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From the Brownian motion to a multifractal process using the Lévy–Ciesielski construction

T. Kleyntssens, S. Nicolay *

Université de Liège, Département de mathématique – zone Polytech 1, 12 allée de la Découverte, Bât. B37, B-4000 Liège, Belgium

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ABSTRACT

We give a construction of a multifractal process with prescribed Hölder exponents starting from the Lévy–Ciesielski construction of a Brownian motion. We also show that this method preserves the law of the iterated logarithm.

1. Introduction

A locally bounded function f belongs to the pointwise Hölder space $\Lambda^\alpha(t_0)$ (with $\alpha \geq 0$ and $t_0 \in \mathbf{R}$) if there exist a constant C and a polynomial P_{t_0} of degree less than α such that $|f(t) - P_{t_0}(t)| < C|t - t_0|^\alpha$, in a neighborhood of t_0. The Hölder exponent of f at t_0 is defined as $h_f(t_0) = \sup\{\alpha \geq 0 : f \in \Lambda^\alpha(t_0)\}$.

For any $t \in \mathbf{R}$, the sample path of a Brownian motion belongs to $\Lambda^{1/2-\varepsilon}(t)$ almost surely for any $\varepsilon > 0$. More precisely, if B denotes a Brownian motion, the Khinchin law of the iterated logarithm (Khintchine, 1924) states that for every $t \in \mathbf{R}$, there exists a positive random variable C such that

$$|B_{t+h} - B_t| \leq C|h|^{1/2}\sqrt{\log\log|h|^{-1}}$$

for h small enough on an event of probability 1.

In this letter, we start from the Lévy–Ciesielski construction of a Brownian motion to build a multifractal process with prescribed Hölder exponents. Given a function $H : [0, 1] \to [0, 1]$ satisfying some conditions, we shape a process B^H on $[0, 1]$ such that, on an event of probability 1, $h_{B^H}(t) = H(t)$ for any t. Moreover, B^H still satisfies the law of the iterated logarithm.

This construction was originally proposed in Kleyntssens (2019), where a multifractal formalism based on the wavelet decomposition is designed to detect the law of the iterated logarithm. This process was preferred over others (see Ayache and Bertrand, 2010 for example) to test the efficiency of such a formalism because one needed to efficiently numerically generate a large amount of signals satisfying the law of the iterated logarithm without using wavelets (in order to avoid using the same functions to both generate and analyze the realizations of the process). The authors suppose that replacing the Schauder functions by wavelets should lead to similar results. However, using a compactly supported piecewise linear basis allows to easily produce numerical realizations at a speed that can be hardly rivaled.

* Corresponding author.
E-mail address: S.Nicolay@uliege.be (S. Nicolay).

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2. The Lévy–Ciesielski construction

Here, we present the decomposition of the Brownian motion in the Schauder basis.

Definition 1. Let us set $F_0(t) = t\mathbf{1}_{[0,1]}(t) + \mathbf{1}_{(1,+\infty)}(t)$ and for any $(j,k) \in \mathbf{N} \times \{0,\ldots,2^j-1\}$,

$$F_{j,k}(t) = \begin{cases} t - k2^{-j} & \text{if } t \in [k2^{-j}, k2^{-j} + 2^{-(j+1)}] \\ -t + (k+1)2^{-j} & \text{if } t \in [k2^{-j} + 2^{-(j+1)}, (k+1)2^{-j}] \\ 0 & \text{else} \end{cases} .$$

These functions are called the Schauder functions.

Remark 1. These functions first appear in the work of Faber (1910) as indefinite integrals of the Haar system. The general construction was carried out by Schauder (1928).

The following result binds the regularity of a function to the decay rate of its coefficients in the Schauder basis (Daoudi et al., 1998).

Proposition 1. *We have the following properties:*

1. *Let $(a_{j,k})_{(j,k)\in\mathbf{N}\times\{0,\ldots,2^j-1\}}$ be a real sequence, $a_0 \in \mathbf{R}$ and $\varepsilon \in (0, 1/2)$. If $\max_{k\in\{0,\ldots,2^j-1\}} |a_{j,k}| = O(2^{j^\varepsilon})$ as $j \to +\infty$, then the function f defined by*

$$t \mapsto a_0 F_0(t) + \sum_{j=0}^{+\infty} \sum_{k=0}^{2^j-1} a_{j,k} 2^{j/2} F_{j,k}(t) \tag{1}$$

is uniformly absolutely-convergent on $[0, 1]$. Besides, f is a real continuous function such that $f(0) = 0$.
2. *Any continuous function f from $[0, 1]$ to $\mathbf{R}$ such that $f(0) = 0$ can be written in the form (1). Besides, if $f \in \Lambda^\alpha(x_0)$ then there exists a constant $C > 0$ such that $|a_{j,k}2^{-j/2}| \leq C(2^{-j} + |k2^{-j} - x_0|)^\alpha$ for any $(j,k) \in \mathbf{N} \times \{0,\ldots,2^j-1\}$.*

The next theorem gives the decomposition of the Brownian motion in the Schauder basis (see Bhattacharya and Waymire (2016)), which will be the starting point of our construction.

Theorem 2. *Let $(Z_{j,k})_{(j,k)\in\mathbf{N}\times\{0,\ldots,2^j-1\}}$ be a sequence of independent real-valued $\mathcal{N}(0, 1)$ Gaussian random variables defined on the probability space Ω. Then, there exists an event $\Omega^* \subset \Omega$ of probability 1 such that, for any $\omega \in \Omega^*$, the function $B_\cdot(\omega)$ defined by*

$$B_\cdot(\omega) : t \mapsto Z_0(\omega)F_0(t) + \sum_{j=0}^{+\infty} \sum_{k=0}^{2^j-1} Z_{j,k}(\omega) 2^{j/2} F_{j,k}(t) \tag{2}$$

is uniformly absolutely-convergent on $[0, 1]$. Besides, the process $B = \{B_t\}_t$ is a Brownian motion.

3. From the Brownian motion to a multifractal process

Let us now define a process with a prescribed local regularity by modifying formula (2).

Let us denote by $\underline{\mathcal{H}}_K$ the set of the functions from $[0, 1]$ to the compact K which are the lower limit of a sequence of continuous functions. Lemma 2 of Daoudi et al. (1998) implies that for any $H \in \underline{\mathcal{H}}_K$, there exists a sequence $(Q_j)_{j\in\mathbf{N}}$ of polynomials such that

$$\begin{cases} H(t) = \liminf_{j\to+\infty} Q_j(t) & \forall t \in [0,1] \\ \|Q_j'\|_\infty \leq j & \forall j \in \mathbf{N} \end{cases} , \tag{3}$$

where Q_j' is the derivative of Q_j. We have a similar result if one replaces the lower limit by a limit in the definition of $\underline{\mathcal{H}}_K$. In this case, the set is denoted by $\mathcal{H}_K$ and the lower limit in relation (3) becomes a limit.

First, let us recall the following classical lemma on the standard Gaussian distribution (see Chatterjee (2014) for example).

Lemma 3. *Let Z be an arbitrary real-valued $\mathcal{N}(0, 1)$ Gaussian random variable. Then, for any $x > 0$, we have*

$$\frac{1}{\sqrt{2\pi}} \frac{x}{x^2+1} e^{-x^2/2} \leq \mathbb{P}(Z > x) \leq \frac{1}{\sqrt{2\pi}} \frac{1}{x} e^{-x^2/2}.$$

This leads to the main result of this section.

Theorem 4. *Let K be a compact of $(-1/2, 1/2)$, $H \in \mathcal{H}_K$ and $(Q_j)_{j\in\mathbf{N}}$ be a sequence of polynomials satisfying Relation (3). For any $(j,k) \in \mathbf{N} \times \{0,\ldots,2^j-1\}$, set $H_{j,k} = Q_j(k/2^j)$, let $(Z_{j,k})_{(j,k)\in\mathbf{N}\times\{0,\ldots,2^j-1\}}$ be a sequence of independent real-valued $\mathcal{N}(0,1)$ Gaussian random variables defined on the probability space Ω and let us define*

$$B_t^H(\omega) = Z_0(\omega)F_0(t) + \sum_{j=0}^{+\infty}\sum_{k=0}^{2^j-1} 2^{-jH_{j,k}} Z_{j,k}(\omega) 2^{j/2} F_{j,k}(t). \tag{4}$$

Then, there exists an event $\Omega^* \subset \Omega$ of probability 1 such that we have the following properties:

1. *For $\omega \in \Omega^*$, the function $t \mapsto B_t^H(\omega)$ is a continuous function defined on $[0,1]$.*
2. *For $\omega \in \Omega^*$, we have $h_{B^H_{\cdot}(\omega)}(t) = 1/2 + H(t)$ for any $t \in [0,1]$.*
3. *If there exists $C > 0$ such that*
$$H(t) - Q_j(t) \le Cj^{-1}, \tag{5}$$
on $[0,1]$ for any $j \in \mathbf{N}$, then there exists a positive random variable C' independent of t such that, for any $\omega \in \Omega^*$,
$$|B_{t+h}^H(\omega) - B_t^H(\omega)| \le C'(\omega)2^C |h|^{1/2+H(t)}\sqrt{\log|h|^{-1}},$$
for any h in a neighborhood of 0.

Proof. First, Relation (3) implies that, for any $\varepsilon' > 0$, there exists $J \in \mathbf{N}$ such that for any $j \ge J$ and for any $t \in [k2^{-j}, (k+1)2^{-j})$, one has

$$H(t) - H_{j,k} = H(t) - Q_j(t) + Q_j(t) - Q_j(k2^{-j}) \le \varepsilon'/2 + j2^{-j} \le \varepsilon'. \tag{6}$$

One thus gets $\inf K - \varepsilon' \le H_{j,k}$ and thus

$$2^{-jH_{j,k}} \le 2^{-j\inf K}2^{j\varepsilon'}. \tag{7}$$

Let us prove the first point of the theorem. Let $\varepsilon \in (0, 1/2)$ and choose a constant $C > 0$. Let us define

$$A_j = \{w \in \Omega : \max_{k\in\{0,\ldots,2^j-1\}} |Z_{j,k}(\omega)| > C2^{j(\varepsilon-\varepsilon'+\inf K)}\}.$$

One has

$$\begin{aligned}\mathbb{P}(A_j) &\le \sum_{k=0}^{2^j-1} \mathbb{P}(\{w \in \Omega : |Z_{j,k}(\omega)| > C2^{j(\varepsilon-\varepsilon'+\inf K)}\})\\ &\le 2^j \mathbb{P}(\{|Z| > C2^{j(\varepsilon-\varepsilon'+\inf K)}\}),\end{aligned}$$

where $Z \sim \mathcal{N}(0,1)$. Using Lemma 3, one has

$$\mathbb{P}(A_j) \le \sqrt{\frac{2}{\pi C^2}}\, 2^{j(1-(\varepsilon-\varepsilon'+\inf K))} e^{-(C^2 2^{2j(\varepsilon-\varepsilon'+\inf K)})/2}.$$

Since $\inf K \in (-1/2, 1/2)$, there exists $\varepsilon \in (0, 1/2)$ such that $\varepsilon - \varepsilon' + \inf K > 0$ for ε' small enough. In this case, $\sum_{j=0}^{+\infty} \mathbb{P}(A_j) < +\infty$. It follows from the Borel–Cantelli lemma that $\mathbb{P}\left(\bigcap_{J\in\mathbf{N}}\bigcup_{j\ge J} A_j\right) = 0$.

Hence, the event

$$\Omega^* = \bigcup_{J\in\mathbf{N}}\bigcap_{j\ge J}\{\omega : \max_{k\in\{0,\ldots,2^j-1\}} 2^{-jH_{j,k}}|Z_{j,k}(\omega)| \le C2^{j\varepsilon}\}$$

has a probability equal to 1. The first point of the theorem is thus proved, thanks to Proposition 1.

Now, let us prove that $h_f(t) \ge 1/2 + H(t)$. The third point is a particular case of this reasoning. From the Borel–Cantelli lemma, for a constant $C > 0$ large enough, the event $\bigcup_{J\in\mathbf{N}}\bigcap_{j\ge J}\{\omega : \max_{k\in\{0,\ldots,2^j-1\}} |Z_{j,k}(\omega)| \le C\sqrt{j}\}$ has probability 1. Let ω be a element of this event; there thus exists $J \in \mathbf{N}$ such that for any $j \ge J$,

$$|Z_{j,k}| \le C\sqrt{j}, \tag{8}$$

for any $k \in \{0,\ldots,2^j-1\}$. Let $t \in [0,1]$, $\varepsilon' > 0$ and suppose that J is large enough in order for Inequality (6) to be satisfied. Let h be a real number such that $t, t+h \in [k2^{-(J+1)}, (k+1)2^{-(J+1)})$ for some $k \in \{0,\ldots,2^{J+1}-1\}$ and let $l > J$ be such that $2^{-l} < |h| \le 2^{-l+1}$. By construction, for any $j \le l$, there exists a unique k_j such that $t, t+h \in [k_j 2^{-j}, (k_j+1)2^{-j})$. One has

$$|B_{t+h}^H - B_t^H| \le Z_0|h| + C_J|h| \tag{9}$$

$$+ \sum_{j=J+1}^{l-1} 2^{-jH_{j,k_j}}|Z_{j,k_j}|2^{j/2}|F_{j,k_j}(t+h) - F_{j,k_j}(t)| \tag{10}$$

$$+\sum_{j=l}^{+\infty}\sum_{k=0}^{2^j-1} 2^{-jH_{j,k}}|Z_{j,k}|2^{j/2}|F_{j,k}(t+h)-F_{j,k}(t)|, \tag{11}$$

for some constant C_J depending only on J.

By eventually reducing h, we can suppose that the terms in the right-hand side of (9) are smaller than $C|h|^{1/2+\sup K}\sqrt{\log|h|^{-1}} \leq C|h|^{1/2+H(t)}\sqrt{\log|h|^{-1}}$, for some constant $C>0$ independent of t and h.

Using Inequality (6) and since the supports of functions F_{j,k_j} are included in $[k_j2^{-j},(k_j+1)2^{-j}]$, each term in (10) is smaller than

$$\begin{aligned}|h|\sum_{j=J+1}^{l-1} 2^{j(1/2-H_{j,k})}\max_{k\in\{0,\ldots,2^j-1\}}|Z_{j,k}(w)| &\leq C|h|\sum_{j=J+1}^{l} 2^{j(1/2-H(t)+\varepsilon')}\sqrt{j}\\ &\leq C'|h|2^{l(1/2-H(t)+\varepsilon')}\sqrt{l}\\ &\leq C'|h|^{1/2+H(t)-\varepsilon'}\sqrt{\log|h|^{-1}},\end{aligned} \tag{12}$$

where $C'>0$ is a constant that does not depend on t or l. The last inequality is true by definition of l.

For any $j\geq l$, there exist unique k_j and k_j' such that $t\in[k_j2^{-j},(k_j+1)2^{-j})$ and $t+h\in[k_j'2^{-j},(k_j'+1)2^{-j})$. Since $|F_{j,k}|\leq 2^{-j-1}$, the terms in (11) are smaller than

$$\begin{aligned}&\sum_{j=l}^{+\infty} 2^{-j/2}\max_{k\in\{0,\ldots,2^j-1\}}|Z_{j,k_j}|(2^{-jH_{j,k_j}}+2^{-jH_{j,k_j'}})\\ &\leq C''2^{-l/2}(2^{-lH_{l,k_l}}+2^{-jH_{l,k_l'}})\sqrt{l}\\ &\leq C''2^{-l(1/2+H(t)-\varepsilon')}\sqrt{l},\end{aligned} \tag{13}$$

where $C''>0$ is a constant independent of t and l. The last inequality is obtained by remarking that $k_l'\in\{k_l-1,k_l,k_l+1\}$. Since the terms in (11) are smaller than $C''|h|^{1/2+H(t)-\varepsilon'}\sqrt{\log|h|^{-1}}$, there exists a constant C''' independent of t such that for any $\varepsilon'>0$, one has

$$|B^H_{t+h}-B^H_t|\leq C'''|h|^{1/2+H(t)-\varepsilon'}\sqrt{\log|h|^{-1}}.$$

We deduce that $h_f(t)\geq 1/2+H(t)$. Now, using hypothesis (5), we obtain $H(t)-H_{l,k_l}\leq (C+1)l^{-1}$. We can directly take $\varepsilon'=(C+1)l^{-1}$ in Inequalities (12) and (13) to obtain the third point of the theorem.

To conclude, it remains to prove that $h_f(t)\leq 1/2+H(t)$. Using Relations (3), for any $J\in\mathbf{N}$ and for any $\varepsilon>0$ there exists $j_J\geq J$ such that

$$Q_{j_J}(t)\leq H(t)+\varepsilon \qquad\text{and}\qquad |H_{j_J,k}-Q_{j_J}(t)|\leq j_J2^{-j_J}\leq\varepsilon.$$

Besides, the Borel–Cantelli lemma applied to the independent events

$$A_{j_J}=\{\omega:|Z_{j_J,k_{j_J}}(\omega)|>2^{-\varepsilon j_J}\}$$

implies that, almost surely, for any $J\in\mathbf{N}$, there exist $J'>J$ such that $|Z_{j_{J'},k_{j_{J'}}}|>2^{-\varepsilon j_{J'}}$. We thus have

$$\begin{aligned}2^{-j_{J'}H_{j_{J'},k}}|Z_{j_{J'},k}|2^{-j_{J'}/2} &\geq 2^{-j_{J'}(j_{J'}2^{-j_{J'}}+\varepsilon+H(t)+1/2)}|Z_{j_{J'},k}|\\ &> 2^{-j_{J'}(1/2+H(t)+3\varepsilon)}.\end{aligned}$$

We can conclude using point (2) of Proposition 1. □

The next proposition studies the local regularity of the process B^H. We first need a lemma from Meyer et al. (1999), Ayache and Taqqu (2003), Ayache and Bertrand (2010).

Lemma 5. *Let $(Z_{j,k})_{(j,k)\in\mathbf{N}\times\{0,\ldots,2^j-1\}}$ be a sequence of independent real-valued $\mathcal{N}(0,1)$ Gaussian random variables defined on the probability space Ω. There exists an event Ω^* of probability 1 and a positive random variable C of finite moment of every order such that, for all $\omega\in\Omega^*$, the inequality*

$$|Z_{j,k}(w)|\leq C(\omega)\sqrt{\log(3+j+|k|)}$$

holds for any $(j,k)\in\mathbf{N}\times\{0,\ldots,2^j-1\}$.

Remark 2. The proof of this lemma relies on the use of a bijection from $\mathbf{N}$ to $\mathbf{N}^2$ in order to index the variables on $\mathbf{N}$ (and then apply the Borel–Cantelli lemma). We will implicitly choose an indexation more suited to our problem in the next proof.

Proposition 6. *Under Hypothesis (5), there exist an event $\Omega^* \subset \Omega$ of probability 1 and a positive random variable C such that, for any $\omega \in \Omega^*$ and for almost every $t \in [0, 1]$,*

$$|B^H_{t+h}(\omega) - B^H_t(\omega)| \leq C(\omega)|h|^{1/2+H(t)}\sqrt{\log\log|h|^{-1}},$$

for any h small enough.

Proof. We will use the same notations as in the proof of previous theorem. Let us fix $t \in [0, 1]$ and recall that for any $j \in J$, there exists a unique $k_j \in \{0, \ldots, 2^j - 1\}$ such that $t \in [k_j 2^{-j}, (k_j + 1)2^{-j})$. That being said, Lemma 5 insures the existence of a positive random variable C_t (of finite moment of every order) such that the inequality $|Z_{j,k}| \leq C_t\sqrt{\log(3 + j + |k - k_j|)}$ holds almost surely for $(j, k) \in \mathbf{N} \times \{0, \ldots, 2^j - 1\}$ (this inequality has to be compared with (8)).

Using the same argument as in the previous proof, we obtain that for any $t \in [0, 1]$, there exists almost surely a constant $C > 0$ (depending on ω), such that

$$|B^H_{t+h} - B^H_t| \leq C|h|^{1/2+H(t)}\sqrt{\log\log|h|^{-1}},$$

for any h small enough. Fubini's theorem allows to conclude. □

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