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Generalized S^ν spaces

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Abstract

In this work, we introduce a generalization of the S^ν spaces underlying a multifractal formalism for non-concave spectra. We prove among other things that the essential topological properties of the S^ν spaces can be transposed in this context; in particular, these new spaces are metric.

KEYWORDS

admissible sequences, S^ν spaces, law of the iterated logarithm

1 | INTRODUCTION

The aim of a multifractal analysis is to scrutinize very erratic signals. They are present in many practical situations such as turbulence [29], physiology [34], biology [31], or geoscience [9]. The regularity of these signals can change widely from a point to another. Information concerning the pointwise smoothness of signals can be grasped via its Hölder spectrum, which relies on the Hölder spaces. A locally bounded function f belongs to $\Lambda^\alpha(x_0)$ (with $\alpha \geq 0$ and $x_0 \in \mathbf{R}^n$) if there exist a constant C and a polynomial P_{x_0} of degree less than α such that

$$|f(x) - P_{x_0}(x)| < C|x - x_0|^\alpha, \tag{1}$$

in a neighborhood of x_0. The Hölder exponent of f at x_0 is defined as

$$h_f(x_0) = \sup\{\alpha \geq 0 : f \in \Lambda^\alpha(x_0)\}.$$

As usual, f belongs to the uniform space $\Lambda^\alpha(\mathbf{R}^n)$ if, for any x_0, Equation (1) is satisfied for any x, the constant C being uniform [22]. For highly irregular functions f, one tries to globally characterize the local regularity of f using the spectrum of regularity d_f:

$$d_f : [0, \infty] \to \{-\infty\} \cup [0, n] \quad h \mapsto \dim_{\mathcal{H}}\{x \in \mathbf{R}^n : h_f(x) = h\},$$

where $\dim_{\mathcal{H}}$ stands for the Hausdorff dimension. Such an approach is particularly suitable for numerical signals, for which the estimation of the Hölder exponent at every point is not conceivable.

In practice, there is no method to systematically obtain the spectrum of regularity d_f, but there exist many heuristic methods, called multifractal formalisms, to estimate this spectrum (see, e.g., [16, 18, 30, 33]). Most of them are based on the same argument and rely on the decomposition of f into its wavelet coefficients. Let us briefly introduce this approach. In most of the cases, one tries, explicitly or not, to compute the function

$$\eta(q) = \sup\{s : f \in B^{s/q}_{q,\infty}(\mathbf{R}^n)\},$$

relying on the Besov spaces $B^{s/q}_{q,\infty}(\mathbf{R}^n)$, using wavelets or an equivalent approach such as box-counting techniques. Considering a heuristic argument, one can expect for the equality

$$d_f(h) = \inf_q \{hq - \eta(q)\} + n \tag{2}$$

to hold in most of the cases.

In practice, these methods give, at best, an upper bound [14], that is, if the locally bounded function f belongs to $\Lambda^\varepsilon(\mathbf{R}^n)$ for some $\varepsilon > 0$, one has

$$d_f(h) \leq \inf_{q \geq q_0} \{hq - \eta(q)\} + n,$$

where q_0 is the value satisfying $\eta(q_0) = n$. Moreover, since d_f is obtained through a Legendre transform from equality (2), these methods only lead, at best, to the concave hull of the regularity spectrum, although there exist functions whose associated spectrum is not concave (see, e.g., [21]).

To circumvent this problem, a method involving the types of histogram of discrete wavelet coefficients was introduced [2]. This method relies on sequence spaces: the S^ν spaces. Since these spaces are the starting point of this work, let us sketch some definitions. Let us set

$$\Lambda = \{(i, j, k) : 1 \leq i < 2^n,\ j \in \mathbf{N},\ k \in \{0, \dots, 2^j - 1\}^n\},$$

and consider sequences $(c^{(i)}_{j,k})_{(i,j,k)\in\Lambda}$ whose indices vary in Λ. For the sake of clarity, we shall omit any reference to the index i in the following; for instance, the set $\{c_{j,k} : (j,k) \in E\}$ has to be understood as the set $\{c^{(i)}_{j,k} : (j,k) \in E,\ 1 \leq i < 2^n\}$. Finally, the sequence $(c^{(i)}_{j,k})_{(i,j,k)\in\Lambda}$ will be denoted by c. Throughout this paper, ν refers to a right-continuous increasing function for which there exists $\alpha_{\min} \in \mathbf{R}$ such that $\nu(\alpha) = -\infty$ if $\alpha < \alpha_{\min}$ and $\nu(\alpha) \in [0, n]$ if $\alpha \geq \alpha_{\min}$. With these notations being fixed, one defines the S^ν space as follows:

$$S^\nu = \big\{c : \forall \alpha \in \mathbf{R}\ \forall \varepsilon > 0\ \forall C > 0$$
$$\exists J > 0\ \forall j \geq J,\ \#E_j(C, \alpha)(c) \leq 2^{(\nu(\alpha)+\varepsilon)j}\big\},$$

where

$$E_j(C, \alpha)(c) = \{k : |c_{j,k}| \geq C2^{-\alpha j}\}. \tag{3}$$

If one considers the sequence c as wavelet coefficients, it can be shown that these spaces allow to deal with non-concave spectra of regularity both in theory and practice [3, 21].

The starting point of this paper is the following: the Besov spaces $B^s_{p,q}(\mathbf{R}^n)$ can be generalized by replacing the dyadic scales appearing in the definition by more general sequences; such spaces have applications in different fields (see, e.g., [6, 10, 11, 17, 25, 32]). For example, concerning the Hölder spaces $\Lambda^\alpha(\mathbf{R}^n)$ (let us recall that we have $\Lambda^\alpha(\mathbf{R}^n) = B^\alpha_{\infty,\infty}(\mathbf{R}^n)$), such a generalization leads to the following definition [23, 26]: given a sequence $\sigma = (\sigma_j)_{j\in\mathbf{N}}$ satisfying some general conditions, a locally bounded function f belongs to the space $\Lambda^{\sigma,\alpha}(\mathbf{R}^n)$ if there exists a constant C such that

$$\inf_{P \in \mathcal{P}_\alpha} \|f - P\|_{L^\infty(x+2^{-j}B)} < C\sigma_j,$$

for any $j \in \mathbf{N}$ and any $x \in \mathbf{R}^n$, where B denotes the unit ball centered at the origin and $\mathcal{P}_\alpha$ the set of polynomials of degree at most α. One can do the same with the S^ν spaces: the sequence $(2^{-\alpha j})_{j\in\mathbf{N}}$ occurring in Equation (3) can be replaced by a sequence $(\sigma_j)_{j\in\mathbf{N}}$.

In this work, we show that this generalization is mathematically sound. The first step consists in checking that the topological properties holding for the usual S^ν spaces are preserved (see Section 2). Moreover, it is crucial to show that different wavelet bases give rise to identical space (see Section 3). Since the S^ν spaces are strongly connected to Besov spaces, it is also natural to link the spaces defined here with the generalised Besov spaces (see Section 4).

These new spaces have been designed to detect the law of the iterated logarithm [19]; the efficiency of this approach has been established in [20, 24]. A random variable X satisfies the law of the iterated logarithm if for every t, there exists a positive random variable C such that

$$|X(t+h)-X(t)| \leq C|h|^{1/2}\sqrt{\log\log|h|^{-1}},$$

for h small enough on an event of probability 1. For example, many financial models are derived from the Brownian motion (e.g., the geometric Brownian motion used in the Black and Scholes model [13], the Hull and White one-factor model [5], etc.) and thus satisfy such an inequality. As $\sigma_j = 2^{-j/2}\sqrt{\log\log 2^j}$ defines an admissible sequence, the corresponding generalized S^ν spaces should prove helpful to check for the existence of logarithmic corrections in order to test if these models are well-founded in practical situations, as shown in [20]. Such results will be statistically validated in a forthcoming work. These concepts could also help to formulate conjectures about the existence of corrections in the Hölderian behavior of functions [20, 24].

2 | DEFINITION AND FIRST PROPERTIES

Let us recall that $c = (c_{j,k}) = (c^{(i)}_{j,k})_{(i,j,k)\in\Lambda}$ stands here for a sequence of complex numbers.

Definition 1. For any $\alpha \in \mathbf{R}$, let $\sigma^{(\alpha)} = (\sigma^{(\alpha)}_j)_{j\in\mathbf{N}}$ be a sequence of positive real numbers. We define

$$\begin{aligned} S^{\nu,\sigma^{(\cdot)}} = \big\{ c : \forall\alpha\in\mathbf{R}\ \forall\varepsilon>0\ \forall C>0 \\ \exists J>0\ \forall j\geq J,\ \#E_j(C,\sigma^{(\alpha)})(c) \leq 2^{(\nu(\alpha)+\varepsilon)j} \big\}, \end{aligned}$$

where

$$E_j(C,\sigma^{(\alpha)})(c) = \left\{ k : |c_{j,k}| \geq C\sigma^{(\alpha)}_j \right\}.$$

To define a topology on these vector spaces, we use the same strategy as for the usual S^ν spaces [3]. For $\alpha \in \mathbf{R}$ and $\beta \in \mathbf{R}\cup\{-\infty\}$, we first define the metric spaces $(E(\sigma^{(\alpha)},\beta), d_{\sigma^{(\alpha)},\beta})$ by

$$E(\sigma^{(\alpha)},\beta) = \big\{ c : \exists C, C' > 0\ \#E_j(C,\sigma^{(\alpha)}) \leq C' 2^{\beta j} \text{ for any } j\in\mathbf{N} \big\}$$

and set

$$d_{\sigma^{(\alpha)},\beta}(c,d) = \inf\big\{ C + C' : C, C' \geq 0\ \#E_j(C,\sigma^{(\alpha)})(c-d) \leq C' 2^{\beta j} \text{ for any } j\in\mathbf{N} \big\}.$$

Let us give the basic properties of these spaces.

Proposition 1. *We have the following properties:*

1. *the space $E(\sigma^{(\alpha)},\beta)$ is a vector space,*
2. *the sum is a continuous operation in $(E(\sigma^{(\alpha)},\beta), d_{\sigma^{(\alpha)},\beta})$, while the product is not necessarily continuous,*
3. *the metric $d_{\sigma^{(\alpha)},\beta}$ is invariant by translation and satisfies the inequality*

$$d_{\sigma^{(\alpha)},\beta}(\lambda c, 0) \leq \sup\{1,|\lambda|\} d_{\sigma^{(\alpha)},\beta}(c,0),$$

for any $\lambda \in \mathbf{C}$,

4. *if $\beta' \leq \beta$ and if there exists $J \in \mathbf{N}$ such that $\sigma^{(\alpha')}_j \leq \sigma^{(\alpha)}_j$ for any $j \geq J$, then $E(\sigma^{(\alpha')},\beta')$ is included in $E(\sigma^{(\alpha)},\beta)$,*

5. suppose that

$$\sigma_j^{(\alpha')}/\sigma_j^{(\alpha)} \to 0 \quad as \quad j \to +\infty$$

and $\beta' < \beta$. If the sequence $(\lambda_m)_{m\in\mathbf{N}}$ converges to λ in $\mathbf{C}$ and if $(c^{(m)})_{m\in\mathbf{N}}$ is a sequence of $E(\sigma^{(\alpha)}, \beta)$ which converges to $c \in E(\sigma^{(\alpha')}, \beta')$ for $d_{\sigma^{(\alpha)},\beta}$, then the sequences $(\lambda_m c^{(m)})_{m\in\mathbf{N}}$ converges to λc for $d_{\sigma^{(\alpha)},\beta}$.

Proof. The first four points are straightforward; let us remark that the product is not necessarily continuous because it was already the case for the S^ν spaces [3]. It remains to prove the last point. From the properties of the metric $d_{\sigma^{(\alpha)},\beta}$, we obtain

$$d_{\sigma^{(\alpha)},\beta}(\lambda_m c^{(m)}, \lambda c) \le \sup\{1, |\lambda_m|\} d_{\sigma^{(\alpha)},\beta}(c^{(m)} - c, 0) + d_{\sigma^{(\alpha)},\beta}((\lambda_m - \lambda)c, 0).$$

So, it suffices to show that $d_{\sigma^{(\alpha)},\beta}((\lambda_m - \lambda)c, 0)$ converges to 0 as $m \to +\infty$. There exist $C, C' \ge 0$ such that

$$\#\left\{k : |c_{j,k}| \ge C\sigma_j^{(\alpha')}\right\} \le C' 2^{\beta' j},$$

for any $j \in \mathbf{N}$. Let us set $\varepsilon > 0$ and $M \in \mathbf{N}$ such that $|\lambda_m - \lambda| < 1$, for any $m \ge M$. There exists $J \ge 0$ such that $C' 2^{\beta' j} \le \varepsilon 2^{\beta j}$ and $C\sigma_j^{(\alpha')} \le \varepsilon \sigma_j^{(\alpha)}$, for any $j \ge J$. In this case, we have

$$\begin{aligned}\#\left\{k : |(\lambda_m - \lambda)c_{j,k}| \ge \varepsilon\sigma_j^{(\alpha)}\right\} &\le \#\left\{k : |c_{j,k}| \ge C\sigma_j^{(\alpha')}\right\} \\ &\le \varepsilon 2^{\beta j},\end{aligned}$$

for any $j \ge J$. Besides, we can suppose that $|(\lambda_m - \lambda)c_{j,k}| < \varepsilon\sigma_j^{(\alpha)}$ for any $j < J$, $k \in \{0, \dots, 2^j - 1\}^n$ and $m \ge M$. With this hypothesis, we get

$$d_{\sigma^{(\alpha)},\beta}((\lambda_m - \lambda)c, 0) \le 2\varepsilon,$$

for any $m \ge M$. □

Proposition 2. *The space $E(\sigma^{(\alpha)}, \beta)$ is complete.*

Proof. Let $(c^{(m)})_{m\in\mathbf{N}}$ be a Cauchy sequence of $E(\sigma^{(\alpha)}, \beta)$. Let us first show that the sequence $(c_{j,k}^{(m)})_{m\in\mathbf{N}}$ is a pointwise Cauchy sequence for any fixed j, k. Let us set $\varepsilon > 0$, $j \in \mathbf{N}$ and $\varepsilon' = \inf\{\varepsilon(\sigma_j^{(\alpha)})^{-1}, 2^{-\beta j - 1}\}$. Since $(c^{(m)})_{m\in\mathbf{N}}$ is a Cauchy sequence, there exists $J > 0$ such that, for any $p, q \ge J$,

$$\#\left\{k : |c_{j,k}^{(p)} - c_{j,k}^{(q)}| \ge \varepsilon'\sigma_j^{(\alpha)}\right\} \le \varepsilon' 2^{\beta j},$$

that is, $|c_{j,k}^{(p)} - c_{j,k}^{(q)}| \le \varepsilon$ for any k.

Therefore, for any j, k, there exists $c_{j,k}$ such that $c_{j,k}^{(m)} \to c_{j,k}$ as $m \to +\infty$. It remains to prove that $c^{(m)} \to c$ in $E(\sigma^{(\alpha)}, \beta)$ as $m \to +\infty$. If $\varepsilon > 0$, there exists M such that

$$\forall j \ge 0,\ \forall p, q \ge M,\ \#\left\{k : |c_{j,k}^{(p)} - c_{j,k}^{(q)}| \ge \varepsilon\sigma_j^{(\alpha)}\right\} \le \varepsilon 2^{\beta j},$$

which implies

$$\forall p, q \ge M,\ c^{(q)} \in \left\{d : \#\left\{k : |c_{j,k}^{(p)} - d_{j,k}| \ge \varepsilon\sigma_j^{(\alpha)}\right\} \le \varepsilon 2^{\beta j}\ \forall j \ge 0\right\}.$$

As these sets are closed for the pointwise convergence, we get that

$$\forall p \geq M,\ c \in \left\{ d : \#\left\{ k : |c_{j,k}^{(p)} - d_{j,k}| \geq \varepsilon\sigma_j^{(\alpha)} \right\} \leq \varepsilon 2^{\beta j}\ \forall j \geq 0 \right\}.$$

We thus have $c \in E(\sigma^{(\alpha)}, \beta)$ (and $c^{(m)}$ converges to c in $E(\sigma^{(\alpha)}, \beta)$). □

The next theorem gives a link between the space $S^{\nu,\sigma^{(\cdot)}}$ and the spaces $E(\sigma^{(\alpha)}, \beta)$. This will allow us to define a topology on $S^{\nu,\sigma^{(\cdot)}}$.

Theorem 1. *Suppose that $\alpha < \alpha'$ implies*

$$\sigma_j^{(\alpha')}/\sigma_j^{(\alpha)} \to 0 \quad as \quad j \to +\infty.$$

For any sequence $(\alpha_l)_{l\in\mathbf{N}}$ dense in $\mathbf{R}$ and any sequence $(\varepsilon_m)_{m\in\mathbf{N}}$ of strictly positive real numbers which converges to 0, we have

$$S^{\nu,\sigma^{(\cdot)}} = \bigcap_{m\in\mathbf{N}} \bigcap_{l\in\mathbf{N}} E(\sigma^{(\alpha_l)}, \nu(\alpha_l) + \varepsilon_m).$$

Proof. We directly have that $S^{\nu,\sigma^{(\cdot)}}$ is included in $E(\sigma^{(\alpha_l)}, \nu(\alpha_l) + \varepsilon_m)$, for any $m, l \in \mathbf{N}$. Let us show that

$$\bigcap_{m\in\mathbf{N}} \bigcap_{l\in\mathbf{N}} E(\sigma^{(\alpha_l)}, \nu(\alpha_l) + \varepsilon_m)$$

is a subset of $S^{\nu,\sigma^{(\cdot)}}$. Let c be an element of the intersection and let $\alpha \in \mathbf{R}$, $\varepsilon > 0$, $C > 0$. For any $m, l \in \mathbf{N}$, there exist $C_{l,m}, C'_{l,m} \geq 0$ such that

$$\#\left\{ k : |c_{j,k}| \geq C_{l,m}\sigma_j^{(\alpha_l)} \right\} \leq C'_{l,m} 2^{(\nu(\alpha_l)+\varepsilon_m)j},$$

for any $j \in \mathbf{N}$. Let $m \in \mathbf{N}$ be such that $2\varepsilon_m < \varepsilon$.

If $\nu(\alpha) = -\infty$ let us take $l \in \mathbf{N}$ such that $\alpha_l \in (\alpha, \alpha_{\min})$. If $\nu(\alpha) \in \mathbf{R}$, let us take $l \in \mathbf{N}$ such that $\alpha_l > \alpha$ and $\nu(\alpha) < \nu(\alpha_l) < \nu(\alpha) + \varepsilon_m$. In any case, there exists $J_{l,m} \in \mathbf{N}$ such that $C_{l,m}\sigma_j^{(\alpha_l)} \leq C\sigma_j^{(\alpha)}$ and $C'_{l,m} 2^{(\nu(\alpha_l)+\varepsilon_m)j} \leq 2^{(\nu(\alpha)+\varepsilon)j}$ for any $m \geq M$ and $j \geq J_{l,m}$. We thus have

$$\begin{aligned} \#\left\{ k : |c_{j,k}| \geq C\sigma_j^{(\alpha)} \right\} &\leq \#\left\{ k : |c_{j,k}| \geq C_{l,m}\sigma_j^{(\alpha_l)} \right\} \\ &\leq 2^{(\nu(\alpha)+\varepsilon)j}, \end{aligned}$$

for any $j \geq J_{l,m}$, which implies $c \in S^{\nu,\sigma^{(\cdot)}}$. □

Let us recall that if (E_m, d_m) is a metric space for any $m \in \mathbf{N}$, then

$$d : (e, f) \mapsto \sum_{m=1}^{+\infty} 2^{-m} \frac{d_m(e, f)}{1 + d_m(e, f)}$$

is a metric on $E = \cap_{m\in\mathbf{N}} E_m$. Moreover, for any $m \in \mathbf{N}$, the projection $i_m : (E, d) \to (E_m, d_m)$ is continuous and the topology induced by d is the weakest topology on E which satisfies this property. Finally, a sequence is a Cauchy sequence (resp. converges to e) in (E, d) if and only if for any $m \in \mathbf{N}$, it is also a Cauchy sequence (resp. it also converges to e) in (E_m, d_m).

Theorem 2. *Under the hypothesis of Theorem 1, if we set*

$$d_{m,l} = d_{\sigma^{(\alpha_l)},\nu(\alpha_l)+\varepsilon_m},$$

then the application

$$d : (c,d) \in S^{\nu,\sigma^{(\cdot)}} \times S^{\nu,\sigma^{(\cdot)}} \mapsto \sum_{m=1}^{+\infty}\sum_{l=1}^{+\infty} \frac{1}{2^{m+l}} \frac{d_{m,l}(c,d)}{1+d_{m,l}(c,d)}$$

is a metric on $S^{\nu,\sigma^{(\cdot)}}$. This application is invariant by translation and the space $(S^{\nu,\sigma^{(\cdot)}}, d)$ is a complete topological vector space. The induced topology is independent of the sequences $(\alpha_l)_{l\in\mathbf{N}}$ and $(\varepsilon_m)_{m\in\mathbf{N}}$.

Proof. We will prove that the product is continuous and that the topology is complete and independent of the sequences $(\alpha_l)_{l\in\mathbf{N}}$ and $(\varepsilon_m)_{m\in\mathbf{N}}$, the other properties being straightforward.

First, let us show that if $\lambda_u \to \lambda$ in $\mathbf{C}$ and if $c^{(u)} \to c$ in $S^{\nu,\sigma^{(\cdot)}}$, then $\lambda_u c^{(u)} \to \lambda c$ in $E(\sigma^{(\alpha_l)}, \nu(\alpha_l)+\varepsilon_m)$, for any $l,m \in \mathbf{N}$. Let us set $l,m\in\mathbf{N}$; if $\nu(\alpha_l) = -\infty$, then there exists $p\in\mathbf{N}$ such that $\alpha_p \in (\alpha_l, \alpha_{\min})$. If $\nu(\alpha_l)\in\mathbf{R}$, there exist $p,q\in\mathbf{N}$ such that $\alpha_p > \alpha_l$ and $\nu(\alpha_p)+\varepsilon_q < \nu(\alpha_l)+\varepsilon_m$. In any case, we have $c^{(u)} \to c$ in $E(\sigma^{(\alpha_l)}, \nu(\alpha_l)+\varepsilon_m)$ and $c \in E(\sigma^{(\alpha_p)}, \nu(\alpha_p)+\varepsilon_q)$. Using Proposition 1, we have $\lambda_u c^{(u)} \to \lambda c$ in $E(\sigma^{(\alpha_l)}, \nu(\alpha_l)+\varepsilon_m)$, which implies that the product is continuous.

Now, let us prove that the topology is independent of the sequences $(\alpha_l)_{l\in\mathbf{N}}$ and $(\varepsilon_m)_{m\in\mathbf{N}}$. We have to show that if $c^{(u)} \to c$ in $S^{\nu,\sigma^{(\cdot)}}$, then $c^{(u)} \to c$ in $E(\sigma^{(\alpha)}, \nu(\alpha)+\varepsilon)$, for any $\alpha\in\mathbf{R}$ and $\varepsilon>0$. Let α be a real number and $\varepsilon > 0$. Since $c^{(u)} \to c$ in $S^{\nu,\sigma^{(\cdot)}}$, there exist $C_{l,m}, C'_{l,m} \geq 0$ such that

$$\#\left\{k : |c^{(u)}_{j,k} - c_{j,k}| \geq C_{l,m}\sigma^{(\alpha_l)}_j\right\} \leq C'_{l,m} 2^{(\nu(\alpha_l)+\varepsilon_m)j},$$

for any $j\in\mathbf{N}$. Let δ be a strictly positive number and take $m\in\mathbf{N}$ such that $2\varepsilon_m < \varepsilon$. Using the same arguments as in the proof of Theorem 1, there exist $l\in\mathbf{N}$ and $J_{l,m}\in\mathbf{N}$ such that $C_{l,m}\sigma^{(\alpha_l)}_j \leq \delta\sigma^{(\alpha)}_j$ and

$$C'_{l,m} 2^{(\nu(\alpha_l)+\varepsilon_m)j} \leq \delta 2^{(\nu(\alpha)+\varepsilon)j},$$

for any $j \geq J_{l,m}$. We get

$$\begin{aligned}\#\left\{k : |c^{(u)}_{j,k} - c_{j,k}| \geq \delta\sigma^{(\alpha)}_j\right\} &\leq \#\left\{k : |c^{(u)}_{j,k} - c_{j,k}| \geq C_{l,m}\sigma^{(\alpha_l)}_j\right\}\\ &\leq \delta 2^{(\nu(\alpha)+\varepsilon)j},\end{aligned}$$

for any $j \geq J_{l,m}$. We have thus obtained $d_{\sigma^{(\alpha)},\nu(\alpha)+\varepsilon}(c^{(u)}_{j,k}, c_{j,k}) \leq 2\delta$.

Finally, let us show that the topology is complete. Let $(c^{(u)})_{u\in\mathbf{N}}$ be a Cauchy sequence of $S^{\nu,\sigma^{(\cdot)}}$. This sequence is also a Cauchy sequence of $E(\sigma^{(\alpha_l)}, \nu(\alpha_l)+\varepsilon_m)$, for any $l,m\in\mathbf{N}$. By Proposition 2, there exists a sequence $c^{(m,l)}$ such that $(c^{(u)})_{u\in\mathbf{N}}$ converges to $c^{(m,l)}$ in $E(\sigma^{(\alpha_l)}, \nu(\alpha_l)+\varepsilon_m)$, for any l,m. Similar arguments as in the first part of the proof of Proposition 2 show that if a sequence $(c^u)_{u\in\mathbf{N}}$ converges to c in $E(\sigma^{(\alpha)},\beta)$, then $(c^{(u)}_{j,k})_{u\in\mathbf{N}}$ converges to $c_{j,k}$ in $\mathbf{C}$ for any j,k. This implies the equality $c^{(m,l)} = c^{(m',l')}$, valid for any $m,m',l,l'\in\mathbf{N}$. □

Let us now give another definition of $S^{\nu,\sigma^{(\cdot)}}$ by introducing a new notion.

Definition 2. The generalized profile of a sequence c is defined by

$$\nu_{c,\sigma^{(\cdot)}} : \alpha\in\mathbf{R} \mapsto \lim_{\varepsilon\to 0^+}\limsup_{j\to+\infty} \frac{\log \#E_j(1,\sigma^{(\alpha+\varepsilon)})(c)}{\log 2^j}.$$

This definition is well-founded if we suppose that for any $\alpha<\alpha'$ there exists $J\in\mathbf{N}$ such that $\sigma^{(\alpha')}_j \leq \sigma^{(\alpha)}_j$ for any $j\geq J$.

The next proposition gives a few properties of the function $\nu_{c,\sigma^{(\cdot)}}$ and leads to an alternative definition of $S^{\nu,\sigma^{(\cdot)}}$.

Proposition 3. *Suppose that $\alpha < \alpha'$ implies $\sigma_j^{(\alpha')}/\sigma_j^{(\alpha)} \to 0$ as $j \to +\infty$. We have the following properties:*

1. *the function $\nu_{c,\sigma^{(\cdot)}}$ is right-continuous and increasing; moreover, we have $\nu_{c,\sigma^{(\cdot)}}(\alpha) \in [0,n] \cup \{-\infty\}$,*
2. *the constant 1 appearing in the definition of $\nu_{c,\sigma^{(\cdot)}}$ is arbitrary,*
3. *a sequence c belongs to $S^{\nu,\sigma^{(\cdot)}}$ if and only if $\nu_{c,\sigma^{(\cdot)}}(\alpha) \leq \nu(\alpha)$ for any $\alpha \in \mathbf{R}$,*
4. *if for any $\alpha < \beta$, we have $\sigma_j^{(\beta)} < \sigma_j^{(\alpha)}$ for any $j \in \mathbf{N}$, then there exists $c \in S^{\nu,\sigma^{(\cdot)}}$ such that $\nu_{c,\sigma^{(\cdot)}} = \nu$.*

Proof. The two first properties are immediate. Let c be a sequence of $S^{\nu,\sigma^{(\cdot)}}$ and $\alpha \in \mathbf{R}$. For any $\varepsilon > 0$, there exists $J \in \mathbf{N}$ such that

$$\#E_j(1,\sigma^{(\alpha+\varepsilon)})(c) \leq 2^{(\nu(\alpha+\varepsilon)+\varepsilon)j},$$

for any $j \geq J$; we thus have $\nu_{c,\sigma^{(\cdot)}}(\alpha) \leq \nu(\alpha)$.

Let c be a sequence such that $\nu_{c,\sigma^{(\cdot)}} \leq \nu$ and let $\alpha \in \mathbf{R}$, $\varepsilon > 0$ and $C > 0$. There exist $\varepsilon' > 0$, $J \in \mathbf{N}$ such that for any $\varepsilon'' < \varepsilon'$ and $j \geq J$, we have

$$\frac{\log \#E_j(1,\sigma^{(\alpha+\varepsilon'')}(c)}{\log 2^j} \leq \nu(\alpha) + \varepsilon,$$

which gives $\#E_j(1,\sigma^{(\alpha+\varepsilon'')}) \leq 2^{(\nu(\alpha)+\varepsilon)j}$. We can suppose that $\sigma_j^{(\alpha+\varepsilon'')} \leq C\sigma_j^{(\alpha)}$ for any $j \geq J$, so that $\#E_j(C,\sigma^{(\alpha)})(c) \leq 2^{(\nu(\alpha)+\varepsilon)j}$.

Now, let us construct a sequence $c \in S^{\nu,\sigma^{(\cdot)}}$ such that $\nu_{c,\sigma^{(\cdot)}} = \nu$. We can work in the one-dimensional case by setting $c_{j,k}^{(i)} = 0$ for $i \neq 1$; let us thus suppose that $n = 1$. Let $(\alpha_m)_{m\in\mathbf{N}}$ be a dense sequence of $[\alpha_{\min}, +\infty)$ and let $(c^{(m)})_{m\in\mathbf{N}}$ be a sequence defined by

$$c_{j,k}^{(m)} = \begin{cases} \sigma_{j+m}^{(\alpha_m)} & \text{for } \lfloor 2^{\nu(\alpha_m)j} \rfloor \text{ values of } k \\ 0 & \text{else} \end{cases}.$$

Next, we define c by

$$c_{j,k} = \begin{cases} 0 & \text{if } k = 0 \\ c_{l,k-2^l}^{(j-l)} & \text{if } k \in \{2^l, \dots, 2^{l+1}-1\} \text{ with } l \in \{0,\dots,j\} \end{cases}.$$

Let us set $\alpha < \alpha_{\min}$. We have

$$\nu_{c,\sigma^{(\cdot)}}(\alpha) = \lim_{\varepsilon\to 0^+} \limsup_{j\to+\infty} \frac{\log \#\left\{k : |c_{j,k}| \geq \sigma_j^{(\alpha)}\right\}}{\log 2^j}$$

and for any j,k, there exists $m \in \{0,\dots,j\}$ such that

$$\begin{aligned} c_{j,k} &= c_{j-m,k-2^{j-m}}^{(m)} \\ &= (\sigma_j^{(\alpha_m)} \text{ or } 0). \end{aligned}$$

So, using the last hypothesis, we get $\nu_{c,\sigma^{(\cdot)}}(\alpha) = -\infty$. To conclude, it remains to prove that $\nu_{c,\sigma^{(\cdot)}}(\alpha_m) = \nu(\alpha_m)$ for any $m \in \mathbf{N}$.

Let m be a natural number and let $\varepsilon > 0$. For any $j \geq m$, we have $\sigma_j^{(\alpha_m)} \geq \sigma_j^{(\alpha_m+\varepsilon)}$ and $c_{j,k}$ is equal to $\sigma_j^{(\alpha_m)}$ for $\lfloor 2^{\nu(\alpha_m)(j-m)} \rfloor$ values of k. So, we obtain

$$\limsup_{j\to+\infty} \frac{\log \lfloor 2^{\nu(\alpha_m)(j-m)} \rfloor}{\log 2^j} \leq \limsup_{j\to+\infty} \frac{\log \#\left\{k : |c_{j,k}| \geq \sigma_j^{(\alpha_m+\varepsilon)}\right\}}{\log 2^j},$$

which gives $\nu(\alpha_m) \leq \nu_{c,\sigma^{(\cdot)}}(\alpha_m)$. Besides, we have, for any $j \geq m$,

$$\begin{aligned}\#\left\{k : |c_{j,k}| \geq \sigma_j^{(\alpha_m+\varepsilon)}\right\} &\leq \sum_{\substack{l=1 \\ \alpha_l \leq \alpha_m+\varepsilon}}^{j} 2^{\nu(\alpha_l)(j-l)} \\ &\leq 2^{\nu(\alpha_m+\varepsilon)j} \sum_{\substack{l=1 \\ \alpha_l \leq \alpha_m+\varepsilon}}^{j} 2^{-l\nu(\alpha_l)} \\ &\leq 2^{\nu(\alpha_m+\varepsilon)j}\, j,\end{aligned}$$

which allows us to write $\nu_{c,\sigma^{(\cdot)}}(\alpha_m) \leq \nu(\alpha_m)$. □

3 | ROBUSTNESS

Until now, we have considered the spaces $S^{\nu,\sigma^{(\cdot)}}$ as sequence spaces, but we should keep in mind that such a sequence represents wavelet coefficients and thus a function. To associate these spaces to functions, we have to check that the definition does not depend on the chosen wavelet basis.

Let us briefly evoke the notion of wavelet basis. Under some general assumptions (for more precisions, see, e.g., [8, 27, 28]), there exist a function ϕ and $2^n - 1$ functions $(\psi^{(i)})_{1\leq i<2^n}$, called wavelets, such that

$$\{\phi(x-k) : k \in \mathbf{Z}^n\} \cup \left\{\psi^{(i)}(2^j x - k) : 1 \leq i < 2^n, k \in \mathbf{Z}^n, j \in \mathbf{N}\right\}$$

form an orthogonal basis of $L^2(\mathbf{R}^n)$. Any function $f \in L^2(\mathbf{R}^n)$ can be decomposed as follows,

$$f(x) = \sum_{k\in\mathbf{Z}^n} C_k \phi(x-k) + \sum_{j\in\mathbf{N}} \sum_{k\in\mathbf{Z}^n} \sum_{1\leq i<2^n} c_{j,k}^{(i)} \psi^{(i)}(2^j x - k),$$

where

$$c_{j,k}^{(i)} = 2^{nj} \int_{\mathbf{R}^n} f(x) \psi^{(i)}(2^j x - k)\, dx$$

and

$$C_k = \int_{\mathbf{R}^n} f(x)\phi(x-k)\, dx.$$

Let us remark that we do not choose the $L^2(\mathbf{R}^n)$ normalization for the wavelets, but rather an $L^\infty(\mathbf{R}^n)$ normalization, which is better fitted to the study of the Hölderian regularity. On the torus $\mathbf{R}^n/\mathbf{Z}^n$, we will use the periodized wavelets

$$\psi_p^{(i)}(2^j x - k) = \sum_{l\in\mathbf{Z}^n} \psi^{(i)}(2^j(x-l) - k) \quad (j \in \mathbf{N},\ k \in \{0, \dots, 2^j - 1\}^n)$$

to form a basis of the one-periodic functions on $\mathbf{R}^n$ which locally belong to $L^2(\mathbf{R}^n)$ [7, 8]. The corresponding coefficients $c_{j,k}^{(i)}$ are naturally called the periodized wavelet coefficients.

We will also need the notion of admissible sequence (see, e.g., [23]).

Definition 3. A sequence $\sigma = (\sigma_j)_{j\in\mathbf{N}}$ of real positive numbers is called admissible if there exists a positive constant C such that

$$C^{-1}\sigma_j \leq \sigma_{j+1} \leq C\sigma_j,$$

for any $j \in \mathbf{N}$.

We set

$$\underline{\Theta}_j = \inf_{k\in\mathbf{N}} \frac{\sigma_{j+k}}{\sigma_k} \quad \text{and} \quad \overline{\Theta}_j = \sup_{k\in\mathbf{N}} \frac{\sigma_{j+k}}{\sigma_k}$$

and define the lower and upper Boyd indices as follows:

$$\underline{s}(\sigma) = \lim_{j\to+\infty} \frac{\log \underline{\Theta}_j}{\log 2^j} \quad \text{and} \quad \overline{s}(\sigma) = \lim_{j\to+\infty} \frac{\log \overline{\Theta}_j}{\log 2^j}.$$

If σ is an admissible sequence, for any $\varepsilon > 0$, there exists a positive constant C such that

$$C^{-1}2^{j(\underline{s}(\sigma)-\varepsilon)} \leq \frac{\sigma_{j+k}}{\sigma_k} \leq C2^{j(\overline{s}(\sigma)+\varepsilon)},$$

for any $j, k \in \mathbf{N}$.

Let us now state definitions and properties related to important classes of linear operators in the context of wavelet bases [15, 28].

Definition 4. For $\gamma > 0$, we set

$$w_\gamma(j, k; j', k') = \frac{2^{-(\gamma+n+1)|j-j'|}}{(1 + 2^{\inf\{j,j'\}}|2^{-j}k - 2^{-j'}k'|)^{\gamma+n+1}},$$

for any $j, j' \in \mathbf{N}$, $k \in \{0, \dots, 2^j - 1\}$ and $k' \in \{0, \dots, 2^{j'} - 1\}$.

We say that

$$A = \big(A(j, k; j', k')\big)_{j,j',k,k'}$$

belongs to $\mathcal{A}_\gamma$ if and only if there exists $C \geq 0$ such that

$$|A(j, k; j', k')| \leq Cw_\gamma(j, k; j', k'),$$

for any j, j', k, k'. The infimum of these constants is denoted by $\|A\|_\gamma$.

The matrix A is almost diagonal (resp. quasidiagonal) if $A \in \mathcal{A}_\gamma$ for any $\gamma > 0$ (resp. A is invertible and $A, A^{-1} \in \mathcal{A}_\gamma$ for any $\gamma > 0$).

It can be shown (see [28]) that the matrix of the operator which maps a C^∞ orthonormal wavelet basis to another C^∞ orthonormal wavelet basis is quasidiagonal.

Definition 5. A property $\mathscr{P}$ is linear robust if the following properties hold:

- the set of c such that $\mathscr{P}(c)$ holds is a vector space;
- if $\mathscr{P}(c)$ holds then, for any almost diagonal operator A, $\mathscr{P}(Ac)$ holds.

Definition 6. A property $\mathscr{P}$ is robust if the following property holds: if $\mathscr{P}(c)$ holds then, for any quasidiagonal operator A, $\mathscr{P}(Ac)$ holds.

Let us first generalize the following classical result of [28]: if $\gamma > |\alpha|$ and $A \in \mathcal{A}_\gamma$ then there exists a constant $\tilde{C}$ (which only depends on the dimension n) such that

$$|c_{j,k}| \leq C2^{-\alpha j}\ \forall j,k \Rightarrow |(Ac)_{j,k}| \leq C\tilde{C}\|A\|_\gamma 2^{-\alpha j}\ \forall j,k. \tag{4}$$

Lemma 1. *Let σ be an admissible sequence and γ be a strictly positive number such that $\gamma > \max\{-\underline{s}(\sigma)-1, \overline{s}(\sigma)\}$. If there exists a constant $C > 0$ such that*

$$|c_{j,k}| \leq C\sigma_j,$$

for any j,k, then there exists a constant $\tilde{C}$ which depends on γ, σ and the dimension n such that for any matrix $A \in \mathcal{A}_\gamma$, we have

$$|(Ac)_{j,k}| \leq C\tilde{C}\|A\|_\gamma \sigma_j,$$

for any j,k.

Proof. Let us set $\varepsilon = \gamma - \max\{-\underline{s}(\sigma)-1, \overline{s}(\sigma)\} > 0$. Since the sequence σ is admissible, there exists a constant $C_{\gamma,\sigma} > 0$ such that

$$\sigma_j \leq C_{\gamma,\sigma} 2^{-(j'-j)(\underline{s}(\sigma)-\varepsilon)}\sigma_{j'},$$

for any $j \leq j'$ and

$$\sigma_j \leq C_{\gamma,\sigma} 2^{(j-j')(\overline{s}(\sigma)+\varepsilon)}\sigma_{j'},$$

for any $j' \leq j$. Let A be a matrix of $\mathcal{A}_\gamma$ and choose a constant D such that $D > \|A\|_\gamma$. Let us remark that

$$|(Ac)_{j',k'}| \leq DC\left(\sum_{j=0}^{j'}\sum_{k=0}^{2^j-1} w_\gamma(j,k;j',k')\sigma_j + \sum_{j=j'+1}^{+\infty}\sum_{k=0}^{2^j-1} w_\gamma(j,k;j',k')\sigma_j\right).$$

If $j \leq j'$, we have

$$\begin{aligned}\sum_{k=0}^{2^j-1} w_\gamma(j,k;j',k') &= \sum_{k=0}^{2^j-1}\left(\frac{1}{1+|k-2^{-(j'-j)}k'|}\right)^{\gamma+n+1} 2^{-(j'-j)(\gamma+n+1)}\\ &\leq \sum_{k=0}^{+\infty}\left(\frac{1}{1+|k-2^{-(j'-j)}k'|}\right)^{n+1} 2^{-(j'-j)(\gamma+n+1)}\\ &\leq C_n 2^{-(j'-j)(\gamma+n+1)},\end{aligned}$$

where C_n is a positive constant that only depends on the dimension n.

If $j > j'$, we have

$$\begin{aligned}\sum_{k=0}^{2^j-1} w_\gamma(j,k;j',k') &= \sum_{k=0}^{2^j-1}\left(\frac{1}{1+|2^{-(j-j')}k-k'|}\right)^{\gamma+n+1} 2^{-(j-j')(\gamma+n+1)}\\ &\leq \sum_{k=0}^{2^j-1}\frac{1}{2^{j-j'}}\left(\frac{1}{1+|2^{-(j-j')}k-k'|}\right)^{n+1} 2^{-(j-j')(\gamma+n)}\\ &\leq C'_n 2^{-(j-j')(\gamma+n)},\end{aligned}$$

where C'_n is a positive constant only depending on the dimension n.

Therefore, there exists a constant $C''_n > 0$ only depending on the dimension n such that

$$|(Ac)_{j',k'}| \leq CC''_n C_{\gamma,\sigma} D\sigma_{j'},$$

for any j', k'. □

Let us remark that the constant $C_{\gamma,\sigma}$ is equal to 1 if $\sigma_j = 2^{-\alpha j}$, which implies that the previous result is a generalization of Equation (4).

The next theorem shows the robustness of the $S^{\nu,\sigma^{(\cdot)}}$ spaces.

Theorem 3. *Suppose that $\alpha < \alpha'$ implies*

$$\sigma_j^{(\alpha')}/\sigma_j^{(\alpha)} \to 0 \quad as \quad j \to +\infty.$$

If for any $\alpha \in \mathbf{R}$, the sequence $\sigma^{(\alpha)}$ is admissible, then $S^{\nu,\sigma^{(\cdot)}}$ is a linear robust space. Besides, for any $c \in S^{\nu,\sigma^{(\cdot)}}$, the function $\nu_{c,\sigma^{(\cdot)}}$ is robust, that is, $\nu_{c,\sigma^{(\cdot)}} = \nu_{Ac,\sigma^{(\cdot)}}$ for any quasidiagonal matrix A.

Proof. Let A be an almost diagonal matrix and take $c \in S^{\nu,\sigma^{(\cdot)}}$. Let us prove that Ac belongs to $S^{\nu,\sigma^{(\cdot)}}$. Let $\alpha \in \mathbf{R}$, $\varepsilon > 0$ and $C > 0$.

If $\alpha < \alpha_{\min}$, then let α' be an element of $(\alpha, \alpha_{\min})$. Since $c \in S^{\nu,\sigma^{(\cdot)}}$, there exists $C' > 0$ such that $|c_{j,k}| \leq C'\sigma_j^{(\alpha')}$, for any j, k. By Lemma 1, there exists a constant $\tilde{C} > 0$ depending on A, α' and the dimension n such that $|(Ac)_{j,k}| \leq C'\tilde{C}\sigma_j^{(\alpha')}$, for any j, k. By hypothesis, there exists $J \in \mathbf{N}$ such that $|(Ac)_{j,k}| < C\sigma_j^{(\alpha)}$ for any $j \geq J$.

It remains to examine the case $\alpha \geq \alpha_{\min}$. Let γ be a strictly positive number such that

$$\gamma > \max\left\{-\underline{s}(\sigma^{(\alpha)}) - 1, \overline{s}(\sigma^{(\alpha)})\right\}$$

and let C' be the constant $\tilde{C}\|A\|_\gamma$ from Lemma 1. For any $j \in \mathbf{N}$, let us define the set

$$Q_j = \left\{k : |c_{j,k}| \geq \frac{C}{2C'}\sigma_j^{(\alpha)}\right\}.$$

Let us remark that we have $Ac = Ac^{(1)} + Ac^{(2)}$, where

$$c_{j,k}^{(1)} = \begin{cases} c_{j,k} & \text{if } k \in Q_j \\ 0 & \text{else} \end{cases} \quad \text{and} \quad c_{j,k}^{(2)} = \begin{cases} c_{j,k} & \text{if } k \notin Q_j \\ 0 & \text{else.} \end{cases},$$

and that $|c_{j,k}^{(2)}| < C/(2C')\,\sigma_j^{(\alpha)}$, for any j, k. Therefore, using Lemma 1, we get $|(Ac^{(2)})_{j,k}| \leq C/2\,\sigma_j^{(\alpha)}$ for any j, k, which implies

$$\#\left\{k : |(Ac)_{j,k}| \geq C\sigma_j^{(\alpha)}\right\} \leq \#\left\{k : |(Ac^{(1)})_{j,k}| \geq \frac{C}{2}\sigma_j^{(\alpha)}\right\}.$$

As in [15], let us define the δ-neighborhood $N^\delta(j,k)$ ($\delta > 0$) of $[k/2^j, (k+1)/2^j)$ as the set of the couples (j', k') such that

$$\begin{cases} |j - j'| \leq \delta j \\ \left|\dfrac{k}{2^j} - \dfrac{k'}{2^{j'}}\right| \leq 2^{2\delta j} 2^{-j} \end{cases}.$$

Let us recall that if $(j',k') \notin N^\delta(j,k)$ then

$$w_{2\delta^{-2}}(j,k;j',k') \leq w_{\delta^{-2}}(j,k;j',k')2^{-j\delta^{-1}}.$$

Let us take (j,k) such that for any $k' \in Q_{j'}$, we have $(j',k') \notin N^\delta(j,k)$. If $\alpha' < \alpha_{\min}$, for any constant C'' large enough and any constant δ small enough, we get

$$\begin{aligned}
|(Ac^{(1)})_{j,k}| &\leq \sum_{j'\in\mathbf{N}}\sum_{k'\in Q_{j'}} \|A\|_{2\delta^{-2}} w_{2\delta^{-2}}(j,k;j',k')|c_{j',k'}| \\
&\leq \|A\|_{2\delta^{-2}} 2^{-j\delta^{-1}} \sum_{j'\in\mathbf{N}}\sum_{k'\in Q_{j'}} w_{\delta^{-2}}(j,k;j',k')|c_{j',k'}| \\
&\leq \|A\|_{2\delta^{-2}} 2^{-j\delta^{-1}} \sum_{j'\in\mathbf{N}}\sum_{k'\in Q_{j'}} w_{\delta^{-2}}(j,k;j',k')C''\sigma_{j'}^{(\alpha')} \\
&\leq \|A\|_{2\delta^{-2}} 2^{-j\delta^{-1}} C'' \sum_{j'\in\mathbf{N}}\sum_{k'=0}^{2^{j'}-1} w_{\delta^{-2}}(j,k;j',k')\sigma_{j'}^{(\alpha')}.
\end{aligned}$$

Using the proof of the previous lemma, there exists a constant C''' depending on δ, α' and the dimension n such that

$$\sum_{j'\in\mathbf{N}}\sum_{k'=0}^{2^{j'}-1} w_{\delta^{-2}}(j,k;j',k')\sigma_{j'}^{(\alpha')} \leq C'''\sigma_j^{(\alpha')}.$$

Since the sequences $\sigma^{(\alpha)}$ and $\sigma^{(\alpha')}$ are admissible, there exists $s > 0$ such that $\sigma^{(\alpha')}/\sigma^{(\alpha)} \leq 2^{sj}$. So, there exists $J \in \mathbf{N}$ such that

$$\begin{aligned}
|(Ac^{(1)})_{j,k}| &\leq \|A\|_{2\delta^{-2}} C''C''' 2^{-j(\delta^{-1}-s)}\sigma_j^{(\alpha)} \\
&\leq \frac{C}{4}\sigma_j^{(\alpha)},
\end{aligned}$$

for any $j \geq J$. We thus have

$$\begin{aligned}
\#\left\{k : |(Ac^{(1)})_{j,k}| \geq \frac{C}{2}\sigma_j^{(\alpha)}\right\} &\leq \#\{k : \exists (j',k') \in N^\delta(j,k),\ k' \in Q_{j'}\} \\
&\leq \sum_{j' : |j-j'|\leq\delta j}\sum_{k'\in Q_{j'}} \#\left\{k : \left|\frac{k}{2^j} - \frac{k'}{2^{j'}}\right| \leq 2^{2\delta j}2^{-j}\right\} \\
&\leq \sum_{j' : |j-j'|\leq\delta j}\sum_{k'\in Q_{j'}} (2^{2\delta j+1}+1)^n \\
&\leq \sum_{j' : |j-j'|\leq\delta j} \#Q_{j'}\,(2^{2\delta j+1}+1)^n,
\end{aligned}$$

for any $j \geq J$. Since c belongs to $S^{\nu,\sigma^{(\cdot)}}$, we have $\#Q_{j'} \leq 2^{(\nu(\alpha)+\varepsilon/2)j'}$ for j' large enough. We get

$$\begin{aligned}
\#\left\{k : |(Ac^{(1)})_{j,k}| \geq \frac{C}{2}\sigma_j^{(\alpha)}\right\} &\leq 2\delta j 2^{(\nu(\alpha)+\varepsilon/2)j(1+\delta)}2^{3\delta jn} \\
&\leq 2^{(\nu(\alpha)+\varepsilon/2)j}2^{(1+\nu(\alpha)+\varepsilon/2+3n)\delta j} \\
&\leq 2^{(\nu(\alpha)+\varepsilon)j},
\end{aligned}$$

for any $j \geq J$ (for large enough J). We thus have shown that the space $S^{\nu,\sigma^{(\cdot)}}$ is linear robust. The second part of the theorem is obtained by using Proposition 3 and the fact that c belongs to $S^{\nu_{c,\sigma^{(\cdot)}},\sigma^{(\cdot)}}$. □

4 | SOME CONNECTIONS WITH GENERALIZED BESOV SPACES

Classical Besov spaces $B^s_{p,q}(\mathbf{R}^n)$ were introduced at the end of the 1950s to fill the gaps between the Hölder–Zygmund spaces and the Sobolev spaces [4, 35]. In the context of the S^ν spaces, one works with periodic Besov spaces $B^s_{p,q}([0,1]^n)$. These spaces can be characterized with wavelet coefficients [28]. The connections between the Besov spaces and the S^ν spaces are given by the following theorem [3]: if $b^s_{p,q}$ are the discrete counterparts of the $B^s_{p,q}([0,1]^n)$ spaces (see Definition 7) then we have

$$S^\nu \subseteq \bigcap_{p>0}\bigcap_{\varepsilon>0} b_{p,\infty}^{\frac{\eta(p)-\varepsilon}{p}},$$

where $\eta(p) = \inf_{\alpha\geq\alpha_{\min}}\{\alpha p - \nu(\alpha) + n\}$. Moreover, the inclusion becomes an equality if and only if ν is concave.

The Besov spaces have been generalized with the help of the admissible sequences (see, e.g., [12]). A characterization with wavelet coefficients is given in [1]. It is thus natural to study the connections between these spaces and the $S^{\nu,\sigma^{(\cdot)}}$ spaces.

First, let us give the definition of the discrete counterparts of the generalized Besov spaces.

Definition 7. Let σ be an admissible sequence and $0 < p, q \leq \infty$. The discrete counterpart of the generalized Besov space $B^\sigma_{p,q}([0,1]^n)$ is defined by

$$b^\sigma_{p,q} = \left\{c : \left(\sum_{i\in\{0,\dots,2^n-1\},j\in\mathbf{N}} (\sigma_j 2^{-jn/p})^q \left(\sum_{k\in\{0,\dots,2^j-1\}^d} |c_{j,k}|^p\right)^{q/p}\right)^{1/q} < \infty\right\}$$

with the usual modification if $p = \infty$ and/or $q = \infty$.

Let us remark that $b^\sigma_{p,q} = b^s_{p,q}$ if $\sigma_j = 2^{sj}$. The next theorem gives a condition under which the $S^{\nu,\sigma^{(\cdot)}}$ spaces are included in an intersection of generalized Besov spaces.

Theorem 4. *For any $\alpha \in \mathbf{R}$, let $\sigma^{(\alpha)}$ be an admissible sequence and let us suppose that*

- *$\alpha < \alpha'$ implies $\sigma^{(\alpha')}_j/\sigma^{(\alpha)}_j \to 0$ as $j \to +\infty$,*
- *$\bar{s}(\sigma^{(\alpha)}) \to -\infty$ as $\alpha \to +\infty$.*

For any $p > 0$, let $\theta^{(p)}$ be an admissible sequence. We have

$$S^{\nu,\sigma^{(\cdot)}} \subseteq \bigcap_{p>0}\bigcap_{\varepsilon>0} b_{p,\infty}^{(\theta^{(p)}_j 2^{-j\varepsilon/p})_j}$$

if and only if for any $p, \varepsilon > 0$ and for any $\alpha \geq \alpha_{\min}$, there exists $C > 0$ such that

$$\theta^{(p)}_j 2^{-j\varepsilon/p} \leq C 2^{jn/p} 2^{-j\nu(\alpha)/p} (\sigma^{(\alpha)}_j)^{-1}, \tag{5}$$

for any j.

Proof. First, let us suppose that the inclusion is satisfied. For any $\alpha \geq \alpha_{\min}$, let $c^{(\alpha)}$ be a sequence defined by

$$c^{(\alpha)} = \begin{cases} \sigma_j^{(\alpha)} & \text{for } \lfloor 2^{\nu(\alpha)j} \rfloor \text{ values of } k \\ 0 & \text{else} \end{cases}.$$

We directly have that $c^{(\alpha)}$ belongs to $S^{\nu,\sigma^{(\cdot)}}$ and thus, for any $p, \varepsilon > 0$, there exists $C > 0$ such that

$$\theta_j^{(p)} 2^{-j\varepsilon/p} 2^{-jn/p} \left(\sum_k |c_{j,k}^{(\alpha)}|^p \right)^{1/p} < C,$$

for any j. From the definition of the sequence $c^{(\alpha)}$, we obtain

$$\theta_j^{(p)} 2^{-j\varepsilon/p} \leq C 2^{jn/p} 2^{-j\nu(\alpha)/p} (\sigma_j^{(\alpha)})^{-1},$$

for any j.

Now, let us suppose that Equation (5) is satisfied and let us show the required inclusion. Let c be a sequence belonging to $S^{\nu,\sigma^{(\cdot)}}$ and let $p, \varepsilon > 0$. Since $\overline{s}(\sigma^{(\alpha)}) \to -\infty$ as $\alpha \to +\infty$, there exists $\alpha' \geq \alpha_{\min}$ such that the sequence $(\theta_j^{(p)} 2^{-j\varepsilon/p} \sigma_j^{(\alpha')})_j$ is bounded. We have

$$\begin{aligned} &\left(\theta_j^{(p)}\right)^p 2^{-j\varepsilon} 2^{-jn} \sum_k |c_{j,k}|^p \\ &\leq \left(\theta_j^{(p)}\right)^p 2^{-j\varepsilon} 2^{-jn} \left(\sum_{k : |c_{j,k}| < \sigma_j^{(\alpha')}} |c_{j,k}|^p + \sum_{k : |c_{j,k}| \geq \sigma_j^{(\alpha')}} |c_{j,k}|^p \right) \\ &\leq \left(\theta_j^{(p)} 2^{-j\varepsilon/p} \sigma_j^{(\alpha')}\right)^p + \left(\theta_j^{(p)}\right)^p 2^{-j\varepsilon} 2^{-jn} \sum_{k : |c_{j,k}| \geq \sigma_j^{(\alpha')}} |c_{j,k}|^p, \end{aligned}$$

for any $j \in \mathbf{N}$. It remains to prove that the second term of the previous inequality is bounded.

Let β be a real number smaller than $\alpha_{\min}$ and let J be such that $|c_{j,k}| < \sigma_j^{(\beta)}$ for any $j \geq J$. We have

$$\left\{ k : |c_{j,k}| \geq \sigma_j^{(\alpha')} \right\} = \left\{ k : \sigma_j^{(\beta)} > |c_{j,k}| \geq \sigma_j^{(\alpha')} \right\},$$

for any $j \geq J$. Besides, for any $\eta > 0$, there exist $\gamma_1, \ldots, \gamma_N \in (\beta - \eta, \alpha')$ such that

$$[\beta, \alpha'] \subset \bigcup_{i=1}^N [\gamma_i, \gamma_i + \eta], \; \left[\sigma_j^{(\alpha')}, \sigma_j^{(\beta)}\right] \subset \bigcup_{j=1}^N \left[\sigma_j^{(\gamma_i + \eta)}, \sigma_j^{(\gamma_i)}\right]$$

and

$$\#\left\{ k : |c_{j,k}^{(i)}| \geq \sigma_j^{(\gamma_i + \eta)} \right\} \leq 2^{(\nu(\gamma_i + \eta) + \eta)j},$$

for any $j \geq J$ (with J large enough). Since ν is a right-continuous function, there exists $\eta > 0$ such that $\nu(\gamma_i + \eta) + \eta < \nu(\gamma_i) + \varepsilon/2$ for any $i \in \{1, \ldots, N\}$. As a consequence, there exists a constant $C' > 0$ such that

$$\begin{aligned} \left(\theta_j^{(p)}\right)^p 2^{-j\varepsilon} 2^{-jn} \sum_{k : |c_{j,k}| \geq \sigma_j^{(\alpha')}} |c_{j,k}|^p &\leq \sum_{i=1}^N \left(\theta_j^{(p)}\right)^p 2^{-j\varepsilon} 2^{-jn} \left(\sigma_j^{(\gamma_i)}\right)^p 2^{(\nu(\gamma_i) + \varepsilon/2)j} \\ &\leq C' N, \end{aligned}$$

for any $j \geq J$. □

Let us recall that $b^{\sigma'}_{p',\infty} \subset b^{\sigma}_{p,\infty}$ if $p' \leq p$ and $\sigma_j 2^{-jn/p} \leq \sigma'_j 2^{-jn/p'}$ for any j large enough; in particular, $b^{\sigma'}_{p,\infty} \subset b^{\sigma}_{p,\infty}$ if $\sigma_j \leq \sigma'_j$ for j large enough [1, 12, 26]. Therefore, to get the equality in the previous theorem, we must choose the "largest" sequences $\theta^{(p)}$ satisfying the condition. To do so, we will introduce a new function $\tilde{\nu}$. Let us first rewrite the condition on the sequences $\theta^{(p)}$. Under the hypothesis of Theorem 4, we have, for j sufficiently large,

$$\begin{aligned}
&\theta^{(p)}_j 2^{-j\varepsilon/p} \leq C 2^{jn/p} 2^{-j\nu(\alpha+\eta/2)/p}\left(\sigma^{(\alpha+\eta/2)}_j\right)^{-1}\\
&\Leftrightarrow \theta^{(p)}_j 2^{-j\varepsilon/p} \leq 2^{jn/p} 2^{-j\nu(\alpha+\eta/2)/p}\left(\sigma^{(\alpha+\eta)}_j\right)^{-1}\\
&\Leftrightarrow 2^{j\nu(\alpha+\eta/2)/p} 2^{-j\varepsilon/p} \leq 2^{jn/p}\left(\theta^{(p)}_j\sigma^{(\alpha+\eta)}_j\right)^{-1}\\
&\Leftrightarrow \nu(\alpha) \leq \lim_{\eta\to 0^+}\inf_{p>0}\limsup_{j\to+\infty} n - p\frac{\log\left(\theta^{(p)}_j\sigma^{(\alpha+\eta)}_j\right)}{\log 2^j}.
\end{aligned}$$

Definition 8. The function $\tilde{\nu}$ is defined by

$$\tilde{\nu}(\alpha) = \begin{cases} \lim_{\eta\to 0^+}\inf_{p>0}\limsup_{j\to+\infty} n - p\dfrac{\log\left(\theta^{(p)}_j\sigma^{(\alpha+\eta)}_j\right)}{\log 2^j} & \text{if } \alpha \geq \alpha_{\min} \\ -\infty & \text{else} \end{cases}.$$

The function $\tilde{\nu}$ is a right-continuous increasing function such that $\tilde{\nu}(\alpha) \geq 0$ for any $\alpha \geq \alpha_{\min}$. Besides, if $\tilde{\nu}$ is an admissible profile (i.e., if $\tilde{\nu} \leq n$) then we directly have

$$S^{\nu,\sigma^{(\cdot)}} \subset S^{\tilde{\nu},\sigma^{(\cdot)}}.$$

Theorem 5. *Under the hypothesis of Theorem 4, if $\tilde{\nu} \leq n$ and if for any $\alpha < \alpha_{\min}$, there exist $p, \varepsilon > 0$ such that $2^{-jn/p}\sigma^{(\alpha)}_j\theta^{(p)}_j 2^{-j\varepsilon/p} \to +\infty$ as $j \to +\infty$, then we have*

$$\bigcap_{p>0}\bigcap_{\varepsilon>0} b_{p,\infty}^{\left(\theta^{(p)}_j 2^{-j\varepsilon/p}\right)_j} \subset S^{\tilde{\nu},\sigma^{(\cdot)}}.$$

Remark 1. Hypothesis $2^{-jn/p}\sigma^{(\alpha)}_j\theta^{(p)}_j 2^{-j\varepsilon/p} \to +\infty$ means that the sequence $\theta^{(p)}_j$ must be "sufficiently large." Besides, if we suppose that

$$\bigcap_{p>0}\bigcap_{\varepsilon>0} b_{p,\infty}^{\left(\theta^{(p)}_j 2^{-j\varepsilon/p}\right)_j} \subset S^{\tilde{\nu},\sigma^{(\cdot)}},$$

then for any $\alpha < \alpha_{\min}$, there exist $p, \varepsilon > 0$ such that

$$\limsup_{j\to+\infty} 2^{-jn/p}\sigma^{(\alpha)}_j\theta^{(p)}_j 2^{-j\varepsilon/p} = +\infty.$$

Now, if there exists $\alpha < \alpha_{\min}$ such that the limit superior is bounded for any p, ε, let us define the sequence c such that for any $j \in \mathbf{N}$, $c_{j,k} = \sigma^{(\alpha)}_j$ for one and only one k and $c_{j,k} = 0$ otherwise. Obviously, c belongs to $\bigcap_{p>0}\bigcap_{\varepsilon>0} b_{p,\infty}^{(\theta^{(p)}_j 2^{-j\varepsilon/p})_j}$ but the sequence c does not belong to $S^{\tilde{\nu},\sigma^{(\cdot)}}$.

Proof. Let c be a sequence belonging to $\bigcap_{p>0}\bigcap_{\varepsilon>0} b_{p,\infty}^{\left(\theta^{(p)}2^{-j\varepsilon/p}\right)_j}$. For any $p,\varepsilon>0$, there exists $C_{p,\varepsilon}>0$ such that

$$\left(\theta_j^{(p)}\right)^p 2^{-j\varepsilon}2^{-jn}\sum_k |c_{j,k}|^p \le C_{p,\varepsilon},$$

for any j.

If $\alpha<\alpha_{\min}$, we have

$$\#\left\{k : |c_{j,k}| \ge \sigma_j^{(\alpha)}\right\} \le C_{p,\varepsilon}\left(2^{-jn/p}\theta_j^{(p)}2^{-j\varepsilon/p}\sigma_j^{(\alpha)}\right)^{-p}.$$

Let us take p,ε as in the hypothesis; there thus exists $J>0$ such that $|c_{j,k}|<\sigma_j^{(\alpha)}$ for any $j\ge J$ and for any k, that is, $\nu_{c,\sigma^{(\cdot)}}(\alpha)=-\infty$.

If $\alpha\ge\alpha_{\min}$, we have

$$\begin{aligned}\#\left\{k : |c_{j,k}| \ge \sigma_j^{(\alpha+\varepsilon/2)}\right\} &\le C_{p,\varepsilon}2^{jn}\left(\theta_j^{(p)}\right)^{-p}2^{\varepsilon j}\left(\sigma_j^{(\alpha+\varepsilon/2)}\right)^{-p}\\ &\le 2^{jn}\left(\theta_j^{(p)}\right)^{-p}2^{\varepsilon j}\left(\sigma_j^{(\alpha+\varepsilon)}\right)^{-p},\end{aligned}$$

for j large enough; we thus have $\nu_{c,\sigma^{(\cdot)}}(\alpha)\le\tilde{\nu}(\alpha)$. This implies that c belongs to $S^{\tilde{\nu},\sigma^{(\cdot)}}$. □

The previous theorem allows us to assert that if $\nu=\tilde{\nu}$ then

$$S^{\nu,\sigma^{(\cdot)}}=\bigcap_{p>0}\bigcap_{\varepsilon>0} b_{p,\infty}^{\left(\theta_j^{(p)}2^{-j\varepsilon/p}\right)_j}.$$

The next corollary shows that when inequalities (5) are satisfied for $\tilde{\nu}$, the previous implication becomes an equivalence.

Corollary 1. *Under the hypothesis of the previous theorem, if for any $p,\varepsilon>0$ and for any $\alpha\ge\alpha_{\min}$, there exists $C>0$ such that*

$$\theta_j^{(p)}2^{-j\varepsilon/p}\le C2^{jn/p}2^{-j\tilde{\nu}(\alpha)/p}\left(\sigma_j^{(\alpha)}\right)^{-1},$$

for any j and if for any $\alpha<\beta$, we have $\sigma_j^{(\beta)}\le\sigma_j^{(\alpha)}$ for any j, then we have

$$S^{\nu,\sigma^{(\cdot)}}=\bigcap_{p>0}\bigcap_{\varepsilon>0} b_{p,\infty}^{\left(\theta_j^{(p)}2^{-j\varepsilon/p}\right)_j}$$

if and only if $\nu=\tilde{\nu}$.

Proof. We directly have

$$\bigcap_{p>0}\bigcap_{\varepsilon>0} b_{p,\infty}^{\left(\theta_j^{(p)}2^{-j\varepsilon/p}\right)_j}=S^{\tilde{\nu},\sigma^{(\cdot)}},$$

so that we can conclude using Proposition 3. □

When $\nu = \tilde{\nu}$, a natural question concerns the link between the topology of the space $(S^{\nu,\sigma^{(\cdot)}}, d)$ defined in Theorem 2 with the topology induced by the intersection in the previous theorem. The canonical topology on the generalized Besov space is the same as the topology on $(S^{\nu,\sigma^{(\cdot)}}, d)$, that is, it is metrizable, complete, and stronger than the pointwise convergence. If the intersection of the generalized Besov spaces can be written as a countable intersection of $b^{\sigma^{(i)}}_{p,\infty}$ spaces ($i \in \mathbf{N}$) then, from the closed graph theorem, the topology τ defined on $S^{\nu,\sigma^{(\cdot)}}$ as the weakest topology such that each identity map $(S^{\nu,\sigma^{(\cdot)}}, \tau) \mapsto b^{\sigma^{(i)}}_{p,\infty}$ is continuous is equivalent to $(S^{\nu,\sigma^{(\cdot)}}, d)$. The next proposition gives some conditions on the sequences $\theta^{(p)}_j$ to have a countable intersection.

Proposition 4. *If the function $p > 0 \mapsto \log_{2^j} \theta^{(p)}_j$ is left continuous uniformly with respect to j then if p_m ($m \in \mathbf{N}$) is a dense sequence of $]0, +\infty[$ and if ε_l ($l \in \mathbf{N}$) is a sequence of strictly positive numbers converging to 0, we have*

$$\bigcap_{p>0}\bigcap_{\varepsilon>0} b_{p,\infty}^{\left(\theta^{(p)}_j 2^{-j\varepsilon/p}\right)_j} = \bigcap_{m\in\mathbf{N}}\bigcap_{l\in\mathbf{N}} b_{p_m,\infty}^{\left(\theta^{(p_m)}_j 2^{-j\varepsilon_l/p_m}\right)_j}.$$

Proof. The inclusion of the first space in the second one is straightforward. Let us prove the other inclusion.

Let us take $p > 0$ and $\varepsilon > 0$. It suffices to find p_m and ε_l such that

$$p_m \le p \text{ and } \theta^{(p_m)}_j 2^{-j(\varepsilon_l+n)/p_m} \ge \theta^{(p)}_j 2^{-j(\varepsilon+n)/p},$$

for j large enough. This is equivalent to ask

$$p_m \le p \text{ and } \frac{\varepsilon}{p} - \frac{\varepsilon_l}{p_m} \ge \frac{n}{p_m} - \frac{n}{p} - \left(\log_{2^j}\theta^{(p_m)}_j - \log_{2^j}\theta^{(p)}_j\right),$$

for j large enough. Since we can choose $p_m \le p$ such that

$$|\log_{2^j}\theta^{(p_m)}_j - \log_{2^j}\theta^{(p)}_j| < \frac{\varepsilon}{3p} \text{ and } \frac{n}{p_m} - \frac{n}{p} < \frac{\varepsilon}{3p},$$

for any $j \in \mathbf{N}$, we can conclude by taking ε_l sufficiently small. □

To end this section, let us show that the preceding results are generalizations of what is known about the S^ν spaces. For $\sigma^{(\alpha)}_j = 2^{-\alpha j}$, let us find an admissible sequence $\theta^{(p)}_j$ such that for any $p, \varepsilon > 0$ and for any $\alpha \ge \alpha_{\min}$, there exists $C > 0$ such that

$$\theta^{(p)}_j 2^{-j\varepsilon/p} \le C 2^{jn/p} 2^{-j\nu(\alpha)/p}\left(\sigma^{(\alpha)}_j\right)^{-1},$$

that is,

$$\theta^{(p)}_j 2^{-j\varepsilon/p} \le C 2^{j(n-\nu(\alpha)+\alpha p)/p}.$$

If we take $\theta^{(p)}_j = 2^{j\eta(p)/p}$ with

$$\eta(p) = \inf_{\alpha\ge\alpha_{\min}}\{\alpha p - \nu(\alpha)\} + n,$$

we get $\tilde{\nu}(\alpha) = \inf_{p>0}\{\alpha p - \eta(p)\} + n$ for any $\alpha \ge \alpha_{\min}$ and the hypothesis of Corollary 1 and Proposition 4 are satisfied. We thus obtain

$$S^\nu = \bigcap_{p>0}\bigcap_{\varepsilon>0} b_{p,\infty}^{\left(\theta^{(p)}_j 2^{-j\varepsilon/p}\right)_j} = \bigcap_{p>0}\bigcap_{\varepsilon>0} b_{p,\infty}^{\frac{\eta(p)-\varepsilon}{p}} = \bigcap_{m\in\mathbf{N}}\bigcap_{l\in\mathbf{N}} b_{p_m,\infty}^{\frac{\eta(p_m)-\varepsilon_l}{p_m}}$$

if and only if $\nu = \tilde{\nu}$, that is, if ν is concave.

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