A Reasoning System of Ternary Projective Relations

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Abstract—**This paper introduces a reasoning system based on a previously developed model for ternary projective relations between spatial objects. The model applies to spatial objects of the kind point and region, is based on basic projective invariants and takes into account the size and shape of the three objects that are involved in a relation. The reasoning system proposes a set of permutation and composition rules, which allow the inference of unknown relations from given ones.**

I. INTRODUCTION

The field of Qualitative Spatial Reasoning (QSR) has experienced a great interest in the spatial data handling community due to its potential applications [1]. An important topic in QSR is the definition of reasoning systems on qualitative spatial relations. For example, regarding topological relations, the 9-intersection model [2] provides formal definitions for the relations and a reasoning system based on composition tables [3] establishes a mechanism to find new relations from a set of given ones.

As discussed in [4], geometric properties can be subdivided in three groups: topological, projective and metric. Most qualitative relations between spatial objects can be defined in terms of topological or projective properties [5]. Qualitative distances are a qualitative interpretation of metric distances [6].

Projective relations are a category of spatial relations that can be described by projective properties of the space without resorting to metric properties [7]. Projective relations are thus qualitative in nature because they do not need exact measures to be explained. Projective relations are more specific than topological relations and can serve as a basis for describing relations that are not captured by topology. At an intermediate rank between metrics and topology, projective relations are as much varied as "right of", "before", "between", "along", "surrounded by", "in front of", "back", "north of", "east of", and so on.

To have a common sense understanding of projective relations, it is helpful to think about different two-dimensional views of a three-dimensional real world scene of objects: changing the point of view, metric aspects such distances and angles among the objects appear to be different, but there are properties that are common in all the views. These common properties are projective properties.

Likewise topological relations, which are defined by using the connectedness topological invariant, projective relations

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can be defined by using the *collinearity* projective invariant, which is the property of three collinear points being still collinear after an arbitrary number of projections. A main difference in the treatment of topological relations and projective relations is that, while basic topological relations are binary, basic projective relations are ternary because they are defined on the collinearity of three points. The definition of collinearity has been extended to regions in [8].

In this paper, we propose a reasoning system for the set of projective relations that was introduced in [9]. Such relations establish a jointly exhaustive and pairwise disjoint (JEPD) set of projective relations among any three regions of the plane. A preliminary version of this model was presented in [10], but the set of relations was not JEPD. A first version of the reasoning system, applied on a subset of the relations, was presented in [11]. The projective relations are ternary relations of the kind $r(A, B, C)$, where A has the role of primary object and B and C have the role of reference objects. This latter terminology derives from the work on positional relations (e.g., [7]), where the position of an object (primary) is stated with respect to the position of one or more other objects acting as reference. Two cases can be distinguished based on whether the convex hulls of the reference objects are disjoint or not disjoint. In the first case, the model, called the *5-intersection*, by using only projective concepts partitions the plane into five acceptance areas with respect to the reference objects; in the second case, the partition of the plane results in two acceptance areas. The model is able to differentiate between 34 different projective relations that are obtained by computing the intersection of the primary object with the acceptance areas that are determined by the reference objects.

The reasoning system establishes rules of permutation and composition of relations in the form of tables. Among the 34 projective relations of the model, we can distinguish single-tile and multi-tile relations, depending whether the primary object intersects one or more of the acceptance areas. The single-tile relations are five for non-intersecting convex hulls of reference objects and two for intersecting convex hulls of reference objects. The permutation rules are of two types: converse and rotation. Regarding the composition table, we initially find it for the composition of single-tile relations with all basic relations (therefore, a 7×34 table). The latter table has been found in two different ways: by manually checking all geometric configurations that satisfy the table and by running a simulation program with a high number of random regions and finding the occurring relations. Afterwards, the full composition table has been found (34×34) with algebraic rules that can be applied to the 7×34 table.

The remainder of the paper is organized as follows. We start in Section II with a brief comparison to existing literature. In Section III, we introduce preliminaries on reasoning with

ternary relations. In Sections IV, we summarize the model of projective relations among points and, in Section V, we describe the reasoning system for points. In Section VI, we summarize the model for projective relations among regions. Section VII presents the main contribution of the paper, consisting in the reasoning system for regions. Section VIII draws conclusions. In the appendix, we include all geometric configurations that have been considered to build the composition tables.

II. RELATED WORK

In the QSR literature, we can find various models for reasoning with projective relations. Freksa's double-cross calculus [12] is similar to our approach in the case of points. Such a calculus, as it has been further discussed in [13], [14], is based on ternary directional relations between points. In Freksa's model, an intrinsic frame of reference centred in a given point partitions the plane into four quadrants that are given by the front-back and right-left dichotomies. This leads to a greater number of qualitative distinctions with different algebraic properties and composition tables. A smaller number of qualitative distinctions and an independence from the specific frame of reference would improve the possibility of extending this model to other spatial types besides points.

Other work on ternary calculi is rather limited. Most approaches consider binary relations to which a frame of reference is associated [15], [16]. Exceptions of ternary relations, such as "between", were considered in [17] and, more recently, in [18]. Projective relations, intended as locative expressions between two objects [19], depend on an underlying frame of reference. The use of ternary relations instead of binary ones allows us to describe the projective relation in a way that is independent from the frame of reference. The ternary model of projective relations can be seen as a geometric abstraction of locative expressions commonly used in the physical world.

Our approach can be compared to various models for orientation relations [20], [21], [22], [23], [24] or cardinal directions between points [25], [26]. Most of them, even when explicitly related to projective geometry, never avoid the use of metric properties (minimum bounding rectangles, angles, etc.) and external frames of reference (such as a grid). To this respect, the main difference in our approach is that we only deal with projective invariants, independently of metric aspects such as distances and angles.

Most work on cardinal directions deals with point abstractions of spatial features and less work has been devoted to extended objects [27], [28], [29], [30], [31]. In [32], the authors use spheres surrounding the objects to take into account the shape of objects in relative orientation. In [27], the authors develop a model for cardinal directions between extended objects, where the partition of the plane is determined by the prolongations of the sides of the minimum bounding rectangle (MBR) of a reference object. The reasoning for such a model has been developed in [33], [34]. Contrary to the double-cross calculus [12] and the direction matrix [27], we use a cone-based model instead of a projection-based model [25].

Ternary projective relations are more general than binary orientation relations or cardinal directions, because they don't need to refer to an external frame of reference. The two reference objects are able to make a partition of the plane to which the position of the primary object is compared. In [35], we find the same approach as ours regarding the reasoning on points, except that authors consider a partition of the plane in seven parts instead of five. Moving from points to extended regions, among the strengths of our model are the facts that it uses projective properties only and that the acceptance areas of relations depend on the shape and size of the reference objects. Independence from a specific frame of reference (both egocentric or allocentric [36]) allows us to specify ternary relations among objects from a purely geometric point of view. A further step is to apply the model of ternary projective relations to specific settings, like robot navigation [35], [37] or cardinal directions on the Earth surface [38].

III. REASONING WITH RELATIONS

Most research on spatial and temporal relations have focused on binary relations and studied algorithms for several useful operators like *converse* and *composition* [39], [3], [25], [40], [33]. The converse of a relation r that holds between A and B, denoted by $r^{\smile}$, specifies the relation between B and A. The composition of a relation r between A and B with a relation q between B and C, denoted by $r \circ q$, specifies the relation between A and C. Converse and composition are used to construct reasoning mechanisms applicable to query processing.

The aforementioned notions can also be extended for ternary relations [14], [41], [13]. Let us consider a set of basic ternary relations $\mathcal{T}$ that contains $|\mathcal{T}|$ jointly exhaustive and pairwise disjoint relations [16]. Elements of $\mathcal{T}$ are used to represent *definite* information. Using these relations, we can define the *powerset* of $\mathcal{T}$ (i.e., the set of all subsets), denoted by $2^{\mathcal{T}}$ that contains $2^{|\mathcal{T}|}$ relations. Elements of $2^{\mathcal{T}}$ can be used to represent definite but also *indefinite* information. For instance, if $t_1, \ldots, t_n$ are basic relations in $\mathcal{T}$, then $\{t_1, \ldots, t_n\}$ is a relation in $2^{\mathcal{T}}$ which is equivalent to $t_1 \vee \cdots \vee t_n$.

Initially, we define converse and rotation. If $r(A, B, C)$ holds, the converse of r specifies the relation between A, C and B while the rotation of r specifies the relation between C, A and B.

Definition 1: Let r be a ternary relation in $\mathcal{T}$. The *converse* of r, denoted by $r^{\smile}$, is a relation in $2^{\mathcal{T}}$ defined as:

$$r^{\smile} = \{t \in \mathcal{T} \mid (\exists A, B, C)\big(\, t(A, C, B) \wedge r(A, B, C) \,\big)\}$$

Definition 2: Let r be a ternary relation in $\mathcal{T}$. The *rotation* of r, denoted by $r^{\frown}$, is a relation in $2^{\mathcal{T}}$ defined as:

$$r^{\frown} = \{t \in \mathcal{T} \mid (\exists A, B, C)\big(\, t(C, A, B) \wedge r(A, B, C) \,\big)\}$$

Notice that the converse and rotation operators as defined above are the ternary counterpart of the binary converse operation [41]. For binary relations, the converse operation is sufficient since there are only 2 permutations between 2 objects A and B, namely (A, B) and (B, A). For the ternary relations case, there are 6 possible permutations between 3

objects A, B and C, namely (A,B,C), (A,C,B), (B,A,C), (B,C,A), (C,A,B) and (C,B,A). It is easy to verify that we need both converse and rotation to move between these permutations. For instance, we may move from (A,B,C) to (C,B,A) by applying the rotation followed by the converse operator.

Next, we define ternary composition.

Definition 3: Let r and q be two ternary relations. The *composition* of r and q, denoted by $r \circ q$, is defined as:

$$r \circ q = \{t \in \mathcal{T} \mid (\exists A,B,C,D)\big(\ r(A,B,C) \wedge q(B,C,D) \wedge \\ t(A,C,D) \ \big) \}$$

Similarly to earlier works in qualitative spatial relations, we use a weak definition of converse and composition [42], [43], [33], [31]. Typically, these operators are expressible for every pair of spatial relations and can be naturally used as a constraint propagation mechanism. On the contrary, the more strict set-theoretic definitions of converse and composition are not always definable [40], [44], [33], [31].

In the following sections, we will define a model for ternary projective relations for points and a model for ternary projective relations for regions. Moreover, we will study the converse, the rotation and the composition operations for the above models.

IV. PROJECTIVE RELATIONS FOR POINTS

Our basic set of projective relations for points is based on the most important geometric invariant in a projective space: the collinearity of three points. Therefore, the nature of projective relations is intrinsically ternary. Let us consider three points P_1, P_2 and P_3. To define the projective relation $r_1(P_1, P_2, P_3)$ of primary point P_1 with respect to reference points P_2 and P_3, we consider the following two cases.

Case 1: $P_2 \neq P_3$. In this case, to define $r_1(P_1, P_2, P_3)$, we use the directed line $\overrightarrow{P_2P_3}$. This directed line partitions the space into 5 parts that correspond to the relations *rightside*, *leftside*, *before*, *after* and *between* and are denoted by $rightside(P_2, P_3)$, $leftside(P_2, P_3)$, $before(P_2, P_3)$, $after(P_2, P_3)$ and $between(P_2, P_3)$, respectively (Figure 1a). In Figure 1a, $rightside(P_2, P_3)$ is the light gray area, $leftside(P_2, P_3)$ is the dark gray area, $before(P_2, P_3)$ is the dotted semi-line, $after(P_2, P_3)$ is the dashed semi-line and $between(P_2, P_3)$ is the thick line segment. Notice that:

- All parts are disjoint.
- The union of all parts is $\Re^2$.
- $P_2 \in between(P_2, P_3)$, $P_3 \in between(P_2, P_3)$, $\overrightarrow{P_2P_3} \notin rightside(P_2, P_3)$ and $\overrightarrow{P_2P_3} \notin leftside(P_2, P_3)$.

If a point P_1 is included (in the set-theoretic sense) in region $rightside(P_2, P_3)$ of some points P_2 and P_3, then we say that P_1 is *rightside* of P_2 and P_3 and we write $rs(P_1, P_2, P_3)$. Similarly, we can define relations ls (*leftside*), bf (*before*), af (*after*) and bt (*between*).

Example 1: For the points of Figure 1b, we have:

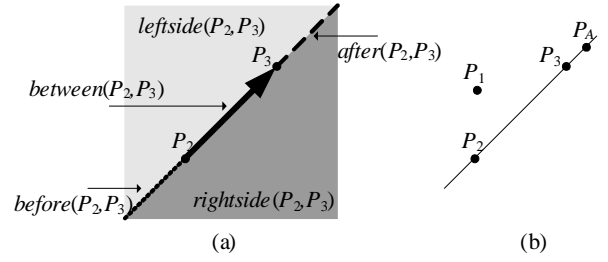$$rs(P_1, P_2, P_3), \ af(P_A, P_2, P_3) \ \text{and} \ bt(P_2, P_2, P_3).$$
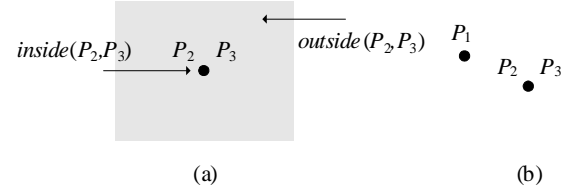

Fig. 1. Ternary relations between points ($P_2 \neq P_3$)

Fig. 2. Ternary relations between points ($P_2 = P_3$)

Case 2: $P_2 = P_3$. In this case, the space is partitioned in a point (P_2) and an open area ($\Re^2 - P_2$). These regions correspond to the relations *inside* and *outside* and are denoted by $inside(P_2, P_3)$ and $outside(P_2, P_3)$ respectively (Figure 2a).

If a point P_1 is included (in the set-theoretic sense) in region $outside(P_2, P_3)$ of some points P_2 and P_3, then we say that P_1 is *outside* of P_2 and P_3 and we write $ou(P_1, P_2, P_3)$. Similarly, we can define relation *inside*, denoted by in.

Example 2: For the points of Figure 2b, we have:

$$ou(P_1, P_2, P_3) \ \text{and} \ in(P_2, P_2, P_3).$$

Summarizing, the set of projective relations between points contains the following 7 (=5+2) relations: rs, ls, bf, af, bt, in and ou. We will use $\mathcal{D}_{point}$ to denote this set. Relations in $\mathcal{D}_{point}$ are jointly exhaustive and pairwise disjoint [9]. Elements of $\mathcal{D}_{point}$ can be used to represent definite information, e.g., $bt(P_1, P_2, P_3)$. To express *indefinite* information, we use the powerset $2^{\mathcal{D}_{point}}$, of $\mathcal{D}_{point}$ which contains $128 = 2^7$ relations. For instance, $\{bt, af\}(P_1, P_2, P_3) \in 2^{\mathcal{D}_{point}}$ denotes that point P_1 is either between or after points P_2 and P_3 (i.e., $bt(P_1, P_2, P_3) \vee af(P_1, P_2, P_3)$ holds). More details about these relations can be found in [9].

V. REASONING WITH PROJECTIVE RELATIONS FOR POINTS

In this section, we will study the converse, rotation and composition operators for the projective relations for points defined in Section IV.

For any projective relation $r(P_1, P_2, P_3)$, Table I presents the relations that correspond to the converse $r^{\smile}(P_1, P_3, P_2)$ and the rotation $r^{\frown}(P_3, P_1, P_2)$ of $r(P_1, P_2, P_3)$. For example, given three points P_1, P_2 and P_3 such that $bf(P_1, P_2, P_3)$ holds, using Table I, we can derive that $af(P_1, P_3, P_2)$ and $af(P_3, P_1, P_2)$ also hold (see also Figure 3a).

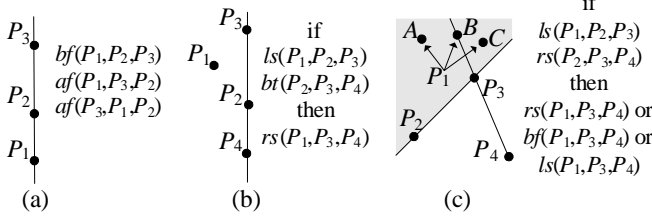| $r(P_1, P_2, P_3)$	$r^\smile(P_1, P_3, P_2)$	$r^\frown(P_3, P_1, P_2)$
bt	bt	bt, bf, ou
rs	ls	rs
bf	af	af
ls	rs	ls
af	bf	bt
in	in	in
ou	ou	bt

TABLE I
CONVERSE AND ROTATION FOR PROJECTIVE RELATIONS BETWEEN POINTS

Fig. 3.   Operations on point relations

Table II illustrates the result of the composition, $r_1 \circ r_2(P_1, P_3, P_4)$, of two basic projective relations $r_1(P_1, P_2, P_3)$ and $r_2(P_2, P_3, P_4)$. For instance, as we can verify in Figure 3b, we have:

$$ls(P_1, P_2, P_3) \circ bt(P_2, P_3, P_4) = rs(P_1, P_3, P_4).$$

Notice that, for some cases, the result of the composition is a relation in $2^{\mathcal{D}_{point}}$. For example, we have:

$$ls(P_1, P_2, P_3) \circ rs(P_2, P_3, P_4) = \{rs, bf, ls\}(P_1, P_3, P_4).$$

This means that, given four points P_1, P_2, P_3 and P_4 such that $ls(P_1, P_2, P_3)$ and $rs(P_2, P_3, P_4)$ hold, the projective relation of P_1 can be rightside or before or leftside P_3 and P_4 (i.e., $rs(P_1, P_3, P_4) \vee bf(P_1, P_3, P_4) \vee ls(P_1, P_3, P_4)$). This can be verified in Figure 3c. If P_1 were placed in point A (respectively B and C) then the first (respectively the second and the third) disjunct would be verified.

Moreover, for some other cases the result is impossible, denoted by IMP. For instance, $in(P_1, P_2, P_3) \circ rs(P_2, P_3, P_4) = IMP$ because we can easily verify that there do not exist points P_1, P_2, P_3 and P_4 such that $in(P_1, P_2, P_3)$ and $rs(P_2, P_3, P_4)$ simultaneously hold.

VI. PROJECTIVE RELATIONS FOR REGIONS

In Sections IV and V, we have defined a model for projective relations for points and we have studied reasoning operators. We will now turn our attention to regions. In this paper, we will consider regions that are formed by finite unions of regions that are homeomorphic to the *closed unit disk* [34]. This set of regions is denoted by REG^*. Regions in REG^* are *regular closed point sets* and can be *disconnected* or have *holes*. However, class REG^* excludes points, lines and regions with emanating lines. Let A be a region in REG^*, we denote the convex hull of A by $CH(A)$.

In this section, we will briefly present the projective model for regions, we refer the interested reader to [9] for a more extended discussion. These relations extend the projective relations for points discussed in Section IV, thus they are also ternary. Let us consider three regions R_1, R_2 and R_3. To define the projective relation $r_1(R_1, R_2, R_3)$ of primary region R_1 with respect to reference regions R_2 and R_3, we consider the following two cases.

Case 1: $CH(R_2) \cap CH(R_3) = \emptyset$. In this case, to define $r_1(R_1, R_2, R_3)$, we use the convex hull of the union of regions R_2 and R_3 and the internal common tangents of regions R_2 and R_3. Since $CH(R_2) \cap CH(R_3) = \emptyset$ holds, we can always define exactly two internal and two external common tangents. For instance, in Figure 4a, we illustrate two regions (R_2 and R_3) and their internal and external common tangents.
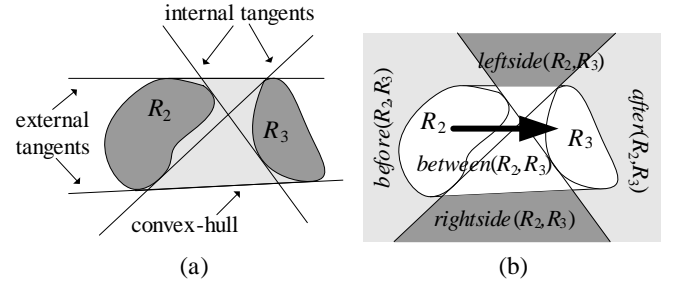

Fig. 4.   Ternary relations between regions $(CH(R_2) \cap CH(R_3) = \emptyset)$

The external common tangents help to find the convex hull of the union of R_2 and R_3. The convex hull and the internal common tangents of regions R_2 and R_3 partition the reference space into 5 regions as in (Figure 4b). Formal definitions of these regions can be found in [9]. Similarly to the point case, these areas correspond to the relations *rightside*, *leftside*, *before*, *after* and *between* and are denoted by $rightside(P_2, P_3)$, $leftside(P_2, P_3)$, $before(P_2, P_3)$, $after(P_2, P_3)$ and $between(P_2, P_3)$ respectively (Figure 4b). To distinguish the above areas, we consider an oriented line from the first reference region (i.e., R_2) to the second reference region (i.e., R_3). Specifically, in Figure 4b, $rightside(P_2, P_3)$ is the lower dark gray area, $leftside(P_2, P_3)$ is the upper dark gray area, $before(P_2, P_3)$ is the light gray area on the left, $after(P_2, P_3)$ is the light gray area on the right and $between(P_2, P_3)$ is the white area on the middle. Notice that:

- The union of all regions is $\Re^2$.
- All areas but $between(R_2, R_3)$ are unbounded.
- Area $between(R_2, R_3)$ is closed.
- The interiors of all areas are disjoint but two areas may share common points in their boundaries. For instance, the areas $before(R_2, R_3)$ and $rightside(R_2, R_3)$ share some points of the internal tangent.

Even though tiles share some points in their borders there is no ambiguity in defining projective relations because the class REG^* does not contain objects that could lie entirely on the borderline (like lines and points).

r_2/r_1	bt	rs	bf	ls	af	in	ou
bt	bt	ls	$\{bt, af\}$	rs	bf	bt	$\{bt, rs, bf, ls, af\}$
rs	rs	$\{bt, rs, ls, af\}$	rs	$\{rs, bf, ls\}$	ls	IMP	IMP
bf	bf	rs	bf	ls	$\{bt, af\}$	IMP	IMP
ls	ls	$\{rs, bf, ls\}$	ls	$\{bt, rs, ls, af\}$	rs	IMP	IMP
af	$\{bt, af\}$	ls	af	rs	bf	IMP	IMP
in	IMP	IMP	IMP	IMP	IMP	in	ou
ou	ou	in, ou	ou	ou	ou	IMP	IMP

TABLE II
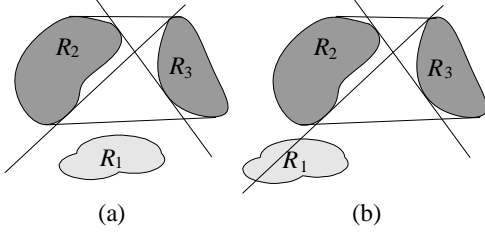COMPOSITION TABLE OF PROJECTIVE RELATIONS BETWEEN POINTS

Fig. 5. Example of projective relations between regions ($CH(R_2) \cap CH(R_3) = \emptyset$)

If a region R_1 is included (in the set-theoretic sense) in region $rightside(R_2, R_3)$ of some reference regions R_2 and R_3 (Figure 5a), then we say that R_1 is *rightside* of R_2 and R_3 and we write $rs(R_1, R_2, R_3)$. Similarly, we can define relations ls (*leftside*), bf (*before*), af (*after*) and bt (*between*).

If a primary region R_1 lies partly in the $rightside(R_2, R_3)$ area and partly in the $before(R_2, R_3)$ area of some reference regions R_2 and R_3 (Figure 5b), then we say that R_1 is partly *rightside* and partly *before* of R_2 and R_3 and we write $rs{:}bf(R_1, R_2, R_3)$.

Case 2: $CH(R_2) \cap CH(R_3) \neq \emptyset$. In this case, the common internal tangents of regions R_2 and R_3 cannot be defined. Thus, we only use the convex hull of regions R_2 and R_3 to partition the reference space into two areas as in Figure 6a. These areas correspond to relations *inside* and *outside* and are denoted by $inside(R_2, R_3)$ and $outside(R_2, R_3)$ respectively.

Region $inside(R_2, R_3)$ is bounded while $outside(R_2, R_3)$ is unbounded. Similarly to Case 1, the union of regions $inside(R_2, R_3)$ and $outside(R_2, R_3)$ is $\Re^2$ and the interiors of these areas are disjoint but they share common points in their boundaries.
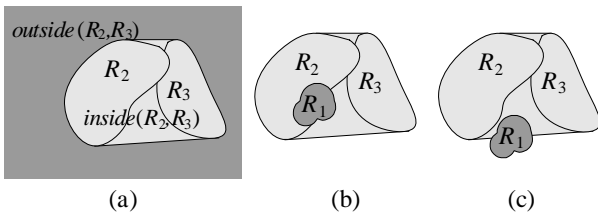

Fig. 6. Ternary relations between points ($CH(R_2) \cap CH(R_3) \neq \emptyset$)

If a region R_1 is included (in the set-theoretic sense) in

region $inside(R_2, R_3)$ of some reference regions R_2 and R_3 (Figure 6b), then we say that R_1 is *inside* of R_2 and R_3 and we write $in(R_1, R_2, R_3)$. Similarly, we can define relation ou (*outside*).

If a primary region R_1 lies partly in the $inside(R_2, R_3)$ area and partly in the $outside(R_2, R_3)$ area of some reference regions R_2 and R_3 (Figure 6c) then we say that A is partly *inside* and partly *outside* of R_2 and R_3 and we write $in{:}ou(R_1, R_2, R_3)$.

Summarizing, the general definition of a basic projective relation in our framework is given as follows:

Definition 4: A *basic projective relation* is an expression $r_1{:}\cdots{:}r_k$ where k and $r_1, \ldots, r_k$ can belong to exactly one of the following cases.

1) $1 \leq k \leq 5$, $r_1, \ldots, r_k \in \{bt, rs, bf, ls, af\}$ and $R_i \neq R_j$ for every i, j such that $1 \leq i, j \leq k$ and $i \neq j$.
2) $1 \leq k \leq 2$, $r_1, \ldots, r_k \in \{in, ou\}$ and $R_i \neq R_j$ for every i, j such that $1 \leq i, j \leq k$ and $i \neq j$.

We refer to $r_1, \ldots, r_k$ as the *tiles* of relation $r_1{:}\cdots{:}r_k$. Moreover, a basic projective relation $r_1{:}\cdots{:}r_k$ is called *single-tile* if $k = 1$; otherwise it is called *multi-tile*.

In order to avoid confusion, we will write the elements of a multi-tile relation according to the following order: bt, rs, bf, ls, af, in and ou. Thus, we always write $rs{:}bf$ and not $bf{:}rs$.

Example 3: Expressions rs, $rs{:}bf$, in and $in{:}ou$ are basic projective relations. The first and the third are single-tile relations, while the second and the forth are multi-tile. Objects involved in these relations are shown in Figures 5a, 5b, 6b and 6c respectively.

The following two definitions formally define basic (single and multi-tile) projective relations.

Definition 5: Let R_1, R_2 and R_3 be three regions in REG^*.

Then, single-tile projective relations are defined as follows.

$$\begin{array}{llll}
bt(R_1, R_2, R_3) & \text{iff} & R_1 \in between(R_2, R_3) \\
& \text{and} & CH(R_2) \cap CH(R_3) = \emptyset \\
rs(R_1, R_2, R_3) & \text{iff} & R_1 \in rightside(R_2, R_3) \\
& \text{and} & CH(R_2) \cap CH(R_3) = \emptyset \\
bf(R_1, R_2, R_3) & \text{iff} & R_1 \in before(R_2, R_3) \\
& \text{and} & CH(R_2) \cap CH(R_3) = \emptyset \\
ls(R_1, R_2, R_3) & \text{iff} & R_1 \in leftside(R_2, R_3) \\
& \text{and} & CH(R_2) \cap CH(R_3) = \emptyset \\
af(R_1, R_2, R_3) & \text{iff} & R_1 \in after(R_2, R_3) \\
& \text{and} & CH(R_2) \cap CH(R_3) = \emptyset \\
in(R_1, R_2, R_3) & \text{iff} & R_1 \in inside(R_2, R_3) \\
& \text{and} & CH(R_2) \cap CH(R_3) \neq \emptyset \\
ou(R_1, R_2, R_3) & \text{iff} & R_1 \in outside(R_2, R_3) \\
& \text{and} & CH(R_2) \cap CH(R_3) \neq \emptyset
\end{array}$$

Definition 6: Let R_1, R_2 and R_3 be three regions in REG^* and $r_1{:}\cdots{:}r_k$ be a multi-tile projective relation. Then, $r_1{:}\cdots{:}r_k(R_1, R_2, R_3)$ holds iff there exist regions $Q_1, \ldots, Q_k \in REG^*$ such that $r_1(Q_1, R_2, R_3), \ldots, r_k(Q_k, R_2, R_3)$ and $R_1 = Q_1 \cup \cdots \cup Q_k$.

Summarizing, the set of basic projective relations between regions contains 34 relations (7 single-tile and 27 multi-tile). We will use $\mathcal{D}_{region}$ to denote this set. Relations in $\mathcal{D}_{region}$ are jointly exhaustive and pairwise disjoint [9]. Elements of $\mathcal{D}_{region}$ can be used to represent definite information about directional information. We will use r, q and p possibly subscripted to denote variables ranging in $\mathcal{D}_{region}$. To also express indefinite information we use the powerset, $2^{\mathcal{D}_{region}}$, of $\mathcal{D}_{region}$. We will use $\bar{r}$, $\bar{q}$ and $\bar{p}$ possibly subscripted to denote variables ranging in $2^{\mathcal{D}_{region}}$

VII. REASONING WITH PROJECTIVE RELATIONS FOR REGIONS

To present our results, we will need the following definitions.

Definition 7: Let $r_1, \ldots, r_k$ be single-tile relations. The δ-*combination* of relations $r_1, \ldots, r_k$, denoted by $\delta(r_1, \ldots, r_k)$, is a short-cut for the projective relation in $2^{\mathcal{D}_{region}}$ that can be constructed by combining single-tile relations $r_1, \ldots, r_k$.

For instance, we have $\delta(bt, rs, ls) = \{bt,\ rs,\ ls,\ bt{:}rs,\ bt{:}ls,\ rs{:}ls,\ bt{:}rs{:}ls\}$.

Definition 8: Let $r_1{:}\cdots{:}r_k$ be a basic projective relation in $\mathcal{D}_{region}$. The δ-combination of $r_1{:}\cdots{:}r_k$ is defined as:

$$\delta(r_1{:}\cdots{:}r_k) = \delta(r_1, \ldots, r_k)$$

For instance, we have $\delta(bt{:}rs{:}ls) = \delta(bt, rs, ls) = \{bt,\ rs,\ ls,\ bt{:}rs,\ bt{:}ls,\ rs{:}ls,\ bt{:}rs{:}ls\}$.

Definition 9: Let $\bar{r}_1 = \{r_1^1, \ldots, r_1^{k_1}\}$, $\ldots$, $\bar{r}_m = \{r_m^1, \ldots, r_m^{k_m}\}$ be m projective relations in $2^{\mathcal{D}_{region}}$ (where $r_1^1, \ldots, r_1^{k_1}, \ldots, r_m^1, \ldots, r_m^{k_m}$ are basic projective relations). The δ-combination of these relations is defined as:

$$\delta(\bar{r}_1, \ldots, \bar{r}_m) = \delta(r_1^1, \ldots, r_1^{k_1}, \ldots, r_m^1, \ldots, r_m^{k_m})$$

Finally, we define

$$\delta\big(\delta(r_1^1, \ldots, r_1^{k_1}), \ldots, \delta(r_m^1, \ldots, r_m^{k_m})\big) = \\ \delta(r_1^1, \ldots, r_1^{k_1}, \ldots, r_m^1, \ldots, r_m^{k_m})$$

Example 4: We have $\delta(\{bt, rs\}, \delta(rs, ls)) = \delta(bt, rs, ls) = \{bt,\ rs,\ ls,\ bt{:}rs,\ bt{:}ls,\ rs{:}ls,\ bt{:}rs{:}ls\}$.

Definition 10: We denote by U_{proj} the universal disjunctive projective relation, i.e.,

$$U_{proj} = \delta(bt, rs, bf, ls, af, in, ou).$$

Moreover, we denote by D_d the universal disjunctive projective relation with reference regions with disjoint convex hulls, i.e., $D_d = \delta(bt, rs, bf, ls, af)$, and by D_c the universal disjunctive projective relation with reference regions with non-disjoint convex hulls, i.e., $D_c = \delta(in, ou)$.

Definition 11: The *tile-union* of basic relations $r_1, \ldots, r_k$, denoted by *tile-union*$(r_1, \ldots, r_k)$, is the basic relation that consists of all the tiles of relations $r_1, \ldots, r_k$.

Example 5: We have *tile-union*$(rs, bf{:}ls) = rs{:}bf{:}ls$ and *tile-union*$(bt, rs, rs{:}ls) = bt{:}rs{:}ls$.

Definition 12: Let $\bar{r}_1 = \{r_1^1, \ldots, r_1^{k_1}\}$, $\ldots$, $\bar{r}_m = \{r_m^1, \ldots, r_m^{k_m}\}$ be m projective relations in $2^{\mathcal{D}_{region}}$. The *product* of relations $\bar{r}_1$, $\ldots$, $\bar{r}_m$, denoted by $\pi(\bar{r}_1, \ldots, \bar{r}_m)$ is a projective relation in $2^{\mathcal{D}_{region}}$ defined as:

$$\pi(\bar{r}_1, \ldots, \bar{r}_m) = \{\text{tile-union}(s_1, \ldots, s_m) \mid \\ s_1 \in \{r_1^1, \ldots, r_1^{k_1}\},\ \ldots,\ s_m \in \{r_m^1, \ldots, r_m^{k_m}\}\}.$$

Example 6: The following are some examples of products.

- $\pi(\{bt, rs\}, \{bf, ls\}) = \{bt{:}bf,\ bt{:}ls,\ rs{:}bf,\ rs{:}ls\}$,
- $\pi(\{rs{:}ls, bt\}, \{bt{:}rs{:}ls, rs{:}bf{:}ls\}) = \{bt{:}rs{:}ls,\ rs{:}bf{:}ls, bt{:}rs{:}bf{:}ls\}$
- $\pi(\{rs{:}ls, bt\}, \{bt{:}rs{:}ls, rs{:}bf{:}ls\}, bf) = \{rs{:}bf{:}ls, bt{:}rs{:}bf{:}ls\}$.

The following is a useful proposition.

Proposition 1: Let $\bar{r}_1$, $\bar{r}_2$ and $\bar{q}$ be projective relations in $2^{\mathcal{D}_{region}}$. Then

$$\pi(\bar{r}_1 \cup \bar{r}_2,\ \bar{q}) = \pi(\bar{r}_1,\ \bar{q}) \cup \pi(\bar{r}_2,\ \bar{q}).$$

The product of relations is commonly used to compute the composition operator for other models of qualitative direction relations [33], [31]. In our work, we also need the following definition that uses the product of two relations.

Definition 13: The *augmentation* of a basic projective relation r in $\mathcal{D}_{region}$ by a projective relation $\bar{q}$ in $2^{\mathcal{D}_{region}}$, denoted by $r/\bar{q}$ is a projective relation in $2^{\mathcal{D}_{region}}$ defined as:

$$r/\bar{q} = r \cup \pi(r, \bar{q}).$$

Example 7: We have $rs{:}ls/\delta(bt, bf) = \{rs{:}ls,\ bt{:}rs{:}ls, rs{:}bf{:}ls,\ bt{:}rs{:}bf{:}ls\}$.

A. Converse and rotation

For any basic projective relation $r(R_1, R_2, R_3)$, Table III presents the relations that correspond to the converse $r^{\smile}(R_1, R_3, R_2)$ and the rotation $r^{\frown}(R_3, R_1, R_2)$ of $r(R_1, R_2, R_3)$.

Example 8: Using Table III, we have that $rs^{\smile} = ls$ and $rs^{\frown} = \{rs, ou\}$. These equations can be verified using regions R_1, R_2 and R_3 of Figure 7. We have $ls \in rs^{\smile}$ since in Figure 7a and b, $rs(R_1, R_2, R_3)$, $ls(R_1, R_3, R_2)$ holds. Similarly, we have *(a)* $rs \in rs^{\frown}$ since in Figure 7a,

$r(R_1,R_2,R_3)$	$r^{\smile}(R_1,R_3,R_2)$	$r^{\frown}(R_3,R_1,R_2)$
bt	bt	$rs{:}ls/\delta(bt,bf) \cup bt/\delta(rs,ls) \cup \delta(bt,bf) \cup bf{:}af/\delta(bt,ls) \cup bf{:}af/\delta(bt,rs) \cup$ $bf{:}ls/bt \cup rs{:}bf/bt \cup bt{:}af/ls \cup bt{:}af/rs \cup D_c$
rs	ls	$\{rs,ou\}$
bf	af	$af/\delta(rs,ls) \cup rs{:}ls \cup ou$
ls	rs	$\{ls,ou\}$
af	bf	$bt/\delta(rs,bf,ls) \cup rs{:}ls/bf$
$bt{:}rs$	$bt{:}ls$	$\delta(bt,bf)/rs \cup bt/\delta(rs,bf,af) \cup bf{:}af/rs \cup D_c$
$bt{:}bf$	$bt{:}af$	$bt/\delta(rs,bf,af) \cup bf{:}af/rs \cup bt/\delta(bf,ls,af) \cup bf{:}af/ls \cup D_c$
$bt{:}ls$	$bt{:}rs$	$\delta(bt,bf)/ls \cup bt/\delta(bf,ls,af) \cup bf{:}af/ls \cup D_c$
$bt{:}af$	$bt{:}bf$	$bt/\delta(rs,bf,ls) \cup rs{:}ls/bf \cup bf/bt \cup bf{:}af/\delta(bt,ls) \cup bf{:}af/\delta(bt,rs) \cup bf{:}ls/bt \cup$ $rs{:}bf/bt \cup bt{:}af/ls \cup bt{:}af/rs \cup D_c$
$rs{:}bf$	$ls{:}af$	$af/rs \cup ou$
$rs{:}ls$	$rs{:}ls$	$af \cup \delta(bt,bf) \cup D_c$
$rs{:}af$	$bf{:}ls$	$bt/\delta(rs,bf) \cup rs{:}bf \cup \{in,in{:}ou\}$
$bf{:}ls$	$rs{:}af$	$af/ls \cup ou$
$bf{:}af$	$bf{:}af$	$bt/\delta(rs,bf,af) \cup bf{:}af/rs \cup bt/\delta(bf,ls,af) \cup bf{:}af/ls \cup D_c$
$ls{:}af$	$rs{:}bf$	$bt/\delta(bf,ls) \cup bf{:}ls \cup \{in,in{:}ou\}$
$bt{:}rs{:}bf$	$bt{:}ls{:}af$	$bt/\delta(rs,bf,af) \cup bf{:}af/rs \cup D_c$
$bt{:}rs{:}ls$	$bt{:}rs{:}ls$	$\delta(bt,bf) \cup D_c$
$bt{:}rs{:}af$	$bt{:}bf{:}ls$	$\delta(bt,bf) \cup rs{:}bf \cup bt/\delta(rs,bf,af) \cup bf{:}af/rs \cup D_c$
$bt{:}bf{:}ls$	$bt{:}rs{:}af$	$bt/\delta(bf,ls,af) \cup bf{:}af/ls \cup D_c$
$bt{:}bf{:}af$	$bt{:}bf{:}af$	$bt/\delta(rs,bf,af) \cup bf{:}af/rs \cup bt/\delta(bf,ls,af) \cup bf{:}af/ls \cup D_c$
$bt{:}ls{:}af$	$bt{:}rs{:}bf$	$\delta(bt,bf) \cup bf{:}ls \cup bt/\delta(bf,ls,af) \cup bf{:}af/ls \cup D_c$
$rs{:}bf{:}ls$	$rs{:}ls{:}af$	$af \cup D_c$
$rs{:}bf{:}af$	$bf{:}ls{:}af$	$bt/\delta(rs,bf,af) \cup bf{:}af/rs \cup D_c$
$rs{:}ls{:}af$	$rs{:}bf{:}ls$	$bt/bf \cup in \cup in{:}ou$
$bf{:}ls{:}af$	$rs{:}bf{:}af$	$bt/\delta(bf,ls,af) \cup bf{:}af/ls \cup D_c$
$bt{:}rs{:}bf{:}ls$	$bt{:}rs{:}ls{:}af$	D_c
$bt{:}rs{:}bf{:}af$	$bt{:}bf{:}ls{:}af$	$bt/\delta(rs,bf,af) \cup bf{:}af/rs \cup D_c$
$bt{:}rs{:}ls{:}af$	$bt{:}rs{:}bf{:}ls$	$bt/bf \cup in \cup in{:}ou$
$bt{:}bf{:}ls{:}af$	$bt{:}rs{:}bf{:}af$	$bt/\delta(bf,ls,af) \cup bf{:}af/ls \cup D_c$
$rs{:}bf{:}ls{:}af$	$rs{:}bf{:}ls{:}af$	$\{in,in{:}ou\}$
$bt{:}rs{:}bf{:}ls{:}af$	$bt{:}rs{:}bf{:}ls{:}af$	$\{in,in{:}ou\}$
in	in	$bt/D_d \cup rs{:}ls/D_d \cup bf{:}af/D_d \cup D_c$
ou	ou	$bt/\delta(rs,ls,af) \cup rs/\delta(bt,ls,af) \cup ls/\delta(bt,rs,af) \cup af/\delta(bt,rs,ls) \cup D_c$
$in{:}ou$	$in{:}ou$	$bt/D_d \cup rs{:}ls/D_d \cup bf{:}af/D_d \cup D_c$

TABLE III
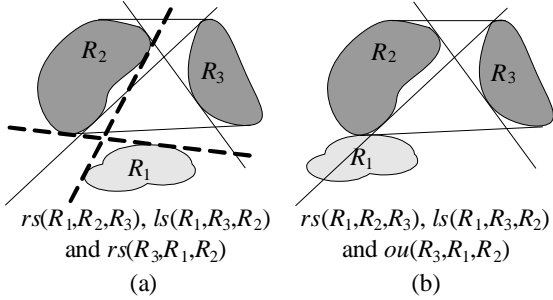CONVERSE AND ROTATION FOR PROJECTIVE RELATIONS BETWEEN REGIONS

Fig. 7. Converse and rotation example

$rs(R_1,R_2,R_3)$ and $rs(R_2,R_3,R_1)$ and *(b)* $ou \in rs^{\frown}$ since in Figure 7b, $rs(R_1,R_2,R_3)$ and $ou(R_3,R_1,R_2)$ holds.

To compute the converse relations presented in Table III, we use the following proposition.

Proposition 2: The converse $r^{\smile}(R_1,R_3,R_2)$ of a relation $r(R_1,R_2,R_3)$ can be computed by performing the following substitutions to the tiles of r.

Tiles of r	Replace with (to get $r^{\smile}$)
bt	bt
rs	ls
ls	rs
bf	af
af	bf

Proof: The proof easily follows from the symmetry of the projective relations. ∎

For instance, the converse of relation rs is relation ls and the converse of relation $bt{:}rs{:}bf$ is relation $bt{:}ls{:}af$ (see also Table III).

The rotation operation is much more involved. To compute the rotation operator, we have implemented Algorithm COM-PUTEROTATION (Figure 8). Given three regions R_1, R_2 and R_3, the above algorithm computes relation $r(R_1,R_2,R_3)$ and its rotation $q(R_3,R_1,R_2)$ (Step 4). Algorithms for computing projective relations were presented in [9]. Regions R_1, R_2 and R_3 are constructed by the union of two rectangles (Step 3) that are taken from a large array of random rectangles $\mathcal{R}$ (Step 1). We consider regions formed by the union of two rectangles because simple rectangles are not general enough to satisfy

multi-tile relations such as $bf{:}af$ or $rs{:}ls$. In our experiments, we have varied the size of array $\mathcal{R}$ from 100,000 to 1,000,000.

Algorithm COMPUTEROTATION
Method:
1. <u>Create</u> an array $\mathcal{R}[1 \ldots n]$ that contains n random rectangles
2. <u>For</u> $i = 1$ <u>To</u> $n - 5$
3.   <u>Set</u> $R_1 = \mathcal{R}[i] \cup \mathcal{R}[i+1]$, $R_2 = \mathcal{R}[i+2] \cup \mathcal{R}[i+3]$, $R_3 = \mathcal{R}[i+4] \cup \mathcal{R}[i+5]$
4.   COMPUTE relations $r(R_1, R_2, R_3)$ and $q(R_3, R_1, R_2)$
5.   ADD q to the rotation entry of r // Since $q \in r^\frown$
6. <u>EndFor</u>

Fig. 8.   Algorithm COMPUTEROTATION

The results that Algorithm COMPUTEROTATION produces are definitely sound since for every result there is an actual configuration of regions satisfying the relation and its rotation. In fact, besides running the algorithm, we checked the results of the table by exhaustively drawing all the configurations, making sure that no results are missing. With another routine, we then manipulate the output of the table to find the compact form with the δ operator. The final result is presented in Table III.

B. Composing a single-tile with a basic relation

For any single-tile relation $r_1(A, B, C)$ and any basic (single or multi-tile) relation $r_2(B, C, D)$, Table IV presents the relations that correspond to their composition $r_1 \circ r_2(A, C, D)$. Similarly to Table II, we use IMP to denote impossible cases.

Fig. 9.   Example of composition

Example 9: Assume that we want to compute the composition of bf with rs. According to Table IV, we have:

$$bf \circ rs = \delta(rs, af) = \{rs, af, rs{:}af\}.$$

To verify this equation, consider Figure 9. We have:

$$
\begin{aligned}
rs \in bf \circ rs &\text{ since } &bf(A_1, B, C), rs(B, C, D) \\
& &\text{and } rs(A_1, C, D). \\
af \in bf \circ rs &\text{ since } &bf(A_2, B, C), rs(B, C, D) \\
& &\text{and } af(A_2, C, D). \\
rs{:}af \in bf \circ rs &\text{ since } &bf(A_1 \cup A_2, B, C), rs(B, C, D) \\
& &\text{and } rs{:}af(A_1 \cup A_2, C, D).
\end{aligned}
$$

Also notice that for all relations $q \notin \delta(rs, af)$ it is impossible to find configurations such that $q(A, C, D)$, $bf(A, B, C)$ and $rs(B, C, D)$ simultaneously hold.

To compute the composition results of Table IV, we proceed as follows. For every pair of a single-tile relation r_1 and a basic relation r_2, we consider every possible basic relation r_3 and check if there exist regions A, B, C and D such that

$$r_1(A, B, C), \ r_2(B, C, D) \text{ and } r_3(A, C, D)$$

hold. If we can find such regions then, according to Definition 3, $r_3 \in r_1 \circ r_2$ holds and thus r_3 is added in the (r_1, r_2) entry of Table IV. Notice that, for each entry of Table IV, in the worst case we have to consider 34 (i.e., the total number of projective relations) configurations involving regions A, B, C and D. To assist this procedure, we have implemented Algorithm COMPUTECOMPOSITION (Figure 10). Given four regions R_1, R_2, R_3 and R_4, the above algorithm computes relation $r_1(R_1, R_2, R_3)$ and $r_2(R_2, R_3, R_4)$ and their composition $r_3(R_1, R_3, R_4)$. Similarly to the rotation operation, regions R_1, R_2, R_3 and R_4 are constructed by the union of two rectangles (Step 3) that are taken from a large array of random rectangles $\mathcal{R}$ (Step 1). In our experiments, we have varied the size of array $\mathcal{R}$ from 100,000 to 1,000,000.

Algorithm COMPUTECOMPOSITION
Method:
1. <u>Create</u> an array $\mathcal{R}[1 \ldots n]$ that contains n random rectangles
2. <u>For</u> $i = 1$ <u>To</u> $n - 7$
3.   <u>Set</u> $R_1 = \mathcal{R}[i] \cup \mathcal{R}[i+1]$, $R_2 = \mathcal{R}[i+2] \cup \mathcal{R}[i+3]$, $R_3 = \mathcal{R}[i+4] \cup \mathcal{R}[i+5]$, $R_4 = \mathcal{R}[i+6] \cup \mathcal{R}[i+7]$
4.   COMPUTE relations $r_1(R_1, R_2, R_3)$, $r_2(R_2, R_3, R_4)$ and $r_3(R_1, R_3, R_4)$.
5.   ADD the r_3 to the composition entry of $r_1 \circ r_2$ // Since $r_3 \in r_1 \circ r_2$
6. <u>EndFor</u>

Fig. 10.   Algorithm COMPUTECOMPOSITION

Similarly to the discussion about rotation, the results that Algorithm COMPUTECOMPOSITION produces are definitely sound since for every result there is an actual configuration of regions satisfying relations and their composition. As part of the proof, we manually drew all the configurations (reported in the Appendix). An empirical support to the completeness of Table IV is given by the fact that continuing to run the experiments no other results were found. The automatic procedure can also be used to find results for the full composition table of two basic relations (34 times 34 table), which is treated in Section VII-C.

C. Composing basic relations

Let us consider two basic relations r and q and let us assume that $r = r_1{:}\cdots{:}r_k$. In this section, we will reduce the computation of $r \circ q$ to the computation of compositions $r_1 \circ q$, ..., $r_k \circ q$. All these expressions denote the composition of a single-tile with a basic relation and can be computed using Table IV (see also Section VII-C). A natural method to perform this reduction is to use the product of relations (Definition 12). Specifically, we may use the expression $\pi(r_1 \circ q, \ldots, r_k \circ q)$. We will refer to this expression as the *product expression*.

The product expression correctly computes the composition of relation $in{:}ou$ as the following lemma demonstrates.

$r_2\backslash r_1$	bt	rs	bf	ls	af	in	ou
bt	bt	$\delta(bt,bf,ls,af)$	$\delta(bt,rs,ls,af) \cup \delta(bt,bf,ls,af) \cup \delta(bt,rs,bf,af)$	$\delta(bt,rs,bf,af)$	$\delta(bt,rs,bf,ls)\cup \delta(bt,rs,bf,af)\cup \delta(bt,bf,ls,af)$	bt	D_d
rs	$\delta(bt,rs,bf)$	$\delta(bt,rs,ls,af)$	$\delta(rs,af)$	$\delta(rs,bf,ls)$	$\delta(bt,bf,ls)$	$\delta(bt,rs,bf)$	D_d
bf	$\delta(bt,bf)$	$\delta(bt,rs,bf,af)$	$\delta(rs,bf,ls) \cup \delta(rs,bf,af) \cup \delta(bf,ls,af)$	$\delta(bt,bf,ls,af)$	D_d	$\delta(bt,bf)$	D_d
ls	$\delta(bt,bf,ls)$	$\delta(rs,bf,ls)$	$\delta(ls,af)$	$\delta(bt,rs,ls,af)$	$\delta(bt,rs,bf)$	$\delta(bt,bf,ls)$	D_d
af	D_d	$\delta(bt,bf,ls,af)$	$\delta(bt,af)$	$\delta(bt,rs,bf,af)$	$\delta(bt,bf)$	IMP	IMP
bt:rs	$\delta(bt,rs,bf,af)$	$\delta(bt,ls,af)$	$\delta(bt,rs,ls,af) \cup \delta(bt,rs,bf,af)$	$\delta(rs,bf)$	$\delta(bt,rs,bf,ls)\cup \delta(bt,bf,ls,af)$	$\delta(bt,rs,bf,af)$	D_d
bt:bf	$\delta(bt,rs,bf,af)\cup \delta(bt,bf,ls,af)$	$\delta(bt,af) \cup bf$	$\delta(bt,rs,bf,af)\cup \delta(bt,bf,ls,af)$	$\delta(bt,af) \cup bf$	$\delta(bt,bf,ls,af)\cup \delta(bt,rs,bf,af)$	D_d	D_d
bt:ls	$\delta(bt,bf,ls,af)$	$\delta(bf,ls)$	$\delta(bt,rs,ls,af) \cup \delta(bt,bf,ls,af)$	$\delta(bt,rs,af)$	$\delta(bt,rs,bf,ls)\cup \delta(bt,rs,bf,af)$	$\delta(bt,bf,ls,af)$	D_d
bt:af	D_d	$\delta(bt,bf,ls,af)$	$\delta(bt,rs,ls,af) \cup \delta(bt,bf,ls,af) \cup \delta(bt,rs,bf,af)$	$\delta(bt,rs,bf,af)$	$\delta(bt,rs,bf,ls)\cup \delta(bt,rs,bf,af)\cup \delta(bt,bf,ls,af)$	D_d	D_d
rs:bf	$\delta(bt,rs,bf)$	$\delta(bt,rs,af)$	$\delta(rs,bf,ls) \cup \delta(rs,bf,af)$	$\delta(bf,ls)$	D_d	$\delta(bt,rs,bf)$	D_d
rs:ls	D_d	$\{rs,ls\}$	$\delta(bt,rs,ls,af) \cup \delta(rs,bf,ls)$	$\{rs,ls\}$	$\delta(bt,rs,ls,af)\cup \delta(rs,bf,ls)$	D_d	D_d
rs:af	D_d	$\delta(bt,ls,af)$	$\delta(bt,rs,af)$	$\delta(rs,bf)$	$\delta(bt,bf,ls)$	D_d	D_d
bf:ls	$\delta(bt,bf,ls)$	$\delta(rs,bf)$	$\delta(rs,bf,ls) \cup \delta(bf,ls,af)$	$\delta(bt,ls,af)$	D_d	$\delta(bt,bf,ls)$	D_d
bf:af	D_d	$\delta(bt,af) \cup bf$	$\delta(bt,rs,bf,af)\cup \delta(bt,bf,ls,af)$	$\delta(bt,af) \cup bf$	$\delta(bt,rs,bf,af)\cup \delta(bt,bf,ls,af)$	D_d	D_d
ls:af	D_d	$\delta(bf,ls)$	$\delta(bt,ls,af)$	$\delta(bt,rs,af)$	$\delta(bt,rs,bf)$	D_d	D_d
bt:rs:bf	$\delta(bt,rs,bf,af)$	$\delta(bt,af)$	$\delta(bt,rs,bf,af)$	bf	$\delta(bt,bf,ls,af)$	D_d	D_d
bt:rs:ls	D_d	ls	$\delta(bt,rs,ls,af)$	rs	$\delta(rs,bf,ls)$	D_d	D_d
bt:rs:af	D_d	$\delta(bt,ls,af)$	$\delta(bt,rs,ls,af) \cup \delta(bt,rs,bf,af)$	$\delta(rs,bf)$	$\delta(bt,rs,bf,ls)\cup \delta(bt,bf,ls,af)$	D_d	D_d
bt:bf:ls	$\delta(bt,bf,ls,af)$	bf	$\delta(bt,bf,ls,af)$	$\delta(bt,af)$	$\delta(bt,rs,bf,af)$	D_d	D_d
bt:bf:af	D_d	$\delta(bt,af)\cup bf$	$\delta(bt,rs,bf,af)\cup \delta(bt,bf,ls,af)$	$\delta(bt,af) \cup bf$	$\delta(bt,rs,bf,af)\cup \delta(bt,bf,ls,af)$	D_d	D_d
bt:ls:af	D_d	$\delta(bf,ls)$	$\delta(bt,rs,ls,af) \cup \delta(bt,bf,ls,af)$	$\delta(bt,rs,af)$	$\delta(bt,rs,bf,ls)\cup \delta(bt,rs,bf,af)$	D_d	D_d
rs:bf:ls	$\delta(bt,rs,bf,ls)$	rs	$\delta(rs,bf,ls)$	ls	$\delta(bt,rs,ls,af)$	D_d	D_d
rs:bf:af	D_d	$\delta(bt,af)$	$\delta(bt,rs,bf,af)$	bf	$\delta(bt,bf,ls,af)$	D_d	D_d
rs:ls:af	D_d	ls	$\delta(bt,rs,ls,af)$	rs	$\delta(rs,bf,ls)$	D_d	D_d
bf:ls:af	D_d	bf	$\delta(bt,bf,ls,af)$	$\delta(bt,af)$	$\delta(bt,rs,bf,af)$	D_d	D_d
bt:rs:bf:ls	IMP	IMP	IMP	IMP	IMP	D_d	D_d
bt:rs:bf:af	D_d	$\delta(bt,af)$	$\delta(bt,rs,bf,af)$	bf	$\delta(bt,bf,ls,af)$	D_d	D_d
bt:rs:ls:af	D_d	ls	$\delta(bt,rs,ls,af)$	rs	$\delta(rs,bf,ls)$	D_d	D_d
bt:bf:ls:af	D_d	bf	$\delta(bt,bf,ls,af)$	$\delta(bt,af)$	$\delta(bt,rs,bf,af)$	D_d	D_d
rs:bf:ls:af	IMP	IMP	IMP	IMP	IMP	D_d	D_d
bt:rs:bf:ls:af	IMP	IMP	IMP	IMP	IMP	D_d	D_d
in	in	D_c	D_c	D_c	D_c	in	D_c
ou	D_c	D_c	ou	D_c	D_c	D_c	D_c
in:ou	D_c	D_c	D_c	D_c	D_c	D_c	D_c

TABLE IV

COMPOSING A SINGLE-TILE WITH A BASIC RELATION

Lemma 1: Let q is a basic relation. The composition of relation $in{:}ou$ and q can be computed using the following formula:

$$in{:}ou \circ q = \pi(\ in \circ q,\ ou \circ q\ ).$$

For example, using Lemma 1, we have

$$in{:}ou \circ rs\ =\ \pi(\ in \circ rs,\ ou \circ rs\ )$$
$$=\ \pi(\ \delta(bt,rs,bf),\ D_d\ )$$
$$=\ D_d.$$

This result can be verified using Figure 11. Notice that $rs(B,C,D)$ holds. To compute the composition we should investigate the possible position of a region A such that $in{:}ou(A,B,C)$ holds. This means that A has a part that is inside and a part that is outside $CH(B \cup C)$. It is not hard to verify that in general $D_d(A,C,D)$ holds. For instance, if $A = A_1 \cup A_2$ then $bt{:}af(A,C,D)$ holds.
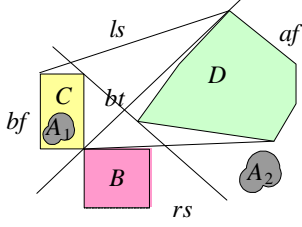
Fig. 11. Composing $in{:}ou$ and rs

Let us now investigate the relation between composition $(r \circ q)$ and the product expression $(\pi(r_1 \circ q, \ldots, r_k \circ q))$ in the case of the remaining ternary projective relations. Consider the following example.

Example 10: Let us consider relations $bt{:}rs$ and $bt{:}bf$. According to Table IV, we have:

$$bt \circ bt{:}bf = \delta(bt, rs, bf, af) \cup \delta(bt, bf, ls, af) \text{ and}$$
$$rs \circ bt{:}bf = \delta(bt, af) \cup bf.$$

Thus, the product expression gives:

$$\pi(bt \circ bt{:}bf, \ rs \circ bt{:}bf) =$$
$$\pi(\delta(bt, rs, bf, af) \cup \delta(bt, bf, ls, af), \delta(bt, af) \cup bf) =$$

$$\pi\left(\left\{\begin{array}{c} bt, rs, bf, ls, af, \\ bt{:}rs, bt{:}bf, bt{:}ls, \\ bt{:}af, rs{:}bf, rs{:}af, \\ bf{:}ls, bf{:}af, ls{:}af, \\ bt{:}rs{:}bf, bt{:}rs{:}af, \\ bt{:}bf{:}ls, bt{:}bf{:}af, \\ bt{:}ls{:}af, rs{:}bf{:}af, \\ bf{:}ls{:}af, bt{:}rs{:}bf{:}af, \\ bt{:}bf{:}ls{:}af \end{array}\right\}, \{bt, bf, af, bt{:}af\}\right) =$$

$$\left\{\begin{array}{c} bt, \ bf, \ af, \ bt{:}rs, \ bt{:}bf, \ bt{:}ls, \ bt{:}af, \ rs{:}bf, \\ rs{:}af, \ bf{:}ls, \ bf{:}af, \ ls{:}af, \ bt{:}rs{:}bf, \\ bt{:}rs{:}af, \ bt{:}bf{:}ls, \ bt{:}bf{:}af, \ bt{:}ls{:}af, \\ rs{:}bf{:}af, \ bf{:}ls{:}af, \ bt{:}rs{:}bf{:}af, \ bt{:}bf{:}ls{:}af \end{array}\right\}.$$

The above expression does not correctly compute the composition. For instance, $bt{:}ls \in \pi(bt \circ bt{:}bf, \ rs \circ bt{:}bf)$ but $bt{:}ls \notin bt{:}rs \circ bt{:}bf$ because there cannot be regions A, B, C and D such that $bt{:}ls(A, C, D)$, $bt{:}rs(A, B, C)$ and $bt{:}bf(B, C, D)$ hold at the same time.

In Example 10, we have seen that the product expression contains relations that do not belong to the composition. We can prove that a possible result of the composition operator is necessarily included in the product expression but the product expression may contain relations that do not belong to the composition. This fact is captured in the following proposition.

Proposition 3: Let $r_1{:}\cdots{:}r_k$ and q be two projective relations. Then, we have:

$$(r_1{:}\cdots{:}r_k) \circ q \subseteq \pi(r_1 \circ q, \ \ldots, \ r_k \circ q).$$

Proof: We will first prove that if $u \in r_1{:}\cdots{:}r_k \circ q$ then $u \in \pi(r_1 \circ q, \ \ldots, \ r_k \circ q)$.

Since $u \in r_1{:}\cdots{:}r_k \circ q$ holds, there are regions A, B, C and D such that

$$r_1{:}\cdots{:}r_k(A, B, C) \wedge q(B, C, D) \wedge u(A, C, D).$$

Since, $r_1{:}\cdots{:}r_k(A, B, C)$ holds, according to Definition 6, there are regions $A_1, \ldots, A_k$ such that $A = A_1 \cup \cdots \cup A_k$ and $r_1(A_1, B, C)$, ..., $r_k(A_k, B, C)$ hold. Thus, we have:

$$r_1(A_1, B, C) \wedge \cdots \wedge r_k(A_k, B, C) \wedge q(B, C, D) \wedge u(A, C, D).$$

Now, let u_1 be the relation that holds between regions A_1, C and D, i.e., $u_1(A_1, C, D)$. Similarly, we define $u_2(A_2, C, D)$, ..., $u_k(A_k, C, D)$. Notice that $u = tile\text{-}union(u_1, \ldots, u_k)$ holds, thus we have:

$$r_1(A_1, B, C) \wedge \cdots \wedge r_k(A_k, B, C) \wedge$$
$$q(B, C, D) \wedge u_1(A_1, C, D) \wedge \cdots \wedge u_k(A_k, C, D)$$

and by rewriting the expression we have:

$$r_1(A_1, B, C) \wedge q(B, C, D) \wedge u_1(A_1, C, D)$$
$$\wedge \cdots \wedge$$
$$r_k(A_k, B, C) \wedge q(B, C, D) \wedge u_k(A_k, C, D).$$

Summarizing, according to Definition 3, we have $u_1 \in r_1 \circ q$, ..., $u_k \in r_k \circ q$ and $u = tile\text{-}union(u_1, \ldots, u_k)$, thus, $u \in \pi(r_1 \circ q, \ \ldots, \ r_k \circ q)$ (see also Definition 12). To conclude this proof, we notice that Example 10 illustrates a case where $u \in \pi(r_1 \circ q, \ \ldots, \ r_k \circ q)$ and $u \notin r \circ q$. ∎

In total the product expression, although it seems to be a natural choice, results in a superset of the composition result. It is interesting to identify the cases where the product expression produces a result that does not belong to the composition. To this end, let us reconsider Example 10. The composition of $bt{:}rs$ with $bt{:}bf$ is given by the following formula.

$$bt{:}rs \circ bt{:}bf = \left\{\begin{array}{c} bt, \ bf, \ af, \ bt{:}rs, \ bt{:}bf, \ bt{:}af, \\ rs{:}af, \ bf{:}ls, \ bf{:}af, \ bt{:}rs{:}bf, \\ bt{:}rs{:}af, \ bt{:}bf{:}ls, \ bt{:}bf{:}af, \\ rs{:}bf{:}af, \ bf{:}ls{:}af, \\ bt{:}rs{:}bf{:}af, \ bt{:}bf{:}ls{:}af \end{array}\right\} \quad (1)$$

To verify this expression we consider Figure 12. In this figure, we present two configurations involving three regions B, C and D such that $bt{:}bf(B, C, D)$. In both configurations of Figure 12, a region A satisfies relation $bt{:}rs(A, B, C)$ iff it has a part that lies in the dark-shaded area and a part that lies in the light-shaded area. It is not hard to verify that all the possible relations that hold between A, C and D are prescribed by the previous expression. Also notice that for all relations q that are not mentioned in the set of Equation 1 it is impossible to find configurations such that $q(A, C, D)$, $bt{:}rs(A, B, C)$ and $bt{:}bf(B, C, D)$ simultaneously hold.

(a) (b)
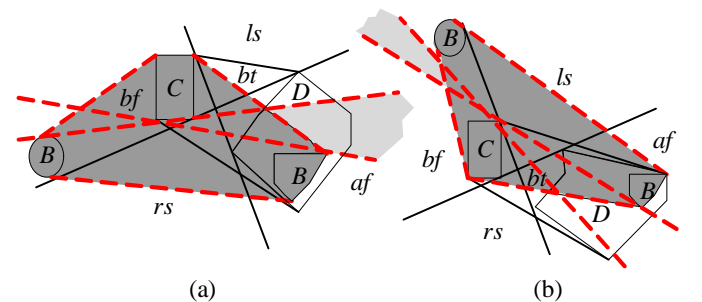
Fig. 12. Composing $bt{:}rs$ with $bt{:}bf$

Equation 1 can be equivalently written as follows.

$$bt{:}rs \circ bt{:}bf = \pi(\ \delta(bt,rs,bf,af), \delta(bt,af)\ )\cup$$
$$\pi(\ \delta(bt,bf,ls,af), bf\ ) \qquad (2)$$

Let us compare the above expression with the product expression of Example 10, i.e.,

$$\pi(bt \circ bt{:}bf,\ rs \circ bt{:}bf) =$$
$$\pi(\ \delta(bt,rs,bf,af)\cup\delta(bt,bf,ls,af),\ \delta(bt,af)\cup bf\ ).$$

Using Proposition 1, we have:

$$\pi(bt \circ bt{:}bf,\ rs \circ bt{:}bf) = \pi(\ \delta(bt,rs,bf,af),\delta(bt,af)\ )\cup$$
$$\pi(\ \delta(bt,bf,ls,af),bf\ )\ \cup\ \pi(\ \delta(bt,rs,bf,af),bf\ )\cup$$
$$\pi(\ \delta(bt,bf,ls,af),\delta(bt,af)\ )$$

and

$$\pi(bt \circ bt{:}bf,\ rs \circ bt{:}bf) = bt{:}rs \circ bt{:}bf\cup$$
$$\pi(\ \delta(bt,rs,bf,af),bf\ )\ \cup\ \pi(\ \delta(bt,bf,ls,af),\delta(bt,af)\ ).$$

In other words, to get the correct composition result we should rule out some combinations that appear in the product expression (see also Example 10 and Proposition 3). These combinations correspond to *impossible* configurations. In our example, it is not possible to find regions A, B, C and D such that $r(A,C,D)$, $bt{:}rs(A,B,C)$ and $bt{:}bf(B,C,D)$ for any relation $r \in \pi(\ \delta(bt,rs,bf,af),bf\ )\cup \pi(\ \delta(bt,bf,ls,af),\delta(bt,af)\ )$. This can be verified using Figure 12.

Inspired by this observation, we refine Table IV that presents the composition of a single-tile with a basic relation. First, for each entry of the table, we consider the possible configurations that produce the composition result. Then, for each row of the table we group the compatible configurations in the same row. The new results appear in Tables V and VI. In total, these tables present the result of the composition of a single-tile relation r_1 with a basic (single or multi-tile) relation r_2. These tables have 34 rows (numbered from 1 to 34) that correspond to the basic relation r_2. Each row is divided into several subrows. For instance, the rows numbered 1 and 13 are divided into 4 and 3 subrows respectively. The composition of relations r_1 and r_2 is computed by the union of the subrows that lie in on the crossing of the row title r_1 with the column title r_2. It is easy to verify that Table IV can be produced by Tables V and VI by unifying the subrows of every cell. For instance, using Table V we have:

$$\begin{aligned} bf \circ bt\ =\ & \delta(bt,af) \cup \delta(bt,rs,ls,af) \cup \delta(bt,bf,ls,af)\cup\\ & \delta(bt,rs,bf,af)\\ =\ & \delta(bt,rs,ls,af) \cup \delta(bt,bf,ls,af)\cup\\ & \delta(bt,rs,bf,af) \end{aligned}$$

This is exactly the result we get by using Table IV.

Each subrow of Tables V and VI corresponds to a possible configuration. Results taken in different subrows of the same row correspond to impossible configurations. For instance, to compute the composition of $bt{:}rs$ and $bt{:}bf$ we consider the elements that lie on the crossing of the 7th row of Table V with columns title bt and rs. To get the result, we take the union of the product of the elements of all subrows. In total, we have:

$$bt{:}rs \circ bt{:}bf = \pi(\ \delta(bt,rs,bf,af),\delta(bt,af)\ )\cup$$
$$\pi(\ \delta(bt,bf,ls,af),bf\ )$$

which is exactly the result of Equation 2.

As another example, the composition of $bt{:}ls{:}af$ with $bt{:}rs{:}af$ is given by the following formula.

$$\begin{aligned} bt{:}ls{:}af \circ bt{:}rs{:}af =\ & \\ \pi(\ & D_d, \delta(rs,bf), \delta(bf,ls)\ )\cup\\ \pi(\ & \delta(bt,rs,bf,af), \delta(rs,bf), \delta(bt,bf,ls)\ )\cup\\ \pi(\ & \delta(bt,rs,bf,af), rs, \delta(bt,rs,bf,ls)\ )\cup\\ \pi(\ & \delta(bt,rs,bf,af), bf, \delta(bt,bf,ls,af)\ )\cup\\ \pi(\ & D_d, rs, \delta(rs,bf,ls)\ ) \end{aligned}$$

To compute Tables V and VI we consider one row at a time. For each row, we consider all different configurations that are able to reproduce all the composition results that appear for the respective row in Table IV. Each configuration corresponds to a subrow. For example, let us consider the first row of Table V, i.e., relation bt. To complete this row, we consider different configurations of regions B, C and D such that $bt(B,C,D)$ holds. To reproduce the results of Table IV, we need four different configurations that correspond to the four different subrows of Table V. These configurations appear in Figure 13. For instance, in Figure 13a:

1) For all regions A such that $bt(A,B,C)$, we have $bt(A,C,D)$.
2) For all regions A such that $rs(A,B,C)$, we have $\delta(bt,bf,ls,af)(A,C,D)$.
3) For all regions A such that $bf(A,B,C)$, we have $\delta(bt,af)(A,C,D)$.
4) For all regions A such that $ls(A,B,C)$, we have $\delta(bt,rs,bf,af)(A,C,D)$.
5) For all regions A such that $af(A,B,C)$, we have $\delta(bt,bf)(A,C,D)$.

All these results are captured in the (a) subrow of bt. Similarly, using Figures 13b-d, we can verify all the other subrows of bt.

The configurations that we consider for each row are a minimal covering set: this assures that no results in the composition are missing and that impossible results are filtered out. The configurations that verify all the other rows of Tables V and VI are presented in the Appendix.

Algorithm COMPOSE
Input: Two basic relations $r = r_1{:}\cdots{:}r_k$ and q.
Output: The composition $r \circ q$.
Method:
$result = \emptyset$
<u>If</u> $r \in \{in{:}ou\}$ <u>Then</u>
   $result = \pi(\ in \circ q,\ ou \circ q\ )$
<u>Else</u>
   <u>For</u> each subrow s of q in Tables V and VI.
      <u>Let</u> $u_1, \ldots, u_k$ be the cells of s that correspond to $r_1, \ldots, r_k$
      $result = result \cup \pi(u_1, \ldots, u_k)$
   <u>EndFor</u>
<u>EndIf</u>
<u>Return</u> $result$

Fig. 14.   Algorithm COMPOSE

Overall, we obtain a procedure to find the composition of two basic relations starting from the knowledge of the composition of a single-tile relation with a basic relation. Such a procedure is summarized in Algorithm COMPOSE presented

	$r_2\backslash r_1$		bt	rs	bf	ls	af
1	bt	a	bt	$\delta(bt,bf,ls,af)$	$\delta(bt,af)$	$\delta(bt,rs,bf,af)$	$\delta(bt,bf)$
		b	bt	$\delta(bt,ls)$	$\delta(bt,rs,ls,af)$	$\delta(bt,rs)$	$\delta(bt,rs,bf,ls)$
		c	bt	$\delta(bt,bf)$	$\delta(bt,bf,ls,af)$	$\delta(bt,af)$	$\delta(bt,rs,bf,af)$
		d	bt	$\delta(bt,af)$	$\delta(bt,rs,bf,af)$	$\delta(bt,bf)$	$\delta(bt,bf,ls,af)$
2	rs	a	$\delta(bt,rs,bf)$	$\delta(bt,rs,ls,af)$	$\delta(rs,af)$	$\delta(rs,bf)$	$\delta(bt,bf,ls)$
		b	$\delta(bt,rs,bf)$	$\delta(bt,rs,ls,af)$	rs	$\delta(rs,bf,ls)$	$\delta(bt,bf,ls)$
3	bf	a	$\delta(bt,bf)$	$\delta(rs,bf)$	$\delta(rs,bf,ls)$	$\delta(bf,ls)$	D_d
		b	$\delta(bt,bf)$	$\delta(bt,rs,bf,af)$	$\delta(rs,bf,af)$	bf	$\delta(bt,bf,ls,af)$
		c	$\delta(bt,bf)$	bf	$\delta(bf,ls,af)$	$\delta(bt,bf,ls,af)$	$\delta(bt,rs,bf,af)$
		d	$\delta(bt,bf)$	$\delta(bt,rs,bf,af)$	bf	$\delta(bt,bf,ls,af)$	$\delta(bt,af)$
4	ls	a	$\delta(bt,bf,ls)$	$\delta(bf,ls)$	$\delta(ls,af)$	$\delta(bt,rs,ls,af)$	$\delta(bt,rs,bf)$
		b	$\delta(bt,bf,ls)$	$\delta(rs,bf,ls)$	ls	$\delta(bt,rs,ls,af)$	$\delta(bt,rs,bf)$
5	af	a	D_d	$\delta(bf,ls,af)$	$\delta(bt,af)$	$\delta(rs,bf,af)$	bf
		b	$\delta(bt,af)$	$\delta(bt,bf,ls,af)$	$\delta(bt,af)$	$\delta(bt,rs,bf,af)$	$\delta(bt,bf)$
		c	$\delta(bt,rs,bf,af)$	$\delta(bt,bf,ls,af)$	$\delta(bt,af)$	$\delta(rs,bf,af)$	$\delta(bt,bf)$
		d	$\delta(bt,bf,ls,af)$	$\delta(bf,ls,af)$	$\delta(bt,af)$	$\delta(bt,rs,bf,af)$	$\delta(bt,bf)$
6	bt:rs	a	$\delta(bt,rs,bf,af)$	$\delta(bt,ls)$	$\delta(bt,rs,ls,af)$	rs	$\delta(bt,rs,bf,ls)$
		b	$\delta(bt,rs,bf,af)$	$\delta(bt,ls,af)$	$\delta(bt,rs,af)$	$\delta(rs,bf)$	$\delta(bt,bf,ls)$
		c	$\delta(bt,rs,bf,af)$	$\delta(bt,af)$	$\delta(bt,rs,bf,af)$	bf	$\delta(bt,bf,ls,af)$
7	bt:bf	a	$\delta(bt,rs,bf,af)$	$\delta(bt,af)$	$\delta(bt,rs,bf,af)$	bf	$\delta(bt,bf,ls,af)$
		b	$\delta(bt,bf,ls,af)$	bf	$\delta(bt,bf,ls,af)$	$\delta(bt,af)$	$\delta(bt,rs,bf,af)$
8	bt:ls	a	$\delta(bt,bf,ls,af)$	ls	$\delta(bt,rs,ls,af)$	$\delta(bt,rs)$	$\delta(bt,rs,bf,ls)$
		b	$\delta(bt,bf,ls,af)$	$\delta(bf,ls)$	$\delta(bt,ls,af)$	$\delta(bt,rs,af)$	$\delta(bt,rs,bf)$
		c	$\delta(bt,bf,ls,af)$	bf	$\delta(bt,bf,ls,af)$	$\delta(bt,af)$	$\delta(bt,rs,bf,af)$
9	bt:af	a	D_d	$\delta(bf,ls,af)$	$\delta(bt,af)$	$\delta(rs,bf,af)$	bf
		b	$\delta(bt,af)$	$\delta(bt,bf,ls,af)$	$\delta(bt,af)$	$\delta(bt,rs,bf,af)$	$\delta(bt,bf)$
		c	$\delta(bt,af)$	$\delta(bt,ls)$	$\delta(bt,rs,ls,af)$	$\delta(bt,rs)$	$\delta(bt,rs,bf,ls)$
		d	$\delta(bt,af)$	$\delta(bt,bf)$	$\delta(bt,bf,ls,af)$	$\delta(bt,af)$	$\delta(bt,rs,bf,af)$
		e	$\delta(bt,af)$	$\delta(bt,af)$	$\delta(bt,rs,bf,af)$	$\delta(bt,bf)$	$\delta(bt,bf,ls,af)$
		f	$\delta(bt,rs,bf,af)$	$\delta(bt,ls)$	$\delta(bt,rs,ls,af)$	rs	$\delta(bt,rs,bf,ls)$
		g	$\delta(bt,bf,ls,af)$	ls	$\delta(bt,rs,ls,af)$	$\delta(bt,rs)$	$\delta(bt,rs,bf,ls)$
10	rs:bf	a	$\delta(bt,rs,bf)$	$\delta(bt,rs,af)$	$\delta(rs,bf,af)$	bf	$\delta(bt,bf,ls,af)$
		b	$\delta(bt,rs,bf)$	rs	$\delta(rs,bf,ls)$	$\delta(bf,ls)$	D_d
		c	$\delta(bt,rs,bf)$	$\delta(bt,rs,af)$	$\delta(rs,bf)$	$\delta(bf,ls)$	$\delta(bt,bf,ls,af)$
11	rs:ls	a	D_d	ls	$\delta(bt,rs,ls,af)$	rs	$\delta(rs,bf,ls)$
		b	$\delta(bt,rs,bf,ls)$	rs	$\delta(rs,bf,ls)$	ls	$\delta(bt,rs,ls,af)$
12	rs:af	a	D_d	$\delta(ls,af)$	$\delta(bt,rs,af)$	$\delta(rs,bf)$	$\delta(bf,ls)$
		b	$\delta(bt,rs,bf,af)$	$\delta(bt,ls,af)$	$\delta(bt,rs,af)$	$\delta(rs,bf)$	$\delta(bt,bf,ls)$
13	bf:ls	a	$\delta(bt,bf,ls)$	bf	$\delta(bf,ls,af)$	$\delta(bt,ls,af)$	$\delta(bt,rs,bf,af)$
		b	$\delta(bt,bf,ls)$	$\delta(rs,bf)$	$\delta(rs,bf,ls)$	ls	D_d
		c	$\delta(bt,bf,ls)$	$\delta(rs,bf)$	$\delta(bf,ls)$	$\delta(bt,ls,af)$	$\delta(bt,rs,bf,af)$
14	bf:af	a	$\delta(bt,rs,bf,af)$	$\delta(bt,af)$	$\delta(bt,rs,bf,af)$	bf	$\delta(bt,bf,ls,af)$
		b	D_d	af	$\delta(bt,rs,bf,af)$	bf	$\delta(bf,ls,af)$
		c	$\delta(bt,bf,ls,af)$	bf	$\delta(bt,bf,ls,af)$	$\delta(bt,af)$	$\delta(bt,rs,bf,af)$
		d	D_d	bf	$\delta(bt,bf,ls,af)$	af	$\delta(rs,bf,af)$
15	ls:af	a	D_d	$\delta(bf,ls)$	$\delta(bt,ls,af)$	$\delta(rs,af)$	$\delta(rs,bf)$
		b	$\delta(bt,bf,ls,af)$	$\delta(bf,ls)$	$\delta(bt,ls,af)$	$\delta(bt,rs,af)$	$\delta(bt,rs,bf)$

TABLE V

COMPOSING A SINGLE-TILE WITH A BASIC RELATION (1/2)

in Figure 14. Furthermore, we tested that the results obtained by Algorithm COMPOSE coincide with the results obtained by Algorithm COMPUTEROTATION.

Theorem 1: Algorithm COMPOSE correctly computes the composition of two basic relations.

Proof: The proof follows from Lemma 1 and the previous discussion. ■

VIII. CONCLUSIONS

In this paper, we developed a reasoning system for ternary projective relations among regions of the plane. The reasoning system is based on three kinds of rules: converse, rotation, and composition. Converse and rotation are given in the form of tables for the 34 basic projective relations of our model. While the converse table is more immediate to find, the rotation table has been experimentally verified with test data. The composition table is the more involved to construct since it contains 34×34 entries. Instead of finding out each entry of the table individually, we tried to discover algebraic rules that were able to compute, at least partially, the results. Though it was impossible to find algebraic rules for the whole table, we could find that, starting from a subset of the table related to the composition of the 7 single-tile relations with all 34 relations, we can compute the remaining part of the table with a so-called product operation. Therefore, we found the 7×34 table both manually and experimentally by running the computation

	$r_2\backslash r_1$		bt	rs	bf	ls	af
16	$bt{:}rs{:}bf$	a	$\delta(bt,rs,bf,af)$	$\delta(bt,af)$	$\delta(bt,rs,bf,af)$	bf	$\delta(bt,bf,ls,af)$
17	$bt{:}rs{:}ls$	a	D_d	ls	$\delta(bt,rs,ls,af)$	rs	$\delta(rs,bf,ls)$
18	$bt{:}rs{:}af$	a	D_d	$\delta(ls,af)$	$\delta(bt,rs,af)$	$\delta(rs,bf)$	$\delta(bf,ls)$
		b	$\delta(bt,rs,bf,af)$	$\delta(bt,ls,af)$	$\delta(bt,rs,af)$	$\delta(rs,bf)$	$\delta(bt,bf,ls)$
		c	$\delta(bt,rs,bf,af)$	$\delta(bt,ls)$	$\delta(bt,rs,ls,af)$	rs	$\delta(bt,rs,bf,ls)$
		d	$\delta(bt,rs,bf,af)$	$\delta(bt,af)$	$\delta(bt,rs,bf,af)$	bf	$\delta(bt,bf,ls,af)$
		e	D_d	ls	$\delta(bt,rs,ls,af)$	rs	$\delta(rs,bf,ls)$
19	$bt{:}bf{:}ls$	a	$\delta(bt,bf,ls,af)$	bf	$\delta(bt,bf,ls,af)$	$\delta(bt,af)$	$\delta(bt,rs,bf,af)$
20	$bt{:}bf{:}af$	a	$\delta(bt,rs,bf,af)$	$\delta(bt,af)$	$\delta(bt,rs,bf,af)$	bf	$\delta(bt,bf,ls,af)$
		b	D_d	af	$\delta(bt,rs,bf,af)$	bf	$\delta(bf,ls,af)$
		c	$\delta(bt,bf,ls,af)$	bf	$\delta(bt,bf,ls,af)$	$\delta(bt,af)$	$\delta(bt,rs,bf,af)$
		d	D_d	bf	$\delta(bt,bf,ls,af)$	af	$\delta(rs,bf,af)$
21	$bt{:}ls{:}af$	a	D_d	$\delta(bf,ls)$	$\delta(bt,ls,af)$	$\delta(rs,af)$	$\delta(rs,bf)$
		b	$\delta(bt,bf,ls,af)$	$\delta(bf,ls)$	$\delta(bt,ls,af)$	$\delta(bt,rs,af)$	$\delta(bt,rs,bf)$
		c	$\delta(bt,bf,ls,af)$	ls	$\delta(bt,rs,ls,af)$	$\delta(bt,rs)$	$\delta(bt,rs,bf,ls)$
		d	$\delta(bt,bf,ls,af)$	bf	$\delta(bt,bf,ls,af)$	$\delta(bt,af)$	$\delta(bt,rs,bf,af)$
		e	D_d	ls	$\delta(bt,rs,ls,af)$	rs	$\delta(rs,bf,ls)$
22	$rs{:}bf{:}ls$	a	$\delta(bt,rs,bf,ls)$	rs	$\delta(rs,bf,ls)$	ls	$\delta(bt,rs,ls,af)$
23	$rs{:}bf{:}af$	a	$\delta(bt,rs,bf,af)$	$\delta(bt,af)$	$\delta(bt,rs,bf,af)$	bf	$\delta(bt,bf,ls,af)$
		b	D_d	af	$\delta(bt,rs,bf,af)$	bf	$\delta(bf,ls,af)$
24	$rs{:}ls{:}af$	a	D_d	ls	$\delta(bt,rs,ls,af)$	rs	$\delta(rs,bf,ls)$
25	$bf{:}ls{:}af$	a	$\delta(bt,bf,ls,af)$	bf	$\delta(bt,bf,ls,af)$	$\delta(bt,af)$	$\delta(bt,rs,bf,af)$
		b	D_d	bf	$\delta(bt,bf,ls,af)$	af	$\delta(rs,bf,af)$
26	$bt{:}rs{:}bf{:}ls$	a	IMP	IMP	IMP	IMP	IMP
27	$bt{:}rs{:}bf{:}af$	a	$\delta(bt,rs,bf,af)$	$\delta(bt,af)$	$\delta(bt,rs,bf,af)$	bf	$\delta(bt,bf,ls,af)$
		b	D_d	af	$\delta(bt,rs,bf,af)$	bf	$\delta(bf,ls,af)$
28	$bt{:}rs{:}ls{:}af$	a	D_d	ls	$\delta(bt,rs,ls,af)$	rs	$\delta(rs,bf,ls)$
29	$bt{:}bf{:}ls{:}af$	a	$\delta(bt,bf,ls,af)$	bf	$\delta(bt,bf,ls,af)$	$\delta(bt,af)$	$\delta(bt,rs,bf,af)$
		b	D_d	bf	$\delta(bt,bf,ls,af)$	af	$\delta(rs,bf,af)$
30	$rs{:}bf{:}ls{:}af$	a	IMP	IMP	IMP	IMP	IMP
31	$bt{:}rs{:}bf{:}ls{:}af$	a	IMP	IMP	IMP	IMP	IMP
32	in	a	in	D_c	D_c	D_c	D_c
33	ou	a	D_c	D_c	ou	D_c	D_c
34	$in{:}ou$	a	D_c	D_c	D_c	D_c	D_c

TABLE VI
COMPOSING A SINGLE-TILE WITH A BASIC RELATION (2/2)

algorithms of the relations on test data. Then, we checked the results of the experiments to verify that the complete composition table that is found by the program corresponds to the table as it can be computed with algebraic rules.

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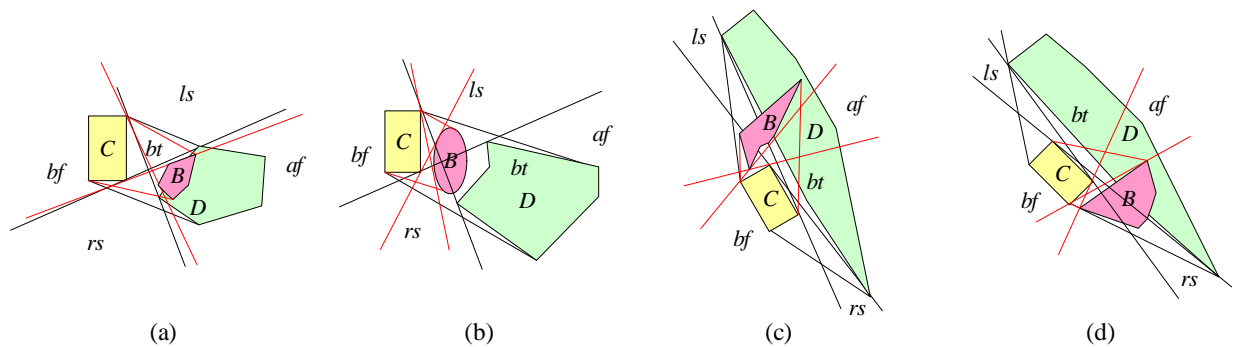[17] K.-P. Gapp, "From vision to language: A cognitive approach to the

Fig. 13. Subrows of row 1 (*bt*)

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