse is not true, as can be seen in the following example.

Example 36. Let σ be the morphism defined by $\sigma(0) = 0110$ and $\sigma(1) = 011$. It is a return morphism for 01. Let X be the image under σ of a Sturmian shift. It is not a dendric shift space since the graph $\mathcal{E}_X(\varepsilon)$ contains a cycle but, by Remark 15, X is eventually dendric with threshold at most 4. We can then show that the only words which are not dendric are ε and 1 thus $G^L(X) = G^L_2(X)$ and $G^R(X) = G^R_2(X)$. These graphs are acyclic for the coloring and connected since 01 (resp., 10) is the only left (resp., right) special word of length 2.

6. Action of morphisms on $G^L(X)$ and $G^R(X)$

In this section, we use the graphs $G^L(X)$ and $G^R(X)$ to study the action of a return morphism on a dendric shift X. We first show that these graphs contain all the information needed to know whether the image Y of X under a return morphism is dendric. This gives us a simpler formulation for Proposition 16. We then explain how to use these graphs to obtain $G^L(Y)$ and $G^R(Y)$.

Definition 37. Let G be a multi-graph (with colored or uncolored edges) with set of vertices $\mathcal{A}$ and $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ be a return morphism for w. For all $s \in \mathcal{T}^L(\sigma)$, we define the graph $G^L_{\sigma,s}$ as the subgraph of G generated by the vertices in $\mathcal{A}^L_{\sigma,s}$. Similarly, for all $p \in \mathcal{T}^R(\sigma)$, the graph $G^R_{\sigma,p}$ is the subgraph of G generated by the vertices in $\mathcal{A}^R_{\sigma,p}$.

If X is an eventually dendric shift over $\mathcal{A}$, we will write $G^L_{\sigma,s}(X)$ instead of $(G^L(X))^L_{\sigma,s}$ and $G^R_{\sigma,s}(X)$ instead of $(G^R(X))^R_{\sigma,s}$.

Proposition 38. *Let X be a dendric shift space and σ a return morphism. The image Y of X under σ is dendric if and only if σ is dendric and for all $s \in \mathcal{T}^L(\sigma)$ and all $p \in \mathcal{T}^R(\sigma)$, the graphs $G^L_{\sigma,s}(X)$ and $G^R_{\sigma,p}(X)$ are connected.*

Proof. As $G^L_n(X) = G^L(X)$ for all large enough n, the graphs $\mathcal{E}_{X,s,\varepsilon}(v)$ and $\mathcal{E}_{X,\varepsilon,p}(v)$ are connected for all $v \in \mathcal{L}(X)$ if and only if $G^L_{\sigma,s}(X)$ and $G^R_{\sigma,p}(X)$ are connected. Indeed, it is a direct consequence of Proposition 26 with $C = \mathcal{A}^L_{\sigma,s}$ and $C = \mathcal{A}^R_{\sigma,p}$ respectively. The conclusion then follows from Proposition 16. □

Definition 39. Let G be the multi-clique $G(\{C_1, \ldots, C_k\})$ with set of vertices $\mathcal{A}$ and $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ be a return morphism. The *left image* of G by σ is the multi-clique

$$\sigma^L(G) = G(\{\varphi^L_{\sigma,s}(C_i) \mid i \le k, s \in \mathcal{T}^L(\sigma)\})$$

where the set of vertices is $\mathcal{B}$ and the *right image* of G by σ is the multi-clique

$$\sigma^R(G) = G(\{\varphi^R_{\sigma,p}(C_i) \mid i \le k, p \in \mathcal{T}^R(\sigma)\})$$

where the set of vertices is $\mathcal{B}$.

Example 40. Let $\beta : 0 \mapsto 0, 1 \mapsto 01, 2 \mapsto 02, 3 \mapsto 032$ be the morphism of Example 10. We have $\mathcal{T}^L(\beta) = \{\varepsilon, 2\}$ and the associated partial maps are given by

$$\varphi^L_{\beta,\varepsilon} : \begin{cases} 0 & \mapsto 0 \\ 1 & \mapsto 1 \\ 2,3 & \mapsto 2 \end{cases} \quad \text{and} \quad \varphi^L_{\beta,2} : \begin{cases} 2 \mapsto 0 \\ 3 \mapsto 3. \end{cases}$$

If $G = G(\{\{0,1\},\{1,2,3\}\})$, then we have

$$\begin{aligned} \beta^L(G) &= G\left(\{\varphi^L_{\beta,\varepsilon}(\{0,1\}), \varphi^L_{\beta,\varepsilon}(\{1,2,3\}), \varphi^L_{\beta,2}(\{0,1\}), \varphi^L_{\beta,2}(\{1,2,3\})\}\right) \\ &= G(\{\{0,1\},\{1,2\},\{0,3\}\}). \end{aligned}$$

The graphs are represented in Figure 5.

Proposition 41. *Let X be an eventually dendric shift and σ a return morphism for w. If $Y = \sigma \cdot X$, then $G^L(Y) = \sigma^L(G^L(X))$ and $G^R(Y) = \sigma^R(G^R(X))$.*

Proof. Let us prove the link between $G^L(X)$ and $G^L(Y)$. By Proposition 19, Y is also eventually dendric. If we write $m = \max\{|\sigma(a)| : a \in \mathcal{A}\}$, let $N \in \mathbb{N}$ (resp., $M \in \mathbb{N}$) be large enough so that N (resp., $M - m$) satisfies the condition 2 of Proposition 17 for the shift X (resp., Y). Let us also assume that the antecedent of any word $u \in \mathcal{L}(Y)$ of length at least M is of length at least N.

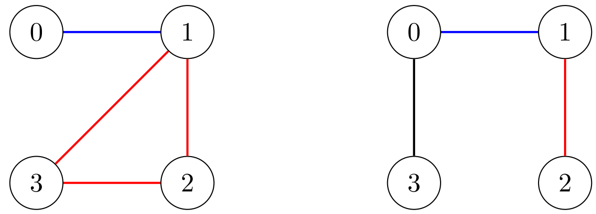


FIGURE 5. Graphs $G = G(\{\{0,1\},\{1,2\},\{0,3\}\})$ (on the left) and $\beta^L(G)$ (on the right)

By Proposition 32, $G^L(X)$ (resp., $G^L(Y)$) is constructed with the left special factors of length N (resp., M).

We define the application f mapping any left special $u \in \mathcal{L}_M(Y)$ to the pair $(s_u, v) \in \mathcal{T}^L(\sigma) \times \mathcal{L}_N(X)$ where (s_u, v_u, p_u) is the triplet associated with u by Proposition 8 and v is the prefix of length N of v_u.

We will prove that f is a bijection between the left special factors of length M in Y and the set

$$D = \{(s,v) \in \mathcal{T}^L(\sigma) \times \mathcal{L}_N(X) \mid \mathrm{Card}(\varphi^L_{\sigma,s}(E^L_X(v))) \geq 2\}.$$

We will also show that, if $(s,v) = f(u)$, then

$$E^L_Y(u) = \varphi^L_{\sigma,s}(E^L_X(v))$$

which will allow us to conclude since

$$\sigma^L(G^L(X)) = G(\{\varphi^L_{\sigma,s}(E^L_X(v)) \mid (s,v) \in D\}).$$

Note that, if $f(u) = (s,v)$, then $s\sigma(v)w$ is a prefix of u, thus, by Proposition 8,

$$E^L_Y(u) \subset E^L_Y(s\sigma(v)w) = \varphi^L_{\sigma,s}(E^L_X(v))$$

and (s,v) is in D.

The application f is injective. Indeed, if $f(u) = f(u') = (s,v)$, then $s_u = s = s_{u'}$ and v is a prefix of both v_u and $v_{u'}$. However, as u and u' are left special, v_u and $v_{u'}$ must be left special. As they have a common prefix of length N and N is chosen large enough, it means that one is prefix of the other. Let us assume that v_u is a prefix of $v_{u'}$. Then $s\sigma(v_u)w$ is a prefix of both u and u'. Since u is of length M, $s\sigma(v_u)w$ is a left special factor of length at least $M - m$ and, as M is large enough, it is prefix of a unique left special factor of length M. This proves that $u = u'$ and that f is injective.

We now prove the surjectivity. For any $(s,v) \in D$, by definition of N, there exists $v' \in v\mathcal{A}^*$ left special such that $|s\sigma(v')w| \geq M$ and $E^L_X(v') = E^L_X(v)$. If u is the prefix of length M of $s\sigma(v')w$, then $s_u = s$ and v_u is a prefix of v' of length at least N. Thus $s\sigma(v)w$ is a prefix of u. In addition, by Proposition 12,

$$E^L_Y(u) \supset E^L_Y(s\sigma(v')w) = \varphi^L_{\sigma,s}(E^L_X(v')) = \varphi^L_{\sigma,s}(E^L_X(v)).$$

In particular, u is left special and $f(u) = (s,v)$. This proves that f is surjective. Moreover, we have both inclusions thus $E^L_Y(u) = \varphi^L_{\sigma,s}(E^L_X(v))$. □

7. S-adic characterization of minimal dendric shifts

We recall the notion of S-adic representation and provide an S-adic characterization of both the dendric and the eventually dendric minimal shift spaces.

7.1. S-adic representations.

Let $\boldsymbol{\sigma} = (\sigma_n : \mathcal{A}^*_{n+1} \to \mathcal{A}^*_n)_{n\geq 0}$ be a sequence of morphisms such that $\max_{a\in\mathcal{A}_n} |\sigma_0 \circ \cdots \circ \sigma_{n-1}(a)|$ goes to infinity when n increases. For $n < N$, we define the morphism $\sigma_{[n,N)} = \sigma_n \circ \sigma_{n+1} \circ \cdots \circ \sigma_{N-1}$. The *language $\mathcal{L}^{(n)}(\boldsymbol{\sigma})$ of level n associated with $\boldsymbol{\sigma}$* is defined by

$$\mathcal{L}^{(n)}(\boldsymbol{\sigma}) = \left\{ w \in \mathcal{A}^*_n \mid w \text{ occurs in } \sigma_{[n,N)}(a) \text{ for some } a \in \mathcal{A}_N \text{ and } N > n \right\}.$$

As $\max_{a \in \mathcal{A}_N} |\sigma_{[n,N)}(a)|$ goes to infinity when N increases, $\mathcal{L}^{(n)}(\boldsymbol{\sigma})$ defines a non-empty shift space $X_{\boldsymbol{\sigma}}^{(n)}$. More precisely, $X_{\boldsymbol{\sigma}}^{(n)}$ is the set of points $x \in \mathcal{A}_n^{\mathbb{Z}}$ such that $\mathcal{L}(x) \subset \mathcal{L}^{(n)}(\boldsymbol{\sigma})$. Note that it may happen that $\mathcal{L}(X_{\boldsymbol{\sigma}}^{(n)})$ is strictly contained in $\mathcal{L}^{(n)}(\boldsymbol{\sigma})$. Also observe that for all n, $X_{\boldsymbol{\sigma}}^{(n)}$ is the image of $X_{\boldsymbol{\sigma}}^{(n+1)}$ under σ_n.

We set $X_{\boldsymbol{\sigma}} = X_{\boldsymbol{\sigma}}^{(0)}$ and call $\boldsymbol{\sigma}$ an *S-adic representation* of $X_{\boldsymbol{\sigma}}$.

We say that the sequence $\boldsymbol{\sigma}$ is *primitive* if, for any n, there exists $N > n$ such that for all $(a, b) \in \mathcal{A}_n \times \mathcal{A}_N$, a occurs in $\sigma_{[n,N)}(b)$. Observe that if $\boldsymbol{\sigma}$ is primitive, then $\min_{a \in \mathcal{A}_n} |\sigma_{[0,n)}(a)|$ goes to infinity when n increases, $\mathcal{L}(X_{\boldsymbol{\sigma}}^{(n)}) = \mathcal{L}^{(n)}(\boldsymbol{\sigma})$, and $X_{\boldsymbol{\sigma}}^{(n)}$ is a minimal shift space (see for instance [20, Lemma 7]).

Given a minimal shift space X over the alphabet $\mathcal{A}$, we can build specific S-adic representations using return words. Let $w \in \mathcal{L}(X)$ be a non-empty word. Recall that a return word to w in X is a non-empty word r such that $rw \in \mathcal{L}(X)$ and rw contains exactly two occurrences of w, one as a prefix and one as a suffix.

We let $\mathcal{R}_X(w)$ denote the set of return words to w in X and we omit the subscript X whenever it is clear from the context. The shift space X being minimal, $\mathcal{R}(w)$ is always finite thus we write $R_X(w) = \{1, \dots, \mathrm{Card}(\mathcal{R}_X(w))\}$. A morphism $\sigma : R(w)^* \to \mathcal{A}^*$ is a *coding morphism* associated with w if $\sigma(R(w)) = \mathcal{R}(w)$. It is trivially a return morphism for w.

Let us consider the set $\mathcal{D}_w(X) = \{x \in R(w)^{\mathbb{Z}} \mid \sigma(x) \in X\}$. It is a minimal shift space, called the *derived shift of X (with respect to w)*. We now show that derivation of minimal shift spaces allows to build primitive $\mathcal{S}$-adic representations with return morphisms. We inductively define the sequences $(w_n)_{n \geq 0}$, $(R_n)_{n \geq 0}$, $(X_n)_{n \geq 0}$ and $(\sigma_n)_{n \geq 0}$ by

- $X_0 = X$, $R_0 = \mathcal{A}$ and $w_0 \in \mathcal{L}(X) \setminus \{\varepsilon\}$;
- for all n, $R_{n+1} = R_{X_n}(w_n)$, $\sigma_n : R_{n+1}^* \to R_n^*$ is a coding morphism associated with w_n, $X_{n+1} = \mathcal{D}_{w_n}(X_n)$ and $w_{n+1} \in \mathcal{L}(X_{n+1}) \setminus \{\varepsilon\}$.

Observe that the sequence $(w_n)_{n \geq 0}$ is not uniquely defined, as well as the morphism σ_n (even if w_n is fixed). However, to avoid heavy considerations when we deal with sequences of morphisms obtained in this way, we will speak about "the" sequence $(\sigma_n)_{n \geq 0}$ and it is understood that we may consider any such sequence.

Theorem 42 (Durand [19])**.** *Let X be a minimal shift space. Using the notation defined above, the sequence of morphisms $\boldsymbol{\sigma} = (\sigma_n : R_{n+1}^* \to R_n^*)_{n \geq 0}$ is a primitive S-adic representation of X using return morphisms. In particular, for all n, we have $X_n = X_{\boldsymbol{\sigma}}^{(n)}$.*

In the case of minimal dendric shifts, we have stronger properties for the S-adic representation $\boldsymbol{\sigma}$. Recall that if $F_{\mathcal{A}}$ is the free group generated by $\mathcal{A}$, an automorphism α of $F_{\mathcal{A}}$ is *tame* if it belongs to the monoid generated by the permutations of $\mathcal{A}$ and by the elementary automorphisms

$$\begin{cases} a \mapsto ab, & \\ c \mapsto c, & \text{for } c \neq a, \end{cases} \quad \text{and} \quad \begin{cases} a \mapsto ba, & \\ c \mapsto c, & \text{for } c \neq a. \end{cases}$$

Theorem 43 (Berthé et al. [9])**.** *Let X be a minimal dendric shift over the alphabet $\mathcal{A} = \{1, \dots, d\}$. For any $w \in \mathcal{L}(X)$, $\mathcal{D}_w(X)$ is a minimal dendric shift over $\mathcal{A}$ and the coding morphism associated with w is a tame automorphism of $F_{\mathcal{A}}$. As a consequence, if $\boldsymbol{\sigma} = (\sigma_n)_{n \geq 0}$ is the primitive directive sequence of Theorem 42, then all morphisms σ_n are tame dendric return morphisms.*

The previous result is true for any sequence $\boldsymbol{\sigma}$ build as above. However, some choices for the words w_n give additional properties to the morphisms σ_n.

In particular, if w_n is a letter, the morphism σ_n is *strongly left proper*, i.e. the image of each letter begins with the letter w_n which does not appear elsewhere. Moreover, we can assume that w_n is a bispecial letter, as shown in the following results. This implies that $w_n \in \mathcal{T}^R(\sigma_n)$ and $\varepsilon \in \mathcal{T}^L(\sigma_n)$.

Lemma 44. *Let X be a minimal shift space over $\mathcal{A}$. If there exist letters $a_1, \dots, a_k$ such that a_{i+1} is the only right (resp., left) extension of a_i for all $i < k$, and a_1 is the only right (resp.,* *left) extension of a_k, then $\mathcal{A} = \{a_1, \dots, a_k\}$. In particular, X does not have any right (resp., left) special letter.*

Proposition 45. *Any minimal shift space X such that the graph $\mathcal{E}_X(\varepsilon)$ is connected has a bispecial letter.*

Proof. First, notice that, because $\mathcal{E}_X(\varepsilon)$ is connected, there is always a left and a right special letter. Let a_1 be a right special letter. If it is bispecial, we can conclude. Otherwise, let a_2 be its unique left extension. Using Lemma 44, we know that $a_2 \neq a_1$. In the graph $\mathcal{E}_X(\varepsilon)$, a_2^L is then the only neighbour of the vertex a_1^R but this graph is connected thus the vertex a_2^L is of degree at least two and a_2 is a right special letter. Once again, if it is left special, we have found a bispecial letter and otherwise it has a unique left extension a_3. By Lemma 44, we must have $a_3 \neq a_1$ and $a_3 \neq a_2$. We can iterate the process to define $a_4, a_5, \dots$. However, since the alphabet is finite, it stops at some point, meaning that we have found a bispecial letter. □

7.2. **Characterization via two graphs.** Similarly to what was done in the case of a ternary alphabet in [28], using the results of Section 6, we can give an S-adic characterization of the minimal dendric shifts on an alphabet $\mathcal{A}$. This characterization uses two graphs corresponding to the left and the right extensions respectively.

The edges of these graphs are given by the left (resp., right) valid triplets, as defined below. Note that we only assume that σ is a return morphism and not a dendric return morphism, even if for the S-adic characterization (Theorem 1), we will restrict ourselves to dendric return morphisms.

Definition 46. Let $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ be a return morphism. The triplet (G', σ, G) is *left (resp., right) valid* if the following conditions are satisfied

1. G is an acyclic for the coloring and connected multi-clique;
2. for all $s \in \mathcal{T}^L(\sigma)$, $G^L_{\sigma,s}$ is connected (resp., for all $p \in \mathcal{T}^R(\sigma)$, $G^R_{\sigma,p}$ is connected);
3. $G' = \sigma^L(G)$ (resp., $G' = \sigma^R(G)$) is an acyclic for the coloring and connected multi-clique.

Note that, if $s \in \mathcal{T}^L(\sigma)$ (resp., $p \in \mathcal{T}^R(\sigma)$) is of minimal length, then $G^L_{\sigma,s} = G$ (resp., $G^R_{\sigma,p} = G$) thus the second item of Definition 46 implies that G is connected. Remark also that, if X is dendric, then item 1 is satisfied for $(G^L(\sigma \cdot X), \sigma, G^L(X))$ and $(G^R(\sigma \cdot X), \sigma, G^R(X))$. Moreover, if σ is a dendric return morphism, then $\sigma \cdot X$ is dendric if and only if item 2 is satisfied for both triplets by Proposition 38, and in that case, item 3 is also satisfied by Proposition 41.

Example 47. The morphism β of Example 40 is a dendric return morphism such that $\mathcal{T}^L(\beta) = \{\varepsilon, 2\}$ and $\mathcal{A}^L_{\beta,\varepsilon} = \{0, 1, 2, 3\}$, $\mathcal{A}^L_{\beta,2} = \{2, 3\}$. Thus the triplet $(\beta^L(G), \beta, G)$, where $G = G(\{0, 1\}, \{1, 2, 3\})$ is the graph of Example 40 is left valid since both graphs are acyclic for the coloring and connected multi-cliques and G contains the edge $(2, 3)$

We can deduce that a shift space is minimal dendric if and only if it has a primitive S-adic representation labeling an infinite path in the graphs built with the left (resp., right) valid triplets for dendric return morphisms.

We do not go into more details as we will prove that, instead of considering all the valid triplets, we can restrict ourselves to the case where the graphs are *colored trees*, i.e. all the edges have a different color and the corresponding uncolored graph is a tree. Note that a colored tree is trivially acyclic for the coloring and connected. We will often say that the colored graph itself is a tree and assimilate the colored and the uncolored version.

For two multi-graphs G and G' with colored edges, we say that G is a subgraph of G' if the uncolored version of G is a subgraph of the uncolored version of G' (but the colors might not coincide).

Lemma 48. *If the multi-clique $G = G(\{C_1, \dots, C_k\})$ is acyclic for the coloring and connected and if D, E are such that*

$$D \cup E = C_1 \quad \textit{and} \quad \mathrm{Card}(D \cap E) = 1,$$

then the multi-clique $G' = G(\{D, E, C_2, \dots, C_k\})$ is an acyclic for the coloring and connected subgraph of G.

Proof. Let $\{c\} = D \cap E$. By construction, G' is a subgraph of G and any lost edge was between a vertex of $D \setminus \{c\}$ and a vertex of $E \setminus \{c\}$. Any two such vertices remain connected through c thus G' is connected. Moreover, any simple cycle of G' corresponds to cycle of G thus only uses edges corresponding to one of the C_i, $i \leq k$, in G. Since c is the only vertex with both ingoing edges corresponding to D and ingoing edges corresponding to E, any simple cycle using edges from $C_1 = D \cup E$ only uses edges corresponding to D or edges corresponding to E. The conclusion follows. □

Proposition 49. *Let (G', σ, G) be a left (resp., right) valid triplet. For any covering tree T' of G', there exists a covering tree T of G such that (T', σ, T) is left (resp., right) valid.*

Proof. We prove the result for left valid triplets. Let $\mathcal{C} = \{C_1, \ldots, C_k\}$ be such that $G = G(\mathcal{C})$. Because (G', σ, G) is left valid, one has $G' = \sigma^L(G) = G(\mathcal{C}')$, where $\mathcal{C}' = \{\varphi^L_{\sigma,s}(C_i) \mid i \leq k, s \in \mathcal{T}^L(\sigma)\}$.

Assume that either G or G' is not a tree. Observe that since (G', σ, G) is a left valid triplet, G and G' are connected multi-cliques. Thus if one of them is not a tree, it must contain a cycle. Whichever of G and G' is not a tree, we start by highlighting a word $s \in \mathcal{T}^L(\sigma)$, some set $C_i \in \mathcal{C}$ and two distinct vertices $a, b \in \varphi^L_{\sigma,s}(C_i)$ as follows.

- If G' is not a tree, it contains a cycle. Since $G' = G(\mathcal{C}')$ is acyclic for the coloring, there exist $i \leq k$ and $s \in \mathcal{T}^L(\sigma)$ such that $C = \varphi^L_{\sigma,s}(C_i)$ contains at least 3 elements. The subgraph T'' of T' generated by the vertices of C is acyclic. Moreover, for any two vertices in C, any path connecting them in G' uses edges corresponding to C as G' is acyclic for the coloring. Thus the path connecting them in T' is in T'' and T'' is connected. We choose $a \in C$ to be a vertex of degree 1 in T'' and let b be its neighbor in T''.
- If G' is a tree but G is not, then similarly to the first case, there exists $i \leq k$ such that C_i contains at least 3 elements. Let s be the longest common suffix to all the $\sigma(d)$, $d \in C_i$; we have $s \in \mathcal{T}^L(\sigma)$. The set $C = \varphi^L_{\sigma,s}(C_i)$ contains exactly two elements. Indeed, it has at least two elements by definition of s and, G' being a tree, it cannot contain more than 2. Let us write $C = \{a, b\}$ with b such that
$$\mathrm{Card}(\{d \in C_i \mid \varphi^L_{\sigma,s}(d) = b\}) \geq 2.$$

In both cases, let $c \in C_i$ be such that $\varphi^L_{\sigma,s}(c) = b$ and let us note
$$D = \{c\} \cup \{d \in C_i \mid \varphi^L_{\sigma,s}(d) = a\}$$
and
$$E = (C_i \setminus D) \cup \{c\}.$$

We now use Lemma 48 to obtain a new left valid triplet (H', σ, H), where

- H is a connected subgraph of G;
- T' is still a covering tree of H';
- the total number of edges in H and H' is less than in G and G'.

This will end the proof by iterating the construction.

If $\mathcal{D} = \{D, E, C_1, \ldots, C_k\} \setminus \{C_i\}$, then by Lemma 48, the multi-clique $H = G(\mathcal{D})$ is an acyclic for the coloring and connected subgraph of G.

Consider now $\mathcal{D}' = \{\varphi^L_{\sigma,s'}(F) \mid F \in \mathcal{D}, s' \in \mathcal{T}^L(\sigma)\}$ and $H' = G(\mathcal{D}')$. We show that the triplet (H', σ, H) is left valid, i.e., we show item 2 and 3 of Definition 46. For any $s' \in \mathcal{T}^L(\sigma)$, we now show that $H^L_{\sigma,s'}$ is connected. Moreover, we describe the sets $\varphi^L_{\sigma,s'}(D)$ and $\varphi^L_{\sigma,s'}(E)$. Observe that for all s', $H^L_{\sigma,s'}$ is a subgraph of $G^L_{\sigma,s'}$ obtained by removing the edges between $D \setminus \{c\}$ and $E \setminus \{c\}$.

- If $s' = s$, then $c \in \mathcal{A}^L_{\sigma,s'}$. The graph $H^L_{\sigma,s'}$ is connected as any pair of vertices is connected in $G^L_{\sigma,s'}$ and a path in $H^L_{\sigma,s'}$ can be deduced from the one in $G^L_{\sigma,s'}$ by replacing an edge $(d, e) \in (D \setminus \{c\}) \times (E \setminus \{c\})$ by the path d, c, e. By definition of D and E,
$$\varphi^L_{\sigma,s'}(D) = \{a, b\} \quad \text{and} \quad \varphi^L_{\sigma,s'}(E) = \varphi^L_{\sigma,s'}(C_i) \setminus \{a\} = C \setminus \{a\}.$$

- If $s \in \mathcal{A}^* ds'$ for some letter d, then $H^L_{\sigma,s'}$ is connected for the same reason and
$$\varphi^L_{\sigma,s'}(D) = \{d\} \quad \text{and} \quad \varphi^L_{\sigma,s'}(E) = \varphi^L_{\sigma,s'}(C_i).$$
- If $s' \in \mathcal{A}^* as$, then $\mathcal{A}^L_{\sigma,s'} \cap E = \emptyset$ and $\mathcal{A}^L_{\sigma,s'} \cap C_i = \mathcal{A}^L_{\sigma,s'} \cap D$ thus no edge was lost from $G^L_{\sigma,s'}$ to $H^L_{\sigma,s'}$. In addition,
$$\varphi^L_{\sigma,s'}(D) = \varphi^L_{\sigma,s'}(C_i) \quad \text{and} \quad \varphi^L_{\sigma,s'}(E) = \emptyset.$$
- Otherwise, as and s' are not suffix comparable thus $\mathcal{A}^L_{\sigma,s'} \cap C_i = \mathcal{A}^L_{\sigma,s'} \cap E$, $\mathcal{A}^L_{\sigma,s'} \cap D \subset \{c\}$ and no edge was lost from $G^L_{\sigma,s'}$ to $H^L_{\sigma,s'}$. We have
$$\varphi^L_{\sigma,s'}(D) = \varphi^L_{\sigma,s'}(\{c\}) \quad \text{and} \quad \varphi^L_{\sigma,s'}(E) = \varphi^L_{\sigma,s'}(C_i).$$

In particular, we have shown that
$$G(\mathcal{D}') = G\left((\mathcal{C}' \setminus \{C\}) \cup \{\{a,b\}, C \setminus \{a\}\}\right)$$
thus, using Lemma 48, the graph
$$H' = G(\mathcal{D}') = \sigma^L(H)$$
is acyclic for the coloring and connected. This proves that the triplet (H', σ, H) is left valid.

We finally show that T' is still a subtree of H' and that the total number of edges decreased. By what have just seen, H' is a subgraph of G' obtained by removing the edges corresponding to C between a and the elements of $C \setminus \{a,b\}$. If G' is not a tree, then H' is a strict subgraph of G' since C contains at least three elements, and a and b where chosen so that T' is a subtree of H'. If G' is a tree, then $H' = G'$ and H is a connected strict subgraph of G as D and E both contain at least two elements. Thus we can iterate the process until both graphs are trees. □

We now define formally the graphs that we will use for the S-adic characterization.

Definition 50. Let $\mathcal{S}$ be a set of return morphisms from $\mathcal{A}^*$ to $\mathcal{A}^*$. The graph $\mathcal{G}^L(\mathcal{S})$ (resp., $\mathcal{G}^R(\mathcal{S})$) is defined by

- each vertex corresponds to a tree whose vertices are the elements of $\mathcal{A}$;
- there is an edge from T' to T labeled by $\sigma \in \mathcal{S}$ if (T', σ, T) is a left (resp., right) valid triplet.

The proof of the main result of this article uses the following lemma.

Lemma 51. *Let X be a dendric shift, σ a dendric return morphism and $Y = \sigma \cdot X$. If there exist a left valid triplet (G', σ, G) and a right valid triplet (H', σ, H) such that G is a covering subgraph of $G^L(X)$ and H is a covering subgraph of $G^R(X)$, then Y is dendric, G' is a covering subgraph of $G^L(Y)$ and H' is a covering subgraph of $G^R(Y)$.*

Proof. For all $s \in \mathcal{T}^L(\sigma)$, the graph $G^L_{\sigma,s}$ is a covering subgraph of $G^L_{\sigma,s}(X)$. Since (G', σ, G) is left valid, $G^L_{\sigma,s}$ is connected and so is $G^L_{\sigma,s}(X)$. The same reasoning proves that $G^R_{\sigma,p}(X)$ is connected for all $p \in \mathcal{T}^R(\sigma)$. By Proposition 38, Y is dendric.

Let us prove that G' is a covering subgraph of $G^L(Y)$. The proof for H' and $G^R(Y)$ is similar. First, notice that $G^L(Y)$ and $G' = \sigma^L(G)$ have the same set of vertices which is the image alphabet of σ. Moreover, they are acyclic for the coloring and multi-clique thus they are simple. Hence, it suffices to prove that if there is an edge in G', it is also an edge in $G^L(Y)$.

Let (a, b) be an edge of G' and let $G = G(\{C_1, \dots, C_k\})$ and $G^L(X) = G(\{C'_1, \dots, C'_l\})$. By definition, $G' = \sigma^L(G)$ thus there exist $i \leq k$ and $s \in \mathcal{T}^L(\sigma)$ such that $a, b \in \varphi^L_{\sigma,s}(C_i)$. In other words, there exist $c, d \in C_i$ such that $\varphi^L_{\sigma,s}(c) = a$ and $\varphi^L_{\sigma,s}(d) = b$. Since (c, d) is an edge of G, it is an edge of $G^L(X)$ and there exists $j \leq l$ such that $c, d \in C'_j$. We then have $a, b \in \varphi^L_{\sigma,s}(C'_j)$ thus there is an edge between a and b in $\sigma^L(G^L(X))$, which is exactly $G^L(Y)$ by Proposition 41. We conclude that G' is a covering subgraph of $G^L(Y)$. □

We can now prove the main result, that we recall here.

Theorem 1. *Let $\mathcal{S}$ be a family of dendric return morphisms from $\mathcal{A}^*$ to $\mathcal{A}^*$ and let X be a shift space having an $\mathcal{S}$-adic representation $\boldsymbol{\sigma} = (\sigma_n)_{n\geq 0}$. Then X is minimal dendric if and only if $\boldsymbol{\sigma}$ is primitive and labels infinite paths in the graphs $\mathcal{G}^L(\mathcal{S})$ and $\mathcal{G}^R(\mathcal{S})$.*

Proof. Assume that X is minimal dendric. The morphisms of $\mathcal{S}$ being return morphisms, the sequence $\boldsymbol{\sigma}$ is as in Theorem 42 and thus is primitive. Let us show that $\boldsymbol{\sigma}$ labels an infinite path in $\mathcal{G}^L(\mathcal{S})$. Similarly, it will label an infinite path in $\mathcal{G}^R(\mathcal{S})$. For all $n \geq 0$, as $X_{\boldsymbol{\sigma}}^{(n+1)}$ and $X_{\boldsymbol{\sigma}}^{(n)}$ are dendric and since $X_{\boldsymbol{\sigma}}^{(n)}$ is the image of $X_{\boldsymbol{\sigma}}^{(n+1)}$ under σ_n, the triplet $(G^L(X_{\boldsymbol{\sigma}}^{(n)}), \sigma_n, G^L(X_{\boldsymbol{\sigma}}^{(n+1)}))$ is left valid by Proposition 38 and Proposition 41. Let T_0 be a covering tree of $G^L(X) = G^L(X_\sigma^{(0)})$. By Proposition 49, there exists a covering tree T_1 of $G^L(X_\sigma^{(1)})$ such that the triplet (T_0, σ_0, T_1) is left valid thus σ_0 labels an edge from T_0 to T_1 in $\mathcal{G}^L(\mathcal{S})$. We iterate the process to obtain an infinite path in $\mathcal{G}^L(\mathcal{S})$.

Now assume that $\boldsymbol{\sigma}$ is primitive and labels a path $\left(G_n^L\right)_{n\geq 0}$ in $\mathcal{G}^L(\mathcal{S})$ and a path $\left(G_n^R\right)_{n\geq 0}$ in $\mathcal{G}^R(\mathcal{S})$. The shift $X_{\boldsymbol{\sigma}}$ is minimal by primitiveness of $\boldsymbol{\sigma}$. Let $u \in \mathcal{L}(X_{\boldsymbol{\sigma}})$ be a factor. If u is an initial factor, then it is dendric since σ_0 is dendric. Otherwise, there exist a unique $k > 0$ and a unique $v \in \mathcal{L}(X_{\boldsymbol{\sigma}}^{(k)})$ initial such that u is a descendant of v by $\sigma_0 \dots \sigma_{k-1}$ by Proposition 8. Let Y be an Arnoux-Rauzy shift and let $Z = \sigma_k \cdot Y$. Since v is initial, $\mathcal{E}_Z(v) = \mathcal{E}_{X_{\boldsymbol{\sigma}}^{(k)}}(v)$. The graph $G^L(Y)$ (resp., $G^R(Y)$) is the complete (simple and monochromatic) graph on $\mathcal{A}$, thus G_{k+1}^L (resp., G_{k+1}^R) is a covering subgraph. By Lemma 51 applied $k+1$ times, we obtain that $\sigma_0 \dots \sigma_{k-1} \cdot Z$ is dendric thus, starting with the extension graph $\mathcal{E}_Z(v)$, v only has dendric descendants under $\sigma_0 \dots \sigma_{k-1}$ and u is dendric. □

By construction, on an alphabet $\mathcal{A}$ of size d, the graphs $\mathcal{G}^L(\mathcal{S})$ and $\mathcal{G}^R(\mathcal{S})$ have d^{d-2} vertices (see sequence A000272 on OEIS) each, meaning that if we take the product of these graphs, we obtain a characterization with a unique graph $\mathcal{G}(\mathcal{S})$ with $d^{2(d-2)}$ vertices.

However, it is possible to reduce the number of vertices with a clever use of permutations. Indeed, the vertices of $\mathcal{G}(\mathcal{S})$ are pairs of labeled trees and some of these pairs are equivalent, in the sense that they are equal up to a permutation of the letters. Thus, by keeping only one pair per equivalence class and adding permutations to the morphisms of the set $\mathcal{S}$, we obtain a smaller graph.

In the case of a ternary alphabet, this allows to replace a graph with 9 vertices by a graph with 2 vertices, as done in [28]. For an alphabet of size 4, we obtain a graph with 14 vertices instead of 256.

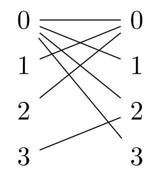
7.3. **An example: $\mathcal{S}$-adic characterization of minimal dendric shifts with one right special factor of each length.** While we do not have an easy description of an explicit set $\mathcal{S}$ for which any minimal dendric shift (over some fixed alphabet) has an $\mathcal{S}$-adic representation, restricting ourselves to subfamilies of minimal dendric shifts allows to build examples. Furthermore, and in accordance to what was stated in the beginning of the previous subsection, if we consider all possible multi-cliques as the vertices of $\mathcal{G}^L(\mathcal{S})$ and $\mathcal{G}^R(\mathcal{S})$ (and not just the covering trees), any primitive $\mathcal{S}$-adic representation $\boldsymbol{\sigma}$ of X will label, among others, the paths $(G^L(X_{\boldsymbol{\sigma}}^{(n)}))_{n\geq 0}$ and $(G^R(X_\sigma^{(n)}))_{n\geq 0}$ (in $\mathcal{G}^L(\mathcal{S})$ and $\mathcal{G}^R(\mathcal{S})$ respectively).

We now use this observation to build a graph characterizing the family F of minimal dendric shift spaces over $\{0, 1, 2, 3\}$ having exactly one right special factor of each length.

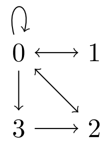
Let X be such a shift space. If $\boldsymbol{\sigma}$ is an S-adic representation of X (with dendric return morphisms) then, for all n, the shift space $X_{\boldsymbol{\sigma}}^{(n)}$ also has a unique right special factor of each length. Indeed, $G^R(X)$ is the complete monochromatic graph K_4 thus, by Proposition 41, $G^R(X_{\boldsymbol{\sigma}}^{(1)})$ contains a complete subgraph of size four and, as $X_{\boldsymbol{\sigma}}^{(1)}$ is dendric, we have $G^R(X_{\boldsymbol{\sigma}}^{(1)}) = K_4$. Hence, $X_{\boldsymbol{\sigma}}^{(1)}$ can only have one right special factor of each length. We iterate for $X_{\boldsymbol{\sigma}}^{(n)}$.

As a consequence, it makes sense to look for an S-adic characterization of F using infinite paths. It also implies that the elements of F have $\mathcal{S}_{1\mathrm{R}}$-adic representations where $\mathcal{S}_{1\mathrm{R}}$ is the set of dendric return morphisms for a letter which has four right extensions. These morphisms can easily be found using the possible extension graphs for the empty word.

Example 52. Assume that the extension graph of the empty word is given by

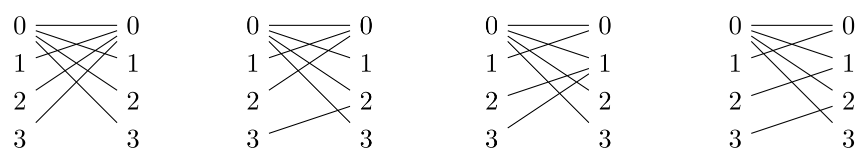


The associated Rauzy graph of order 1 is



and the return words for 0, corresponding to paths from and to the vertex 0, are given by 0, 01, 02 and 032. Thus, the morphisms of $\mathcal{S}$ associated with this extension graph are the elements of $\beta\Sigma_4$ where β is the morphism of Example 40, i.e. $\beta : 0 \mapsto 0, 1 \mapsto 01, 2 \mapsto 02, 3 \mapsto 032$, and Σ_4 is the set of permutations over $\{0, 1, 2, 3\}$.

Up to a permutation of the alphabet, the possible extension graphs (with a unique right special letter) and their associated morphisms are given in Figure 6. Notice that, as we take the return morphisms for the only right special letter, there is, up to a permutation, only one morphism corresponding to each extension graph. The set $\mathcal{S}_{1\mathrm{R}}$ is then finite and given by $\Sigma_4\{\alpha, \beta, \gamma, \delta\}\Sigma_4$, where $\alpha, \beta, \gamma, \delta$ are defined in Figure 6.



$$\alpha : \begin{cases} 0 \mapsto 0 \\ 1 \mapsto 01 \\ 2 \mapsto 02 \\ 3 \mapsto 03 \end{cases} \quad \beta : \begin{cases} 0 \mapsto 0 \\ 1 \mapsto 01 \\ 2 \mapsto 02 \\ 3 \mapsto 032 \end{cases} \quad \gamma : \begin{cases} 0 \mapsto 0 \\ 1 \mapsto 01 \\ 2 \mapsto 021 \\ 3 \mapsto 031 \end{cases} \quad \delta : \begin{cases} 0 \mapsto 0 \\ 1 \mapsto 01 \\ 2 \mapsto 021 \\ 3 \mapsto 0321 \end{cases}$$

FIGURE 6. Possible extension graphs of the empty word in a minimal dendric shift over $\{0, 1, 2, 3\}$ and their associated return morphisms.

We now build the graph that will give us the characterization. For the right side, we consider that the graph $\mathcal{G}^R(\mathcal{S}_{1\mathrm{R}})$ is built with all the right valid triplets. However, as we have observed before, any $\mathcal{S}_{1\mathrm{R}}$-adic representation $\boldsymbol{\sigma}$ of $X \in F$ labels the path $(G^R(X_{\boldsymbol{\sigma}}^{(n)}))_{n\geq 0} = (K_4)_{n\geq 0}$. We can thus assume that the right side graph is reduced to the vertex K_4 without losing $\mathcal{S}_{1\mathrm{R}}$-adic representations.

For the left side, we consider the usual graph $\mathcal{G}^L(\mathcal{S}_{1\mathrm{R}})$ (with trees as vertices). When taking the product with the right side graph, we can use permutations and, since K_4 is symmetric, this gives us a graph with only two vertices (G_1, K_4) and (G_2, K_4) where G_1 and G_2 are represented in Figure 7.

In the end, we obtain that each minimal shift space in F has a primitive $\mathcal{S}_{1\mathrm{R}}$-adic representation in the graph represented in Figure 8. Note that, to reduce the number of morphisms, we added a loop on (G_1, K_4) (resp., (G_2, K_4)) labeled with the permutations that fix the graph G_1 (resp., G_2). We will then consider that $\mathcal{S}_{1\mathrm{R}}$ also contains the permutations. To improve the readability of the graph, we also introduce the following notations: $\Sigma_{\{1,2,3\}}$ is the subset of permutations of

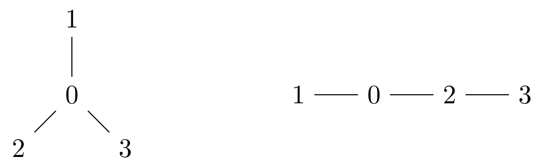


FIGURE 7. Graphs G_1 on the left and G_2 on the right

Σ_4 fixing 0, π_{ab} stands for the cyclic permutation $(a \ \ b)$ and we define the following morphisms

$$\beta' = \beta\pi_{23}, \quad \delta' = \delta\pi_{23}, \quad \gamma' = \gamma\pi_{12}, \quad \gamma'' = \pi_{23}\gamma\pi_{23}.$$

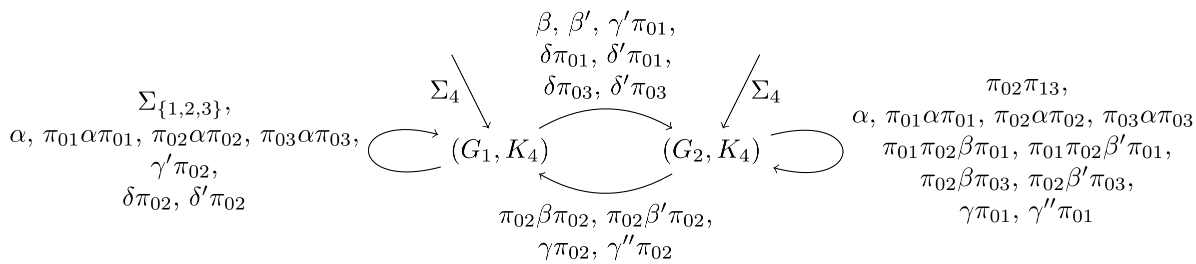


FIGURE 8. A shift space over $\{0, 1, 2, 3\}$ is minimal, dendric and has exactly one right special factor of each length if and only if it has a primitive $\mathcal{S}_{1\mathrm{R}}$-adic representation labeling an infinite path in this graph.

The converse result is also true, i.e. if a shift space X has a primitive $\mathcal{S}_{1\mathrm{R}}$-adic representation $\boldsymbol{\sigma}$ labeling an infinite path in the graph of Figure 8, then it is a minimal dendric shift space over $\{0, 1, 2, 3\}$ having exactly one right special factor of each length. Indeed, it is minimal dendric by the proof of Theorem 1. To conclude, we show that we can find infinitely many factors having four right extensions in X. This will imply that, for each length, there is a factor having four right extensions and it must be the unique right special factor since X is dendric.

It follows from this observation: if v has a left extension a such that av has four right extensions and if $\sigma \in \mathcal{S}_{1\mathrm{R}}$ is a return morphism for ℓ, then the extended image $\sigma(v)\ell$ has a left extension b such that $b\sigma(v)\ell$ has four right extensions. Indeed, it suffices to take $b = \varphi^L_{\sigma,\varepsilon}(a)$ by Proposition 12 and by definition of the morphisms of $\mathcal{S}_{1\mathrm{R}}$. For any $k > 0$, starting with ε in $X^{(k)}_{\boldsymbol{\sigma}}$, we can then find a word $u \in \mathcal{L}(X)$ of length at least k having four right extensions.

7.4. **Characterization of eventually dendric shifts.** Using the S-adic characterization obtained for minimal dendric shifts, we can deduce an S-adic characterization of minimal eventually dendric shifts. It is based on the fact that any eventually dendric shift space can be obtained from a dendric shift space in the following way.

Proposition 53. *Let X be a minimal eventually dendric shift space with threshold N. For any non empty $w \in \mathcal{L}(X)$, the derived shift $\mathcal{D}_w(X)$ is eventually dendric with threshold at most* $\max\{0, N - |w|\}$.

Proof. The proof is almost exactly the same as in the dendric case (see [9, Theorem 5.13]), replacing [9, Proposition 5.7] by [17, Lemma 9.3]. □

By [17, Theorem 7.3], if X is eventually dendric and w is of length at least N, $\mathcal{D}_w(X)$ is on an alphabet of size $1 + p_X(N + 1) - p_X(N)$.

The next result directly follows from the previous one and from Proposition 19.

Theorem 54. *A minimal shift X is eventually dendric if and only if it has a dendric derived shift, if and only if every derived shift with respect to a long enough factor of X is dendric.*

The S-adic characterization of minimal eventually dendric shifts is now a direct consequence of Theorem 1.

Theorem 55. *A shift space is minimal eventually dendric if and only if it has a primitive S-adic representation $(\sigma_n)_{n\geq 0}$ where σ_0 is a return morphism, σ_n is a dendric return morphism for all $n \geq 1$ and $(\sigma_n)_{n\geq 1}$ labels infinite paths in $\mathcal{G}^L(\mathcal{S})$ and $\mathcal{G}^R(\mathcal{S})$ where $\mathcal{S} = \{\sigma_n : n \geq 1\}$.*

7.5. **Decidability of ultimate dendricity for minimal substitutive shifts.** A deeply studied class of shift spaces is the one of substitutive (or morphic) shifts that roughly correspond to S-adic spaces with an eventually periodic S-adic representation. A shift space X is said to be morphic if there exist morphisms $\sigma : \mathcal{B}^* \to \mathcal{B}^*$ and $\tau : \mathcal{B}^* \to \mathcal{A}^*$ such that

$$X = \{x \in \mathcal{A}^{\mathbb{Z}} \mid \forall u \in \mathcal{L}(x), \exists b \in \mathcal{B}, n \in \mathbb{N} : u \in \mathcal{L}(\tau(\sigma^n(b))\}.$$

We let $X(\sigma, \tau)$ denote such a shift. In this section, we show that, under minimality, (eventual) dendricity is decidable for this class of shift spaces. Whenever τ is the identity, the minimality of X is also decidable [11].

It is classical [15] that we can always suppose that σ is non-erasing ($\sigma(a) \neq \varepsilon$ for every letter) and τ is a coding ($|\tau(a)| = 1$ for every letter) so we always make such an assumption.

Observe that one usually considers the restricted version of morphic shifts consisting in shifts generated by a morphic word. The shift space generated by an infinite word $x \in \mathcal{A}^{\mathbb{N}}$ is the shift space $X_x = \{y \in \mathcal{A}^{\mathbb{Z}} \mid \mathcal{L}(y) \subset \mathcal{L}(x)\}$. Whenever x is *uniformly recurrent* (that is, every finite word $u \in \mathcal{L}(x)$ occurs infinitely many times in x and with bounded gaps), the shift space X_x is minimal. An infinite word x is *morphic* if there exist morphisms $\sigma : \mathcal{B}^* \to \mathcal{B}^*$ and $\tau : \mathcal{B}^* \to \mathcal{A}^*$ such that σ is *prolongable* on some letter $b \in \mathcal{B}$ (that is, $\sigma(b) \in b\mathcal{B}^*$ and $\lim_{n\to+\infty} |\sigma^n(b)| = +\infty$) and $x = \tau(\sigma^\omega(b)) = \lim_{n\to+\infty} \tau(\sigma^n(b^\omega))$. It is also decidable whether a morphic word is uniformly recurrent [22]. Though, the shift space X_x could be minimal with x not uniformly recurrent. Thus, deciding if $X(\sigma, \tau)$ is minimal is an open problem.

A key argument to decide ultimate dendricity in a minimal morphic shift is to make more deterministic the S-adic construction by return words described in Section 7.1. More precisely, using the notation of the S-adic construction of Section 7.1 (on Page 13), we want to algorithmically chose the words w_n for which we consider return words as well as the coding morphisms $\sigma_n : R_{n+1}^* \to R_n^*$. This can be achieved by fixing some morphic word generating $X(\sigma, \tau)$.

Lemma 56. *If $X(\sigma, \tau)$ is minimal, then it is generated by a uniformly recurrent morphic word x and one can compute morphisms σ', τ' such that $x = \tau'(\sigma'^\omega(b'))$.*

Proof. If $X(\sigma, \tau)$ is finite, then it is generated by $x = u^\omega$, where u is a finite word. It is direct to check that $x = \sigma'^\omega(b')$, where b' is the first letter of u and $\sigma(a) = u^2$ for every letter a.

Assume now that $X(\sigma, \tau)$ is aperiodic and let B_g, B_b be the subalphabets of growing and bounded letters, i.e.,

$$B_g = \{b \in \mathcal{B} \mid \lim_{n\to+\infty} |\sigma^n(b)| = +\infty\}$$
$$B_b = \{b \in \mathcal{B} \mid (|\sigma^n(b)|)_{n\geq 0} \text{ is bounded}\}.$$

The set B_g is non-empty by aperiodicity, so let $a \in B_g$. For every $n \in \mathbb{N}$, we set $\sigma^n(a) = u_n a_n v_n$, where $a_n \in B_g$ and $u_n \in B_b^*$. By the pigeonhole principle, we can find $k < \ell$ such that $a_k = a_\ell = b'$. We thus have $\sigma^{\ell-k}(b') = ub'v$ for some word $u \in B_b^*$. Since σ, τ are non-erasing and since $X(\sigma, \tau)$ is aperiodic and minimal, we must have $u = \varepsilon$. In other words, $\sigma' = \sigma^{\ell-k}$ is prolongable on b' and, by minimality, the word $\tau(\sigma'^\omega(b'))$ generates $X(\sigma, \tau)$. □

Assume now that $x \in \mathcal{A}^{\mathbb{N}}$ is a uniformly recurrent morphic word generating $X(\sigma, \tau)$. In what follows, we omit the subscript $X(\sigma, \tau)$ to alleviate notation. If u is a non-empty prefix of x, we set $R(u) = \{1, \ldots, \mathrm{Card}(\mathcal{R}(u))\}$ and we define $\theta_u : R(u)^* \to \mathcal{A}^*$ so that $\theta_u(R(u)) = \mathcal{R}(u)$ and according to the order of appearance of return words to u in x. That is, for all $i \in R(u)$, if k is the first occurrence of $\theta_u(i)u$ in x, then $x_{[0,k)}$ belongs to $\theta_u(\{1, \ldots, i-1\}^*)$. Then there exists a unique infinite word $\mathcal{D}_u(x)$, called the *derived sequence of x (with respect to u)* such that $\theta_u(\mathcal{D}_u(x)) = x$.

Theorem 57 (Durand [22])**.** *Let $x = \tau(\sigma^\omega(b))$ be an aperiodic uniformly recurrent morphic word in $\mathcal{A}^\mathbb{N}$.*

(1) *For every non-empty prefix u of x, the morphism θ_u is computable and there exist some computable morphisms $\sigma_u : \mathcal{C}^* \to \mathcal{C}^*$ and $\tau_u : \mathcal{C}^* \to R(u)^*$ such that $\mathcal{D}_u(x) = \tau_u(\sigma_u^\omega(1))$.*
(2) *There is computable constant D such that the set $\{(\sigma_u, \tau_u) \mid u \text{ non-empty prefix of } x\}$ has cardinality at most D. In particular, the number of derived sequences of x (on its non-empty prefixes) is at most D.*

Proof. The existence of σ_u and τ_u is [22, Proposition 28] and the fact that θ_u, σ_u and τ_u are computable is explained in [22, Section 4]. The bound on the number of possible pairs (σ_u, τ_u) is [22, Theorem 29]. □

Lemma 58 (Durand [19])**.** *Let x be an aperiodic uniformly recurrent infinite word and let u be a non-empty prefix of x. For every non-empty prefix v of $\mathcal{D}_u(x)$, we have $\mathcal{D}_v(\mathcal{D}_u(x)) = \mathcal{D}_w(x)$, where $w = \theta_{x,u}(v)u$*[1]*.*

Theorem 59. *Let $x = \tau(\sigma^\omega(b))$ be an aperiodic uniformly recurrent morphic word. One can algorithmically compute (from (σ, τ, b)) two return morphisms $\theta : \mathcal{C}^* \to \mathcal{C}^*$ and $\lambda : \mathcal{C}^* \to \mathcal{A}^*$ such that θ is primitive and $x = \lambda(\theta^\omega(1))$.*

Proof. We define a sequence of derivated sequence of x as follows. Fix $x^{(0)} = x$ and $w_0 = x_0^{(0)}$, i.e., w_0 is the first letter of $x^{(0)}$. For every integer $n \geq 1$, we set $x^{(n)} = \mathcal{D}_{w_{n-1}}(x^{(n-1)})$ and $w_n = 1$. In other words, $x^{(1)}$ is the derived sequence of x with respect to the first letter of x and, for every $n \geq 2$, $x^{(n)}$ is the derived sequence of $x^{(n-1)}$ with respect to the letter 1 (which is the first letter of $x^{(n-1)}$). By iterating Lemma 58, we deduce that there is a sequence of words $u_n \in \mathcal{L}(x)$ such that $\lim_{n\to+\infty} |u_n| = +\infty$ and $x^{(n)}$ is the derived sequence of x with respect to u_n.

Every sequence $x^{(n)}$ is uniformly recurrent and, by Theorem 57, we can compute some morphisms $\theta_n, \sigma_n, \psi_n$ such that $x^{(n)} = \psi_n(\sigma_n^\omega(1)) = \theta_n(x^{(n+1)})$. Still by Theorem 57, there exists $1 \leq m < n$ bounded by a computable constant such that $(\sigma_m, \psi_m) = (\sigma_n, \psi_n)$, so that $x^{(m)} = x^{(n)}$. The construction being deterministic, this implies that for all $k \geq m$, we have $x^{(k+m-n)} = x^{(k)}$, so that $\theta_{k+m-n} = \theta_k$. In particular, setting $\lambda = \theta_0 \circ \cdots \circ \theta_{m-1}$ and $\theta = \theta_m \circ \cdots \circ \theta_{n-1}$, we have $x = \lambda(x^{(m)})$ and $x^{(m)} = \theta(x^{(m)})$. In particular, $\theta(1)$ starts with 1. Furthermore, both λ and θ are return morphisms and, using Theorem 42, θ is primitive, hence prolongable on 1. We finally get $x = \lambda(\theta^\omega(1))$ □

The next result directly follows from Theorem 1, Theorem 54 and Theorem 55.

Theorem 60. *Let $x = \tau(\sigma^\omega(b))$ be an aperiodic uniformly recurrent morphic word and let λ, θ be the computable morphisms given by Theorem 59.*

(1) *The shift space X_x is eventually dendric if and only if θ is dendric and the sequence $(\theta, \theta, \dots)$ labels an infinite path in $\mathcal{G}^L(\{\theta\})$ and in $\mathcal{G}^R(\{\theta\})$.*
(2) *The shift space X_x is dendric if and only if λ and θ are dendric and the sequence $(\lambda, \theta, \theta, \dots)$ labels an infinite path in $\mathcal{G}^L(\{\lambda, \theta\})$ and in $\mathcal{G}^R(\{\lambda, \theta\})$.*

Corollary 61. *Given two morphisms σ, τ such that $X(\sigma, \tau)$ is minimal, it is decidable whether $X(\sigma, \tau)$ is (eventually) dendric.*

Proof. By Lemma 56, we can compute morphisms σ', τ' such that $X(\sigma, \tau) = X_x$, with $x = \tau'(\sigma'^\omega(b))$. It is decidable whether x is periodic and, if so, one can compute the minimal period [21]. If it is, then $X(\sigma, \tau)$ is eventually dendric and it is dendric if and only if the minimal period is 1. If not, then we use Theorem 60 to conclude. □

[1]Note that we added a subscript x to $\theta_{x,u}$ to emphasize that it is a coding morphism for return words to u in x.

8. Interval exchanges

An interval exchange transformation is a pair $(\{I_a \mid a \in \mathcal{A}\}, T)$, where $\mathcal{A}$ is an alphabet, $\{I_a \mid a \in \mathcal{A}\}$ is a partition of $[0,1)$ into left-closed and right-open intervals and $T : [0,1) \to [0,1)$ is a bijection whose restriction to each interval I_a is a translation. Another way of defining an interval exchange transformation is by means of a pair $\binom{\leq_1}{\leq_2}$ of total orders on $\mathcal{A}$. An interval exchange transformation is said to be *regular* if the orbits of the non-zero left endpoints of the intervals I_a are infinite and disjoint. We refer to [26] for more precise definitions.

8.1. Extension graphs in codings of IET.

With an interval exchange transformation is associated a shift space, called the *natural coding* of T and defined as the set of infinite words $(x_n)_{n\in\mathbb{Z}}$ over $\mathcal{A}$ such that for every factor $u = u_0 \cdots u_{n-1}$ of x, the set $\bigcap_{0\leq i<n} T^{-i}(I_{u_i})$ is non-empty. The definition using total orders allows to give the following characterization of natural codings of regular interval exchange transformations.

Theorem 62 (Ferenczi-Zamboni [26])**.** *A shift space X over $\mathcal{A}$ is the natural coding of a regular interval exchange transformation with the pair of orders $\binom{\leq_1}{\leq_2}$ if and only if it is minimal, $\mathcal{A} \subset \mathcal{L}(X)$ and it satisfies the following conditions for every $w \in \mathcal{L}(X)$:*

1. *for all $(a_1,b_1),(a_2,b_2) \in E_X(w)$, if $a_1 <_2 a_2$, then $b_1 \leq_1 b_2$;*
2. *for all $a_1, a_2 \in E_X^L(w)$, if a_1, a_2 are consecutive for $\leq_2$, then $E_X^R(a_1w) \cap E_X^R(a_2w)$ is a singleton.*

In particular, for every $w \in \mathcal{L}(X)$, $E_X^L(w)$ is an interval for $\leq_2$ and $E_X^R(w)$ is an interval for $\leq_1$.

This result can be reformulated as follows. Let $\leq^L$ and $\leq^R$ be two total orders on $\mathcal{A}$. A factor w of a shift space $X \subset A^{\mathbb{Z}}$ is said to be *planar* for $(\leq^L, \leq^R)$ if item 1 is satisfied for $\binom{\leq^R}{\leq^L}$. This implies that, placing the left and right vertices of $\mathcal{E}_X(w)$ on parallel lines and ordering them respectively by $\leq^L$ and $\leq^R$, the edges of $\mathcal{E}_X(w)$ may be drawn as straight noncrossing segments, resulting in a planar graph. In particular, $\mathcal{E}_X(w)$ is acyclic.

If w is planar for $(\leq^L, \leq^R)$, item 2 is then equivalent to the fact that $\mathcal{E}_X(w)$ is a tree. In particular, this proves that codings of regular interval exchanges are dendric shift spaces [7].

A shift space over $\mathcal{A}$ is said to be *planar* for $(\leq^L, \leq^R)$ if every $w \in \mathcal{L}(X)$ is planar for $(\leq^L, \leq^R)$. Theorem 62 then states that a shift space is the natural coding of a regular interval exchange transformation for the orders $\binom{\leq^R}{\leq^L}$ if and only if it is a minimal dendric shift space planar for $(\leq^L, \leq^R)$. Note that, in this case, it is known that the pair $(\leq^L, \leq^R)$ is *irreducible*, i.e. for any $0 < k < \mathrm{Card}(\mathcal{A})$, the k $\leq^L$-smallest letters are not the k $\leq^R$-smallest letters.

The next result is classical when dealing with interval exchange transformations. We give a combinatorial proof.

Proposition 63. *Let X be a minimal dendric shift over $\mathcal{A}$ which is planar for $(\leq^L, \leq^R)$.*

1. *Any long enough left special factor w is such that $E_X^L(w)$ is equal to $\{a,b\}$, where a,b are consecutive for $\leq^L$. Furthermore, for every two $\leq^L$-consecutive letters a,b and for all $n \in \mathbb{N}$, there is a (unique) left special factor w of length n such that $\{a,b\} \subset E_X^L(w)$.*
2. *The same holds on the right, i.e., any long enough right special factor w is such that $E_X^R(w)$ is equal to $\{a,b\}$, where a,b are consecutive for $\leq^R$. Furthermore, for every two $\leq^R$-consecutive letters a,b and for all $n \in \mathbb{N}$, there is a (unique) right special factor w of length n such that $\{a,b\} \subset E_X^R(w)$.*

Proof. We prove the first item. Since X is dendric, there exists N satisfying the conditions of Proposition 17. Let $w \in \mathcal{L}_{\geq N}(X)$ be a left special factor. By Theorem 62, the letters of $E_X^L(w)$ are consecutive for $\leq^L$ and it suffices to prove that $\mathrm{Card}(E_X^L(w)) = 2$. Assume that there exist $a, b, c \in E_X^L(w)$ such that $a <^L b <^L c$.

We claim that bw is not right special. Otherwise, by definition of N, $E_X^R(bw) = E_X^R(w)$ and, as there exist a', c' such that $E_X^R(aw) = \{a'\}$ and $E_X^R(cw) = \{c'\}$, wa' and wc' are left special. However, this contradicts the definition of N thus $a' = c'$. Using item 1 of Theorem 62, this implies that $E_X^R(bw) = \{a'\}$, a contradiction.

Thus bw is not right special and there exists a unique letter b' such that $bwb' \in \mathcal{L}(X)$. By definition of N, $E^L(wb') = E^L_X(w)$ and we can iterate the reasoning to show that no word of $bw\mathcal{A}^* \cap \mathcal{L}(X)$ is right special. This is a contradiction as X is minimal dendric.

We now prove the second part of item 1 by induction on n. It is true for $n = 0$ and if u is the unique word of $\mathcal{L}_n(X)$ such that $a, b \in E^L_X(u)$ then, by item 2 of Theorem 62, there exists c such that $E^R_X(au) \cap E^R_X(bu) = \{c\}$. Thus, uc is the unique word of $\mathcal{L}_{n+1}(X)$ having a and b as left extensions. □

8.2. **S-adic characterization of regular interval exchange transformations.** From Theorem 62, we know that the natural coding of a regular interval exchange transformation is a minimal dendric shift. By Theorem 1, any of its $\mathcal{S}$-adic representation made of return morphisms labels a path in $\mathcal{G}^L(\mathcal{S})$ and in $\mathcal{G}^R(\mathcal{S})$. In this section, we characterize those labeled paths.

Definition 64. A *line graph* on $\mathcal{A}$ is a graph G such that, if $\mathcal{A} = \{a_1, \dots, a_n\}$, the edges are exactly the pairs (a_i, a_{i+1}), $i < n$. This graph is associated with the orders

$$a_1 < a_2 < \cdots < a_n \quad \text{and} \quad a_n \prec a_{n-1} \prec \cdots \prec a_1.$$

We then define $G(\leq)$ and $G(\preceq)$ as the graph G.

The following result is a direct consequence of Proposition 63. Its converse is false, as seen in Example 36.

Corollary 65. *If X is a minimal dendric shift planar for the orders $(\leq^L, \leq^R)$, then $G^L(X) = G(\leq^L)$ and $G^R(X) = G(\leq^R)$.*

If $\preceq$ and $\leq$ are two total orders on $\mathcal{A}$, a partial map $\varphi : \mathcal{A} \to \mathcal{A}$ is *order preserving* from $\preceq$ to $\leq$ if, for all $x, y \in \mathrm{dom}(\varphi)$, we have

$$x \prec y \Rightarrow \varphi(x) \leq \varphi(y).$$

A return morphism $\sigma : \mathcal{A}^* \to \mathcal{A}^*$ is *left order preserving* from $\preceq$ to $\leq$ if, for all $s \in \mathcal{T}^L(\sigma)$, $\varphi^L_{\sigma,s}$ is order preserving from $\preceq$ to $\leq$. Similarly, σ is *right order preserving* from $\preceq$ to $\leq$ if, for all $p \in \mathcal{T}^R(\sigma)$, $\varphi^R_{\sigma,p}$ is order preserving from $\preceq$ to $\leq$.

Lemma 66. *For every return morphism $\sigma : \mathcal{A}^* \to \mathcal{A}^*$ and every total order $\leq$ on $\mathcal{A}$, there exists a unique total order $\preceq$ on $\mathcal{A}$ such that σ is left (resp., right) order preserving from $\preceq$ to $\leq$.*

Proof. We prove the result for left order preserving. Let us begin with the existence of such an order $\preceq$. For all $s \in \mathrm{Suff}(\sigma(\mathcal{A}))$, we will build an order $\preceq_s$ on $\mathcal{B}_s = \{a \in \mathcal{A} \mid \sigma(a) \in \mathcal{A}^* s\}$[2] such that, for all $s' \in \mathcal{A}^* s$, the map $\varphi^L_{\sigma,s'}$ is order preserving from $\preceq_s$ to $\leq$. The conclusion will follow with $s = \varepsilon$.

We proceed by induction on the length of s, starting with $|s|$ maximal. If s is maximal, i.e. for all $a \in \mathcal{A}$, $\mathcal{B}_{as}$ is empty, then $s \in \sigma(\mathcal{A})$ and $\mathcal{B}_s$ contains a unique element thus $\preceq_s$ is a trivial order.

Assume now that s is not of maximal length and that we have the orders $\preceq_{as}$ for all $a \in \varphi^L_{\sigma,s}(\mathcal{A})$. Since $\sigma(\mathcal{A})$ is a suffix code, the sets $\mathcal{B}_{as}$ form a partition of $\mathcal{B}_s$ thus, we can define the order $\preceq_s$ on $\mathcal{B}_s$ by $x \prec_s y$ if

(1) $x, y \in \mathcal{B}_{as}$ and $x \prec_{as} y$, or
(2) $x \in \mathcal{B}_{as}$, $y \in \mathcal{B}_{bs}$ and $a < b$.

For all $s' \in \mathcal{A}^* as$, $\varphi^L_{\sigma,s'}$ is order preserving from $\preceq_{as}$ to $\leq$ thus it is order preserving from $\preceq_s$ to $\leq$. In addition, if $x, y \in \mathcal{B}_s$ are as in case 1, then

$$\varphi^L_{\sigma,s}(x) = a = \varphi^L_{\sigma,s}(y)$$

and if they are as in case 2, then

$$\varphi^L_{\sigma,s}(x) = a < b = \varphi^L_{\sigma,s}(y).$$

Thus, $\varphi^L_{\sigma,s}$ is also order preserving from $\preceq_s$ to $\leq$.

[2] Note the difference with $\mathcal{A}^L_{\sigma,s} = \{a \in \mathcal{A} \mid \sigma(a) \in \mathcal{A}^+ s\}$.

We now prove the uniqueness. Assume that σ is left order preserving from $\preceq$ to $\leq$ and from $\preceq'$ to $\leq$ and let $x, y \in \mathcal{A}$ be such that $x \prec y$ and $y \prec' x$. If $s = s_\sigma(x, y)$, then $\varphi^L_{\sigma,s}(x) \neq \varphi^L_{\sigma,s}(y)$. Since σ is left order preserving from $\preceq$ to $\leq$, we have $\varphi^L_{\sigma,s}(x) < \varphi^L_{\sigma,s}(y)$. Since σ is also left order preserving from $\preceq'$ to $\leq$, we have the converse inequality, which is a contradiction. □

Example 67. Let us consider the morphism $\beta : 0 \mapsto 0, 1 \mapsto 01, 2 \mapsto 02, 3 \mapsto 032$ of Example 40 and let us build the order $\preceq$ such that β is left order preserving from $\preceq$ to $\leq$ where $3 < 0 < 2 < 1$. Using the notations of the previous proof, we have $3 \prec_2 2$ since $3 < 0$. We then have $0 \prec_\varepsilon 3 \prec_\varepsilon 2 \prec_\varepsilon 1$ since $0 < 2 < 1$, and $\preceq$ is given by the order $\preceq_\varepsilon$.

We say that a return morphism σ is *planar preserving* from $(\preceq^L, \preceq^R)$ to $(\leq^L, \leq^R)$ if it is left order preserving from $\preceq^L$ to $\leq^L$ and right order preserving from $\preceq^R$ to $\leq^R$. And it is $(\leq^L, \leq^R)$-*planar* if all of its initial factor are planar for $(\leq^L, \leq^R)$. Note that this definition is completely independent on the choice of the word w for which σ codes the return words.

The terminology planar preserving comes from the following results.

Proposition 68. *Let X be a shift space and σ be a return morphism planar preserving from $(\preceq^L, \preceq^R)$ to $(\leq^L, \leq^R)$. A word $v \in \mathcal{L}(X)$ is planar for $(\preceq^L, \preceq^R)$ if and only if every extended image of v under σ is planar for $(\leq^L, \leq^R)$.*

Proof. We prove that the graph $\mathcal{E}_X(v)$ has two crossing edges for $(\preceq^L, \preceq^R)$ if and only if there exists an extended image u such that $\mathcal{E}_{\sigma\cdot X}(u)$ has two crossing edges for $(\leq^L, \leq^R)$. Let (x_1, y_1) and (x_2, y_2) be two bi-extensions of v such that $x_1 \neq x_2$ and $y_1 \neq y_2$. If $s = s_\sigma(x_1, x_2)$ and $p = p_\sigma(y_1, y_2)$, we denote

$$x'_1 = \varphi^L_{\sigma,s}(x_1), \quad x'_2 = \varphi^L_{\sigma,s}(x_2), \quad y'_1 = \varphi^R_{\sigma,p}(y_1) \quad \text{and} \quad y'_2 = \varphi^R_{\sigma,p}(y_2).$$

By construction, (x'_1, y'_1) and (x'_2, y'_2) are two bi-extensions of the extended image $u := s\sigma(v)p$ and are such that $x'_1 \neq x'_2$ and $y'_1 \neq y'_2$. Remark that, for any such pair of bi-extensions of an extended image u' of v, it is possible to find a corresponding pair of bi-extensions of v.

To conclude, it suffices to prove that (x_1, y_1) and (x_2, y_2) are crossing edges in $\mathcal{E}_X(v)$ if and only if (x'_1, y'_1) and (x'_2, y'_2) are crossing edges in $\mathcal{E}_{\sigma\cdot X}(u)$. Assume that $x_1 \prec^L x_2$. Since σ is planar preserving, $\varphi^L_{\sigma,s}$ is order preserving from $\preceq^L$ to $\leq^L$ and we have $x'_1 <^L x'_2$. Moreover, $\varphi^R_{\sigma,p}$ is order preserving from $\preceq^R$ to $\leq^R$ thus

$$y_1 \prec^R y_2 \Leftrightarrow y'_1 <^R y'_2.$$

This ends the proof. □

Corollary 69. *Let X and Y be two dendric shifts such that $Y = \sigma \cdot X$ where σ is $(\leq^L, \leq^R)$-planar and planar preserving from $(\preceq^L, \preceq^R)$ to $(\leq^L, \leq^R)$. The shift X is planar for $(\preceq^L, \preceq^R)$ if and only if Y is planar for $(\leq^L, \leq^R)$.*

We can now prove a result similar to Theorem 1 but in the case of interval exchanges. This time, the characterization only uses one graph.

Definition 70. Let a, b be two distinct letters in $\mathcal{A}$. Let $\mathcal{S}$ be a set of dendric return morphisms from $\mathcal{A}^*$ to $\mathcal{A}^*$. The graph $\mathcal{G}_{IET}(\mathcal{S})$ is such that its vertices are the irreducible pairs of orders $(\leq^L, \leq^R)$ such that $a <^L b$, and there is an edge from $(\leq^L, \leq^R)$ to $(\preceq^L, \preceq^R)$ labeled by $\sigma \in \mathcal{S}$ if

(1) $(G(\leq^L), \sigma, G(\preceq^L))$ is an edge of $\mathcal{G}^L(\mathcal{S})$;
(2) $(G(\leq^R), \sigma, G(\preceq^R))$ is an edge of $\mathcal{G}^R(\mathcal{S})$;
(3) σ is $(\leq^L, \leq^R)$-planar;
(4) σ is planar preserving from $(\preceq^L, \preceq^R)$ to $(\leq^L, \leq^R)$, or from $((\preceq^L)^*, (\preceq^R)^*)$ to $(\leq^L, \leq^R)$, where $\leq^*$ denotes the reverse order of $\leq$.

By Lemma 66, the graph $\mathcal{G}_{IET}$ is deterministic in the sense that, for each vertex and each morphism, there is at most one edge labeled by the morphism leaving the vertex.

Theorem 71. *Let $\mathcal{S}$ be a family of dendric return morphisms from $\mathcal{A}^*$ to $\mathcal{A}^*$ and let X be a shift space having an $\mathcal{S}$-adic representation $\boldsymbol{\sigma} = (\sigma_n)_{n\geq 0}$. Then X is the coding of a regular interval exchange if and only if $\boldsymbol{\sigma}$ is primitive and labels an infinite path in the graph $\mathcal{G}_{IET}(\mathcal{S})$. Moreover,* *if this path starts in the pair* $(\leq^L, \leq^R)$, *then* X *is the coding of a regular interval exchange for the orders* $\binom{\leq^R}{\leq^L}$.

Proof. Let us assume that X is the coding of a regular interval exchange for the orders $\binom{\leq_0^R}{\leq_0^L}$, we can moreover assume that $a <_0^L b$, where a and b are the letters used to define $\mathcal{G}_{IET}(\mathcal{S})$. Then, σ_0 is $(\leq_0^L, \leq_0^R)$-planar, and by Lemma 66, there exist two orders $\leq_1^L$ and $\leq_1^R$ such that $a <_1^L b$ and σ is planar preserving from $(\leq_1^L, \leq_1^R)$ to $(\leq_0^L, \leq_0^R)$ or from $((\leq_1^L)^*, (\leq_1^R)^*)$ to $(\leq_0^L, \leq_0^R)$.

As $X_{\boldsymbol{\sigma}}^{(1)}$ is dendric, by Corollary 69, it is the coding of a regular interval exchange for the orders $\binom{\leq_1^R}{\leq_1^L}$. In particular, the pair $(\leq_1^L, \leq_1^R)$ is irreducible. Moreover, σ_0 labels an edge from $G(\leq_0^L)$ to $G(\leq_1^L)$ in $\mathcal{G}^L(\mathcal{S})$ and from $G(\leq_0^R)$ to $G(\leq_1^R)$ in $\mathcal{G}^R(\mathcal{S})$. Therefore, σ_0 labels an edge from $(\leq_0^L, \leq_0^R)$ to $(\leq_1^L, \leq_1^R)$ in $\mathcal{G}_{IET}(\mathcal{S})$. We then iterate this reasoning to show that $\boldsymbol{\sigma}$ labels an infinite path in $\mathcal{G}_{IET}(\mathcal{S})$.

Assume now that $\boldsymbol{\sigma}$ is primitive and labels a path $((\leq_n^L, \leq_n^R))_{n\geq 0}$ in $\mathcal{G}_{IET}(\mathcal{S})$. Since $\boldsymbol{\sigma}$ also labels an infinite path in $\mathcal{G}^L(\mathcal{S})$ and $\mathcal{G}^R(\mathcal{S})$, X is minimal dendric. Let us prove that it is planar for $(\leq_0^L, \leq_0^R)$. The initial factors of X are planar for $(\leq_0^L, \leq_0^R)$ since σ_0 is $(\leq_0^L, \leq_0^R)$-planar. Let $u \in \mathcal{L}(X)$ be a non-initial factor of X. There exist a unique $k > 0$ and a unique $v \in \mathcal{L}(X_{\boldsymbol{\sigma}}^{(k)})$ initial such that u is a descendant of v by $\sigma_0 \dots \sigma_{k-1}$ by Proposition 8. However, σ_k is $(\leq_k^L, \leq_k^R)$-planar thus v is planar for these orders. Iterating Proposition 68 k times allows us to conclude that u is planar for $(\leq_0^L, \leq_0^R)$. As this is true for any $u \in \mathcal{L}(X)$, X is the coding of a regular interval exchange for the orders $\binom{\leq_0^R}{\leq_0^L}$. □

As in the dendric case, we can reduce the number of vertices of this graph using permutation. As a first step, we can completely fix the right (or top) order. However we can reduce this graph even more. For example, on the alphabet $\{1, 2, 3, 4\}$, the irreducible pairs $\binom{1<2<3<4}{2<4<3<1}$ and $\binom{4<2<1<3}{1<2<3<4}$ are equivalent up to permutation even if they have the same top order. Indeed, $\binom{1<2<3<4}{2<4<3<1}$ is equivalent to $\binom{4<3<2<1}{3<1<2<4}$ which becomes $\binom{1<2<3<4}{2<4<3<1}$ with a simple permutation of the letters. In particular, this then gives a graph with 2 vertices on a ternary alphabet, as done in [28], and with 9 vertices for an alphabet of size 4.

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