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A NEW SOLUTION TO THE DYSON-SCHWINGER EQUATION FOR THE GLUON PROPAGATOR†

by

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Abstract

A new analytic solution is found for the Dyson-Schwinger equation for the gluon propagator in the axial gauge. This propagator $D_{\mu\nu}(q^2)$ has a cut rather than a pole as $q^2 \to 0$. It can be thought of as describing a gluon which is not confining but rather confined. This solution is shown to be relevant for processes in which soft gluons are exchanged, such as total and elastic cross sections, and constitutes a first step towards the building of the QCD pomeron.

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1. INTRODUCTION

Hadron scattering at low momentum transfer is remarkably well reproduced by Regge exchanges, which reduce at high-energy to the pomeron. This means that hadronic amplitudes are proportional to $s^{\alpha(t)}$ where $\alpha(t)$ is the pomeron trajectory. It is essentially linear $\alpha(t)=\alpha(0)+\alpha' t$ with its intercept $\alpha(0)\approx 1$ and its slope $\alpha'\approx 0.25$ GeV^{-2}. Also, quark counting and factorisation seem to hold to about 10% and suggest that the pomeron trajectory is a single pole. This behaviour is well reproduced in models where the pomeron is described by the exchange of a single object coupled to quarks via a γ_μ vertex and with the quantum numbers of the vacuum[1].

One then has to see how QCD can give rise to such an object. Low and Nussinov[2] suggested that the lowest order QCD construction possessing the correct quantum numbers was two gluon exchange. Explicit calculations[3-4] using such a model in the context of perturbative QCD are infrared finite. Yet, their exact predictions are dependent on the form assumed for the hadronic wave functions and more importantly, they cannot reproduce the details of the elastic differential distributions[5]. However, such perturbative estimates surprisingly give the correct order of magnitude for the total cross section. So the message from these is that perturbative QCD is not very far off, but the precise details of the process cannot be accounted for.

Landshoff and Nachtmann[6] showed that a process-independent smoothing of the gluon propagator near the origin would result in factorising hadronic amplitude and in the quark counting rule. Such a simple idea turned out to lead to a very successful phenomenology[7], predicting diffractive scattering, the gluon structure function and elastic scattering from the gluon condensate and the total cross section via two-gluon exchange.

The lowest order estimate would then give the bulk of the hadronic amplitudes. The s-dependence and the real part would be generated by higher order contributions. Perturbative QCD seems to have further problems here, as it produces a cut[8] or a series of poles[9] (depending on how one treats α_S), which violate factorisation. Yet, D. Ross has shown[10] that a modification of the gluon propagator at all orders similar to that proposed by Landshoff and Nachtmann could in principle solve these problems.

The question then arises whether such a behaviour can be obtained theoretically. We know gluons do not couple to colour singlets at infinite distances. This picture would make a stronger assumption, namely that gluons do not propagate at infinite distances. Can the gluon self-interaction prevent propagation to infinity, or equivalently, can it remove the pole in the propagator?

The simplest way to investigate this question is to use the Dyson-Schwinger equation for the gluon propagator. It is an exact result from field theory, and it can be truncated by making use of the Ward-Slavnov-Taylor identities (another exact result) into a closed form for the gluon propagator and longitudinal vertex. Baker, Ball and Zachariasen (BBZ) derived such a truncated form in the axial gauge[11]. The solution they found, behaving like $1/q^4$ near the origin, is more singular than the

perturbative one, and was interpreted as a *confining* propagator. Such a behaviour makes the above-mentioned problems worse, and does not seem to provide any insight into the problem of scattering.

As the equation is highly nonlinear, we expect it to have other solutions. It is indeed known, in the light cone gauge, to possess at least two solutions (both of them with a pole at the origin). Hints of a softer behaviour at the origin can be found in lattice calculations: both in the Landau[12] and in the axial[13] gauges, the gluon propagator seems to behave like that of a massive particle and therefore vanishes at large distances. We recently succeeded in showing that a softer solution does exist[14]. It is this solution - and its derivation - for which we found a simpler analytic parametrization, that will be described here.

2. THE BAKER-BALL-ZACHARIASEN EQUATION

We shall only sketch the main steps of the derivation of this truncated equation, as the details are already present in the literature[11]. One first assumes that the gluon propagator $D_{\mu\nu}(q^2)$ can be well approximated by looking at gluon interactions only. This can be thought of as a first step towards a more precise study in which quarks would be included. Secondly, one considers the gluon propagator in the axial gauge. Assumptions regarding ghosts are thus absent from the calculation. On the other hand, one has an extra kinematic variable n.q, and the propagator in principle can depend on it. Furthermore, in the axial gauge, the propagator in general depends on two independent scalar functions[15], and the Dyson-Schwinger equation can be written as a set of two coupled equations for these two functions. We choose to investigate the class of solutions that have the same spin structure as the free propagator and the same n.q dependence. In Euclidean space, they take the form:

$$D_{\mu\nu}(q^2) = -\frac{Z(q^2)}{q^2}\left(\delta_{\mu\nu} - \frac{q_\mu n_\nu + n_\mu q_\nu}{q.n} + \frac{q_\mu q_\nu\, n^2}{(q.n)^2}\right) \equiv Z(q^2)\, D^0_{\mu\nu}(q^2) \qquad (1)$$

and in this case, the inverse propagator is:

$$\Pi_{\mu\nu}(q^2) = -\frac{q^2}{Z(q^2)}\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) \qquad (2)$$

BBZ show that eq. (1), implying $n_\mu D^{\mu\nu}=0$, leads to a Dyson-Schwinger equation for $n_\mu n_\nu \Pi^{\mu\nu}$ that involves only Z and the triple gluon vertex. The latter can be decomposed into a longitudinal and a transverse part. Ward-Slavnov-Taylor identities relate the longitudinal part to the gluon propagator, and the transverse part is assumed to be negligible. One then gets a closed equation for the gluon propagator. One Wick rotates it into Euclidean space and further simplify it using the approximations of Schoenmaker[16] for the angular integrals. The equation then can be schematically written as:

$$\frac{1}{Z(q^2)} - 1 = \alpha_S^0 \int_0^{q^2} dy\, F^l_<(q^2,y) + \frac{1}{Z(q^2)} F^q_<(q^2,y) + \alpha_S^0 \int_{q^2}^{\infty} dy\, F^l_>(q^2,y) + \frac{1}{Z(q^2)} F^q_>(q^2,y) \qquad (3)$$

where < (>) refer to different functional forms, α_S^0 is the bare coupling and F^q is quadratic in Z and F^l linear. The explicit forms of these kernels can be found in reference 16.

The equation then contains both quadratic and logarithmic divergences. The linear divergences are q^2 independent, and BBZ have shown[11] that they correspond to $\Pi_{\mu\nu}(0)$ being a surface term, or equivalently:

$$\frac{1}{Z(0)} \sim \int_0^{\infty} dk^2 \frac{d}{dk^2}(k^2 Z(k^2)) \qquad (4)$$

In the case of a solution softer than a pole near the origin, one sees that the contribution from this term comes entirely from the perturbative region: a gauge-invariant regularization would give zero as an answer. Thus one should assume that $\Pi_{\mu\nu}(0) = 0$. Subtracting both sides of eq. (3) by their value at zero then removes the quadratic divergence. Note the in the case of a $1/q^4$ solution, one needs an extra subtraction to ensure $\Pi_{\mu\nu}(0) = 0$. We thus do not solve the same equation as in reference 11.

To deal with the logarithmic divergence, one needs to perform charge renormalization by choosing a renormalization point m where, by definition: $Z(q^2) \equiv Z(m^2)\ Z_R(q^2)$, or $Z_R(m^2) \equiv 1$. To absorb all the $Z(m^2)$ dependence, the renormalized coupling needs to be defined as:

$$\alpha_s(m^2) = \frac{\alpha_S^0 Z(m^2)}{1+\alpha_S^0 Z(m^2)\left[\int_0^{q^2} F^l_< + \int_0^{q^2} F^l_>\right]_{Z=Z_R}} \qquad (5)$$

With this definition, and after some rearranging of terms, the net effect of charge renormalization is to replace all Z by Z_R, the coupling α_S^0 by $\alpha_S(m^2)$ and to subtract from the right hand side of the equation its value at $q^2=m^2$. This leads to the final equation that we shall solve, and that is spelled out in reference 14. Note that the renormalized coupling, which is given by eq.(5), is not the usual axial gauge running coupling constant, but instead corresponds to another subtraction scheme, which we shall call the BBZ scheme. With this in mind, we can now turn to the description of the solution we have obtained.

3. A SOLUTION WITHOUT A POLE

The BBZ equation relies on a series of approximations that are supposed to become exact in the infrared regime. The equation however involves integrals to infinite momenta, so one needs to use an ansatz that for all q^2, but we shall trust it only for low q^2 values. In our previous work[14], we showed that one could use a functional form for low q^2 and match it by hand to the asymptotic behaviour advocated by BBZ. This procedure has the apparent advantage of defining the onset of the asymptotic region. The solution however has none of the analytic properties a propagator should have. We show here another form that solves the equation at small q^2 to a better accuracy and that is analytic everywhere:

$$D(q^2) = \frac{Z(q^2)}{q^2} = \frac{m^{-2}}{C_1\left[\frac{q^2}{m^2}\right]^{\gamma_1} + C_2\left[\frac{q^2}{m^2}\right]^{\gamma_2} + L\, q^2 \log\left(\lambda_1 \frac{q^2}{m^2} + \lambda_2\right)} \qquad (6)$$

As we showed previously[14], the exact behaviour at $q^2 \to \infty$ does not influence the infrared part very much, and we take for it the true physical one. Although we have in mind solutions softer than a pole, in the axial gauge we saw that $\Pi_{\mu\nu}(0)=0$. *This means that a cut is the minimum singularity that a solution can have: the propagator must be infinite at $q^2=0$, although this infinity does not need to be a pole.* We thus restrict one of the γ_i to be positive in eq. (6).

Our strategy to solve the equation is to input Z_{in} given by eq. (6) into the right hand side of the renormalized eq.(3). The left hand side gives then Z_{out} and we choose the input parameters (including the coupling $\alpha_S(m^2)$) so that they minimize the difference between Z_{in} and Z_{out}. We give in Table 1 the values of the parameters for such a solution.

parameter	value	parameter	value
$\alpha_S(m^2)$	0.70	L	0.59
C_1	0.88	C_2	−0.95
γ_1	0.22	γ_2	0.86
λ_1	2.1	λ_2	4.1

TABLE 1
Set of parameters for our solution.

We show in figure 1 Z_{in} and Z_{out}, for which the maximum difference between $q^2/m^2 = 0$ and 1 is 1.4%.

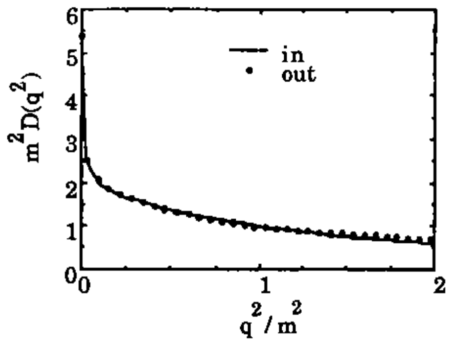

FIGURE 1
Plot of the propagator as a function of the squared momentum q^2 scaled by m^2. The solid line is our ansatz (3) with the parameters of Table 1. The circles are the output obtained when this propagator is inserted into the right hand side of the equation.

Note that the exact behaviour as $q^2 \to 0$ cannot be determined numerically with great accuracy: one can only say that the propagator near zero behaves like $(q^2)^{-\gamma_1}$. Although left free, the power γ_1 always converges to a value between 0.01 and 0.3. More singular propagators are clearly ruled out. Different solutions can be found, for which the couplings $\alpha_S(m^2)$ takes values in the range 0.2 to 1.2. We give here the solution which has a coupling $\alpha_S(m^2)$ closest to the one previously published[14].

The fit is established for $q^2<m^2$, and one can see from figure 1 that although the assumed asymptotic form is not the one prescribed by BBZ, our solution automatically satisfies eq. (3) up to values of q^2 beyond m^2. Clearly, Z_{out} falls off less rapidly than Z_{in} at large q^2, as expected. The onset of perturbative QCD, which we artificially fixed at m in our previous analysis, is now less well defined as the solution goes smoothly from the nonperturbative to the perturbative region: m does not have any special meaning: it is simply the renormalization point and also sets the scale of the problem. One can think of $m/\sqrt{\lambda_1}$ as the effective Λ_{QCD}. However, as was previously mentioned, the coupling renormalization is different from the usual one, and so the value of Λ_{QCD} is not directly related to that of the $\overline{MS}$ scheme.

The analytic properties of this solution should not be misleading. It is determined numerically in the t channel and it is known only to some accuracy. Within that precision, it is always possible to find an analytic function that will be negligible in the t channel but that will grow very fast outside of it. Such a function can always be added to our solution, forbidding the extrapolation to the s channel. To obtain a solution there, one will need to write the truncated Dyson-Schwinger equation and solve it in both channels simultaneously.

Z being dimensionless, it is a function of q^2/m^2, so that a rescaling of all the momenta produces another solution. To decide what the actual value of m is, we need to calculate some dimensional quantity, such as the total pp cross section, and we shall examine this in the next section.

4. SOME PHENOMENOLOGICAL CONSEQUENCES

We shall now give a rough estimate of the scale of the problem by studying the total and elastic cross sections. Although the gluon propagator is known, the quark-gluon vertices have not been calculated yet. Until a proper evaluation of these has been done, this part of our work has to be considered as tentative.

The main assumptions that enter this estimate are as follows: firstly, at very high energy the main role of confinement is to modify the gluon propagator. Otherwise, confinement effects can be absorbed in quark wave functions. Secondly, the lowest order Born approximation is used. This is admittedly very crude, and one expects large corrections from higher order terms (i.e., the exchanged pomeron should be built out of ladders involving nonperturbative gluons). Finally, as the gluon propagator gets modified, the quark propagator as well as the quark-gluon and gluon-gluon vertices will also receive nonperturbative contributions. Here, as a first approximation these are neglected, and the quark-gluon vertex is taken to be γ_μ. These questions will be treated in a future publication, but the present estimate will show that the phenomenology, even at lowest order, is roughly right for a range of values of m.

Nevertheless, the use of our solution, which includes some of the nonperturbative effects of QCD, should be better than the usual purely perturbative ansätze. A special feature of the axial gauge is that the BBZ equation determines the value of the coupling, which is kept fixed, and the running is included in the propagator. This totally avoids the Landau pole. As we have assumed the physical asymptotic form for the ansatz (6), the matching to perturbative QCD is automatic.

We shall now calculate the elastic cross section $d\sigma/dt$ for pp collisions, and its corollary, the total cross section. We use the formalism of Gunion and Soper[3] to evaluate two-gluon exchange, giving the following amplitude:

$$A\ (q^2) = i\, s\, \frac{8}{9} n_p^2 \alpha_S^2\ (T_1 - T_2) \qquad (7)$$

with

$$T_1 = \int_0^s d^2k\, D_g(\frac{q}{2}+k)\, D_g(\frac{q}{2}-k)\, [G_p(q,0)]^2$$

$$T_2 = \int_0^s d^2k\, D_g(\frac{q}{2}+k)\, D_g(\frac{q}{2}-k)\, [G_p(q,k-\frac{q}{2})] \times [2\, G_p(q,0) - G_p(q,k-\frac{q}{2})]$$

where $n_p=3$ is the number of quarks in the proton, $D_g(q)=Z(q^2)/q^2$ is the gluon propagator and $G_p(q,l)$ is a convolution of proton wave functions: $G_p(q,l)=\int d^2p\, d\alpha\, \Psi(\alpha,p)\, \Psi(\alpha,p-l-\alpha\, q)$ with the wavefunction $\Psi(\alpha,p)$ being the amplitude for the quark to have transverse momentum p and fraction α of the longitudinal momentum. T_1 comes from diagrams where both gluons are attached to the same quark within one proton, whereas T_2 comes from diagrams in which the gluons are attached to different quarks. $G_p(q,0)$ is equal to the electromagnetic form factor, $F_1(q^2)$, for which we shall use the parametrization of reference 1. For proton-proton scattering, the second term cannot be related to F_1. To estimate its magnitude, the wavefunction is assumed to be peaked at $\alpha=1/3$ and the form for G_p extends that found for pion-pion scattering:

$$G_p(q,l) = F_1(q^2 + \frac{n_p^2}{4}|4k^2-q^2|) \tag{8}$$

This is by no means a unique prescription and guesses in the wavefunction lead to uncertainties of at least a factor of two in the cross section.

From this amplitude, one can get the total cross section $\sigma_T=A(0)/is$ and the elastic cross section $d\sigma/dt=|A(t)|/(16\pi s^2)$. The latter behaves like $e^{-B(t)}$ and the logarithmic slope B also can be estimated. We show in figures 2 and 3 our results for the total cross section and the slope B at t=0 as a function of the renormalization scale m.

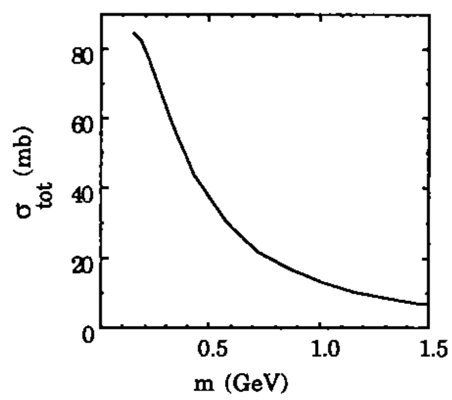

FIGURE 2
The total cross section for pp scattering as a function of m.

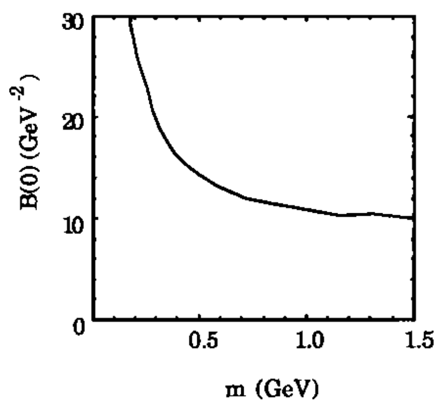

FIGURE 3
The logarithmic slope B at t=0 for the differential cross section in pp elastic scattering.

The pomeron fit of Donnachie and Landshoff[1] gives $\sigma_{tot}=22.7\, s^{0.08}$. If one assumes that higher-order corrections are responsible for the s dependence, $(s/m)^{0.08}$, the above estimates agree when m is about 0.75 GeV. This leads to a slope B(0) of 11.5 GeV^{-2}, consistent with data, B_{exp}= 12 GeV^{-2}. This value of m would correspond to a value of $\Lambda_{QCD}=m/\sqrt{\lambda_1}$ of about 0.5 GeV. This is close to the value of the momentum subtraction scheme, which is similar to the one we are using here.

In perturbative QCD, the total cross section is of the right order of magnitude, but the shape of the elastic one is wrong, in particular $B(0)=\infty$: both T_1 and T_2 diverge, and although their difference gives a finite total cross section, this is not true for their logarithmic derivative B(0). On the other hand, Landshoff and Nachtmann[6] supposed that T_2 is negligible, and regulated the other terms by assuming a finite propagator near the origin. The propagator discussed here gives intermediate results: using the optimum value of m=0.75 GeV, $T_1/(T_1+T_2) = 0.23$.

This suppression leads to a reasonable agreement with the quark counting rule. For the pion, we assume that the wavefunction is peaked around $\alpha=1/2$ and take the estimate of the pion form factor from reference 4. We find $\sigma_{p\pi}/\sigma_{pp} \approx 0.54$. A more precise agreement with the experimental number of 0.62 would require some tuning of the wavefunctions. Note that if we assume, as in reference 4, that the proton wavefunction is peaked at $\alpha=1/2$, the ratio becomes 0.6.

5. CONCLUSION

The Dyson-Schwinger equation in the truncated approximation proposed by Baker Ball and Zachariasen[11] and simplified by Schoenmaker[16] has been shown to possess a solution softer than a pole. In its present form, the solution is an analytic function that matches smoothly to perturbative QCD at high momentum transfer. Lowest order phenomenology leads to a value for the renormalization scale m, which we take as a typical scale in the problem, of about 1 GeV.

These results explain why perturbative calculations have been very successful down to momentum transfers of the order of 1 GeV, and simultaneously incorporate some of the phenomenological ideas of Landshoff and Nachtmann[6] and put them on a theoretical footing. Besides, the method presented here provides a theoretical framework that can easily be extended to explore nonperturbative effects on the quark propagators and the QCD vertices, and can in principle lead to a consistent framework for the evaluation of nonperturbative effects in high energy amplitudes. One can also, using analyticity argument, address the question of the gluon and quark propagators in the s channel, and thus estimate their density of states, as well as their fragmentation properties.

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