# On some generalisations of the Fréchet functional equations

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1905 Hamel provided such a solution using Zorn's lemma to get a basis (of $\mathbb{R}$ as a $\mathbb{Q}$-vector space) which now bears his name.

# Fréchet functional equation

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In general, when studying such an equation, one tries to obtain the weakest hypothesis under which the solutions are polynomials.

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although other conditions exist.

## Theorem

Let $m \in \mathbb{N}$ and $f : \mathbb{R} \to \mathbb{R}$ be a functions that is bounded almost everywhere on a neighborhood of a point $x_0 \in \mathbb{R}$.
If $\Delta_h^m f = 0$ is satisfied on a neighborhood of $x_0$ for almost every $h$ sufficiently small, then $f$ can be written as a polynomial of degree at most $m - 1$ on a neighborhood of $x_0$.

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- The proof becomes much simpler if one assumes the measurability of $f$.
- This result is still valid in $\mathbb{R}^n$, using $f = g * \Phi$ with $\Phi \in D(\mathbb{R}^n)$, $\int \Phi dx = 1$ and $g = f\chi_{B(x_0,\epsilon)}$.

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- Newton's interpolation formula:

$$f(x + qrh) = \sum_{j=0}^{q} \frac{\Delta_{rh}^j f(x)}{j!} (q)_j,$$

where $(q)_j$ is the falling factorial ($\prod_{k=0}^{j-1}(q-k)$ for $j > 0$).

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- Using the stirling numbers of the first kind $s(j,k)$ and of second kind $S(k,l)$ ($l \in \{0, \dots, m-1\}$),

$$\frac{\Delta_h^l f(x)}{l!} = \sum_{m=l-1}^{m-1} \sum_{j=k}^{q} \frac{\Delta_{rh}^j f(x)}{j!} s(j,k) r^{-k} S(k,l),$$

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- so that $|\frac{\Delta_h^l f(x)|}{l!}| \leq C' \sum_{k=l}^{m-1} \frac{|h|^k}{(\epsilon - h)^k}$ and $f$ is continuous near $x_0$ (take $l = 1$).

For $T \in D'(\mathbb{R}^n)$, let us define $\Delta_h^m T(\phi) = T(\Delta_{-h}^m \phi)$, for $h \in \mathbb{R}^n$ and $\phi \in D(\mathbb{R}^n)$.

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Since $\lim_{h \to 0} \frac{\Delta_h^m T}{h^m} = T^{(m)}$ for $T \in D'(\mathbb{R})$, if $T$ satisfies $\Delta_h^m T = 0$ for $h$ sufficiently small, $T$ is associated to a polynomial of order at most $m - 1$.

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**Theorem**

For $T \in D'(\mathbb{R}^n)$, the solutions of $\Delta_h^m T = 0$ for almost every $h \in \mathbb{R}^d$ are the distributions associated to a polynomial of degree at most $m-1$.

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In particular, if $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ satisfies $\Delta_h^m f = 0$ for almost every $h$, then $f$ is a polynomial of degree at most $m-1$.

If $G$ is a connected Lie group, let $L_x$ and $R_x$ denote the left and right translations by $x$: $R_x : y \mapsto xy$ on $G$.

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We naturally set $\Delta_h = R_h^* - I$ where $R_h$ is the pullback of $R_h$ and $\Delta^m_{h_1,\ldots,h_m} = \Delta_{h_m} \circ \cdots \circ \Delta_{h_1}$.

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In the same way, ${}_h\Delta = L_h^* - I$.

We will consider the solutions locally satisfying

$$\Delta^m_{h_1,\ldots,h_m} f(x) = 0 \qquad (*)$$

at $x_0$ (i.e. for $x$ in a neighborhood of $x_0$) for $h_1, \ldots, h_m$ near the identity

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at $x_0$, for $h$ near the identity.

We will only consider solutions that are bounded a.e. in a neighborhood of $x_0$.

**Proposition**

A function $f : G \to \mathbb{R}$ satisfies $\Delta^m_{h_1,\ldots,h_m} f(x) = 0$ at 1 iff it satisfies ${}_{h_1,\ldots,h_m}\Delta^m f(x) = 0$ at 1.

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## Theorem

If $f : G \to \mathbb{R}$ is bounded a.e. in a neighborhood of $x_0$ and satisfies the local Fréchet equation $(*')$ at $x_0$ then $f$ is smooth in a neighborhood of $x_0$.

Each $X \in \mathfrak{g}$ has a natural action on $C^1$ functions by the action of the associated left-invariant vector field as a derivation; $\mathcal{L}_X$ will denote the left-invariant vector field associated to $X$.

### Theorem

Let $\mathfrak{g}$ be the Lie algebra of $G$ and $E_1, \dots, E_n$ be a basis of $\mathfrak{g}$;
A function $f : G \to \mathbb{R}$ that is bounded a.e. in a neighborhood of $x_0$ is a solution of the local Fréchet equation $(*')$ at $x_0$ of order $m$ iff there exists a neighborhood $U$ of $x_0$ in $G$ and a neighborhood $V$ of 0 in $\mathfrak{g}$ s.t.

$$
\begin{aligned}
f(x \exp X) &= \sum_{j=0}^{m-1} \frac{\mathcal{L}_X^j f(x)}{j!} \\
&= f(x) + \sum_{j=1}^{m-1} \sum_{1 \leq i_1, \cdots, i_j \leq n} \frac{\mathcal{L}_{E_{i_1}} \cdots \mathcal{L}_{E_{i_j}} f(x)}{j!} X_{i_1} \cdots X_{i_j},
\end{aligned}
$$

for all $X \in V$ and $x \in U$.

We will say that $f$ is locally right-abelian at $x_0$ if $f(xyz) = f(xzy)$ for $x$ near $x_0$ and $y, z$ near the identity.

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### Theorem

If $f : G \to \mathbb{R}$ is locally bounded a.e. in a neighborhood of $x_0$ and locally right-abelian at $x_0$, then it is a solution of the local Fréchet equation $(*)$ or $(*')$ at $x_0$ of order $m$ iff there exist $f_1, \dots f_k \in \mathrm{Hom}_{\mathrm{loc}}(G, \mathbb{R})$, real numbers $a_\alpha$ for $\alpha \in \mathbb{N}^k$ and a neighborhood $U$ of the identity such that

$$f(x_0 h) = \sum_{|\alpha|<m} a_\alpha f_1(h)^{\alpha_1} \cdots f_k(h)^{\alpha_k},$$

for all $h \in U$, where $k = \dim(\mathrm{Hom}_{\mathrm{loc}})$.

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$$f(x_0 h) = \sum_{|\alpha| < m} a_\alpha f_1(h)^{\alpha_1} \cdots f_k(h)^{\alpha_k},$$

for all $h \in U$, where $k = \dim(\mathrm{Hom}_{\mathrm{loc}})$.

If $G$ is simply connected, $f_j$ belongs to $\mathrm{Hom}(G, \mathbb{R})$ and a global solution is given by the previous identity, with $x_0 = 1$ and any $h \in G$.

## Theorem

If $G$ is unimodular, $T \in D'(G)$ is a solution of the local Fréchet equation $(*')$ at $x_0$ of order $m$ iff there exists a neighborhood $U$ of $x_0$ such that $T|_U$ is a distribution associated to the smooth function $f : G \to \mathbb{R}$ such that

$$f(x \exp X) = \sum_{j=0}^{m-1} \frac{\mathcal{L}_X^j f(x)}{j!},$$

for $x \in U$ and $X$ in a neighborhood of 0 in $\mathfrak{g}$.

If $G$ is a connected Abelian Lie group, then $G = \mathbb{R}^n \times (S^1)^k$ for some $n$ and some $k$.

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**Proposition**

On $\mathbb{R}^n \times (S^1)^k$, the global solutions of the Fréchet equation $(*)$ or $(*')$ of order $m$ are of the form

$$f(x, y) = \sum_{|\alpha|<m} a_\alpha x^\alpha,$$

for $x \in \mathbb{R}^n$ and $y \in (S^1)^k$.

### Proposition

For the $(ax + b)$-group, the solutions of the local Fréchet equation $(*)$ or $(*')$ of order $m$ at 1 are given by $f(x, y) = \sum_{j=0}^{m-1} a_j \ln^j x$.

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### Proposition

For the $SL(2,\mathbb{R})$, the solutions of the local Fréchet equation $(*)$ or $(*')$ of order $m$ at 1 are constant near the identity.

**Proposition**

If $G$ is a two-step nilpotent Lie group, $f : G \to \mathbb{R}$ is a solution of the local Fréchet equation $(*)$ or $(*')$ at $x_0$ of order $m$ iff $f \circ L_{x_0} \circ \exp$ is a polynomial of degree at most $m - 1$.

If $G$ is not nilpotent, we have

$$\mathfrak{g} \supset \mathfrak{g}^{(1)} \supset \mathfrak{g}^{(2)} \supset \cdots \supset \mathfrak{g}^{(N)} = \mathfrak{g}^{(N+n)},$$

with $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{g}^{(j)} = [\mathfrak{g}, \mathfrak{g}^{j-1}]$.

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**Definition**

Let $N$ be the smallest integer s.t. $\mathfrak{g}^{(N)} = \mathfrak{g}^{(N+1)}$; the space of fundamental monomials of on $G$ is $(\mathfrak{g}^{(N)})^{\perp}$.
The set of functions $f : G \to \mathbb{R}$ s.t. $f \circ \exp$ is a polynomial in a neighborhood of 0 generated by a basis of fundamental monomials form will be denoted $P(G)$.

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Elements of $P(G)$ are solutions of the local Fréchet equation $(*')$.

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In particular, the solutions for semi-simple Lie groups are constant in a neighborhood of 1.


Not sure if thank you for your attention
or thank you for not sleeping during the presentation