

Low Mach Number Local Preconditioning For Unsteady Viscous Finite Volumes Simulations On 3D Unstructured Grids

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Introduction

Design of a density-based solver for unsteady compressible viscous flows.

- Time and space discretization schemes
- Numerical advective flux function
- Solver for large algebraic systems of non linear equations (Pseudo-transient Newton-GMRes)

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But what about these at the limit of **incompressibility** ?

$$\lim_{M \rightarrow 0} CN = \lim_{M \rightarrow 0} \frac{1 + M}{M} = \infty$$

Introduction

Use of an artificial speed of sound c' by altering the transient behaviour of Navier-Stokes equations

$$c' \sim \|\mathbf{u}\| \longrightarrow CN \sim 1$$

Local preconditioning methods for steady viscous flows

(Chorin [*JCP* 1967], Turkel [*JCP* 1987], Choi & Merkle [*JCP* 1993], Weiss & Smith [*AIAAJ* 1995])

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Local preconditioning methods for **steady** viscous flows

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What about **unsteady** flows ?

(Venkateswaran & Merkle [*AIAAP* 1995], Turkel [*AIAAP* 2003])

- Use of a dual time stepping technique
- Does the condition number depend on the time step ?
- Should the numerical flux function be modified ?

Choice of variables

Use of primitive variables w rather than conservative ones s

$$p = \frac{p_d - p_0}{\rho_0 U_0^2} \quad \mathbf{u} = \frac{\mathbf{u}_d}{U_0} \quad T = \frac{T_d}{T_0} \quad \rightarrow \quad \rho(p, T) = \frac{\rho_d(p_d, T_d)}{\rho_0}$$

$$Str \frac{\partial}{\partial t} \mathbf{s} + \nabla \cdot \mathbf{F}_a = \frac{1}{Re} \nabla \cdot \mathbf{F}_d$$

$$\mathbf{s} = \begin{pmatrix} \rho \\ \rho \mathbf{u} \\ \rho H - \chi p \end{pmatrix} \quad \mathbf{F}_a = \begin{pmatrix} \rho \mathbf{u}^T \\ \rho \mathbf{u} \mathbf{u}^T + p \mathbf{I} \\ \rho \mathbf{u}^T H \end{pmatrix} \quad \mathbf{F}_d = \begin{pmatrix} \mathbf{0}^T \\ \mathbf{T} \\ -\frac{1}{Pr} \mathbf{q}^T + \chi \mathbf{u}^T \mathbf{T} \end{pmatrix}$$

$$Str = \frac{L_0}{U_0 t_0} \quad Re = \frac{\rho_0 U_0 L_0}{\mu_0} \quad \chi = \frac{U_0^2}{Cp_0 T_0}$$

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For a **perfect gas** :

$$H = T + \frac{1}{2} \chi \|\mathbf{u}\|^2 \quad \rho = \frac{1 + \frac{\gamma}{\gamma-1} \chi p}{T} \quad \chi = (\gamma - 1) M_0^2$$

Space and time discretization

- Cell-centered finite volumes
- First order of accuracy on diffusive terms
- Second order of accuracy on advective terms

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$$CFL_c = M^{-1} CFL_u$$

\implies Second order **implicit** Crank-Nicolson scheme

Implicit pseudo-transient method

The Crank-Nicolson scheme is written as

$$Str \frac{s(\mathbf{w}^{l+1}) - s(\mathbf{w}^l)}{\Delta t^l} = \theta \mathbf{Rhs}(\mathbf{w}^{l+1}) + (1 - \theta) \mathbf{Rhs}(\mathbf{w}^l)$$

Fully implicit pseudo-transient iterations are applied

$$\frac{s_p(\mathbf{w}^{n+1}) - s_p(\mathbf{w}^n)}{\Delta \tau^n} + \mathbf{f}(\mathbf{w}^{n+1}) = -\mathbf{g}(\mathbf{w}^l)$$

At each pseudo-time step, the following system is solved by GMRes iterations

$$\begin{aligned} \left[\frac{1}{\Delta \tau^n} \mathbf{P}^n + \mathbf{J}^n \right] \Delta \mathbf{w}^n &= -(\mathbf{f}(\mathbf{w}^n) + \mathbf{g}(\mathbf{w}^l)) \\ \longrightarrow \mathbf{w}^{n+1} &= \mathbf{w}^n + \Delta \mathbf{w}^n \end{aligned}$$

Parameters of preconditioning

- The preconditioning matrix is written as

$$\mathbf{P} = \begin{pmatrix} \rho'_p & \mathbf{0}^T & \rho_T \\ \rho'_p \mathbf{u} & \rho \mathbf{I} & \rho_T \mathbf{u} \\ \rho'_p H + \rho h_p - \chi & \chi \rho \mathbf{u}^T & \rho_T H + \rho h_T \end{pmatrix}$$

- Density thermodynamical properties are modified : ρ'_p
- The artificial speed of sound c' is defined by

$$c'^2 = \frac{\rho h_T}{\rho h_T \rho'_p + \rho_T (\chi - \rho h_p)} \longrightarrow \rho'_p = \frac{1}{c'^2} - \frac{(\chi - \rho h_p) \rho_T}{\rho h_T}$$

- Finally the only parameter to be fixed is $M_* = \frac{c'}{c}$

Eigenvalues analysis

Does the condition number depend on the $CF L_c$ number?

- Pseudo-transient system in its quasi-linear form

$$\mathbf{P} \frac{\partial \mathbf{w}}{\partial \tau} + \frac{Str}{\Delta t} \mathbf{S} \mathbf{w} + \theta \left(\mathbf{A}_x \frac{\partial \mathbf{w}}{\partial x} + \mathbf{A}_y \frac{\partial \mathbf{w}}{\partial y} + \mathbf{A}_z \frac{\partial \mathbf{w}}{\partial z} \right) = \mathbf{0}$$

- Introducing a Fourier mode

$$\mathbf{w}(\tau, x, y, z) = \mathbf{w}_0 \exp(\mathbf{k}^T \mathbf{x} - i \omega \tau)$$

- With \mathbf{k} limited by the grid spacing :

$$\|\mathbf{k}\| = \phi / \Delta \text{ with } \phi \in [0, \pi]$$

Eigenvalues analysis

If $u_k = \frac{\mathbf{u}^T \mathbf{k}}{\|\mathbf{k}\|}$ and $\lambda = \frac{\omega}{\|\mathbf{k}\|}$, the eigenvalues are found to be

$$\lambda_{1,2,3} = \theta u_k (1 - i CFL_u^{-1})$$

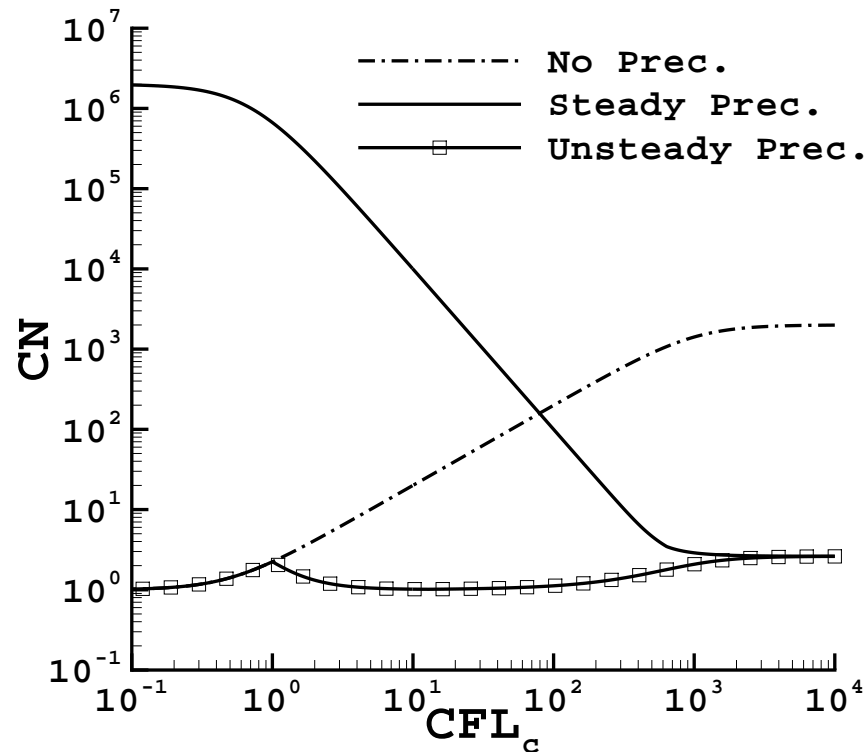
$$\lambda_{4,5} = \theta u_k (1 - i CFL_u^{-1}) T_{\pm}$$

$$T_{\pm} = \frac{1}{2} (1 + M_*^2) \pm \frac{1}{2} \sqrt{(1 - M_*^2)^2 - \frac{4 M_*^2}{(i M + CFL_c^{-1})^2}}$$

- Wave Speed $\longrightarrow \text{Re}(\lambda)$
- Wave Damping $\longrightarrow \text{Im}(\lambda)$

The condition number is $CN = \frac{\max(1, |T_+|, |T_-|)}{\min(1, |T_+|, |T_-|)}$

Preconditioning techniques



No Prec.

$$M_* = 1$$

Steady Prec.

$$M_* = \min [1, \max (M, M_\epsilon)]$$

Unsteady Prec. $M_* = \min \left[1, \max \left(\sqrt{M^2 + CFL_c^{-2}}, M_\epsilon \right) \right]$

AUSM+up scheme

Should the numerical flux function be modified ?

AUSM+up scheme

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$$\widetilde{\mathbf{F}}_a \cdot \mathbf{n} = c_{\frac{1}{2}} M_{\frac{1}{2}} \begin{pmatrix} \rho \\ \rho \mathbf{u} \\ \rho H \end{pmatrix}_{L/R} + p_{\frac{1}{2}} \begin{pmatrix} 0 \\ \mathbf{n} \\ 0 \end{pmatrix}$$

$$M_{\frac{1}{2}} = \mathcal{M}_{(4)}^+(M_L) + \mathcal{M}_{(4)}^-(M_R) - M_p$$

$$p_{\frac{1}{2}} = \mathcal{P}_{(5)}^+(M_L) p_L + \mathcal{P}_{(5)}^-(M_R) p_R - p_u$$

$$M_p = K_p \max \left(1 - \overline{M}^2, 0 \right) \frac{p_R - p_L}{f_c \left(\frac{2}{\gamma M_0^2} + p_R + p_L \right)}$$

$$p_u = K_u \mathcal{P}_{(5)}^+(M_L) \mathcal{P}_{(5)}^-(M_R) (\rho_L + \rho_R) f_c c_{\frac{1}{2}} (\mathbf{u}_R \cdot \mathbf{n} - \mathbf{u}_L \cdot \mathbf{n})$$

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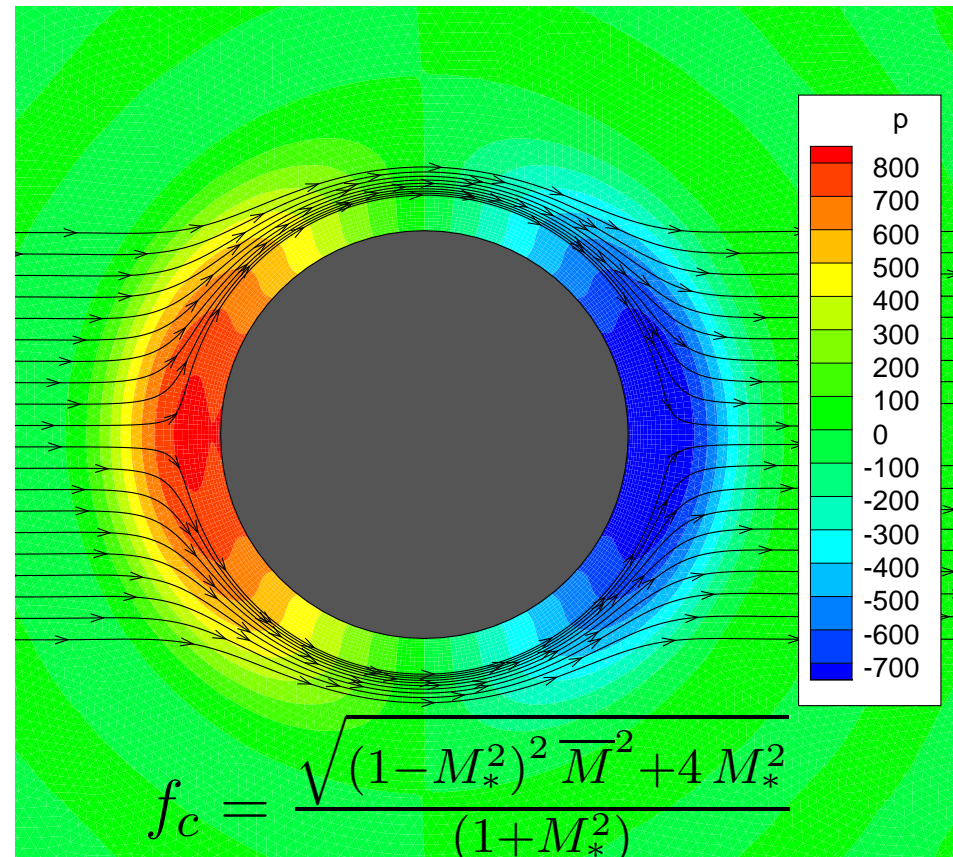
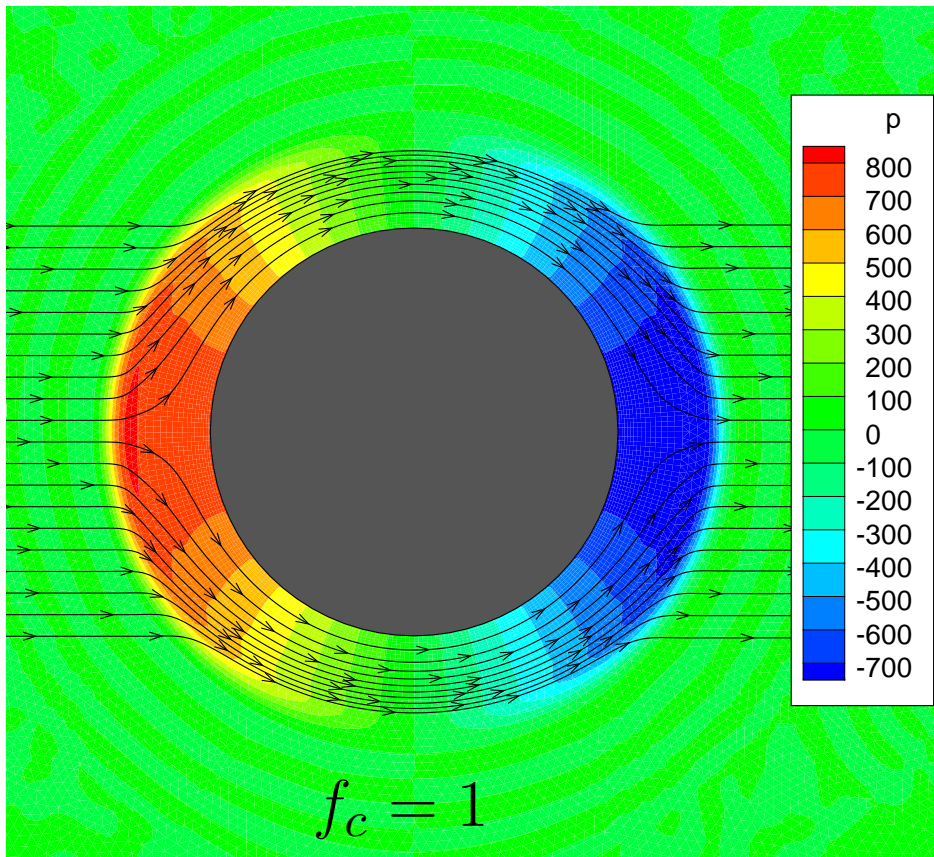
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No Prec. $\rightarrow f_c = 1$

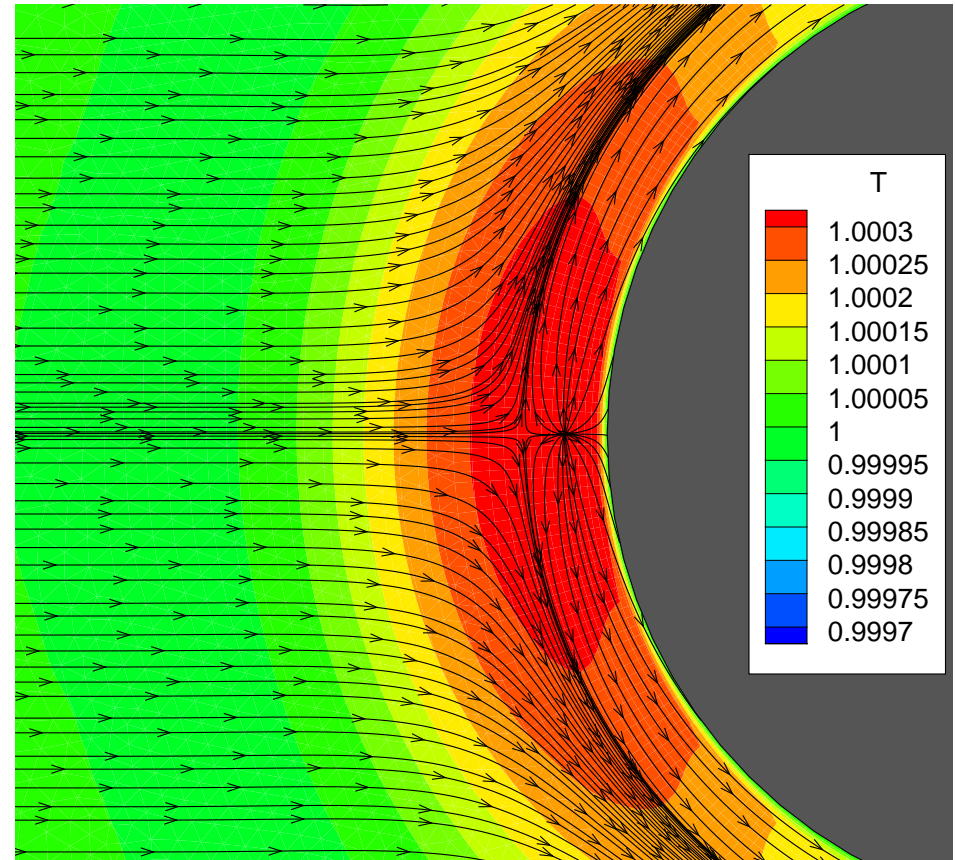
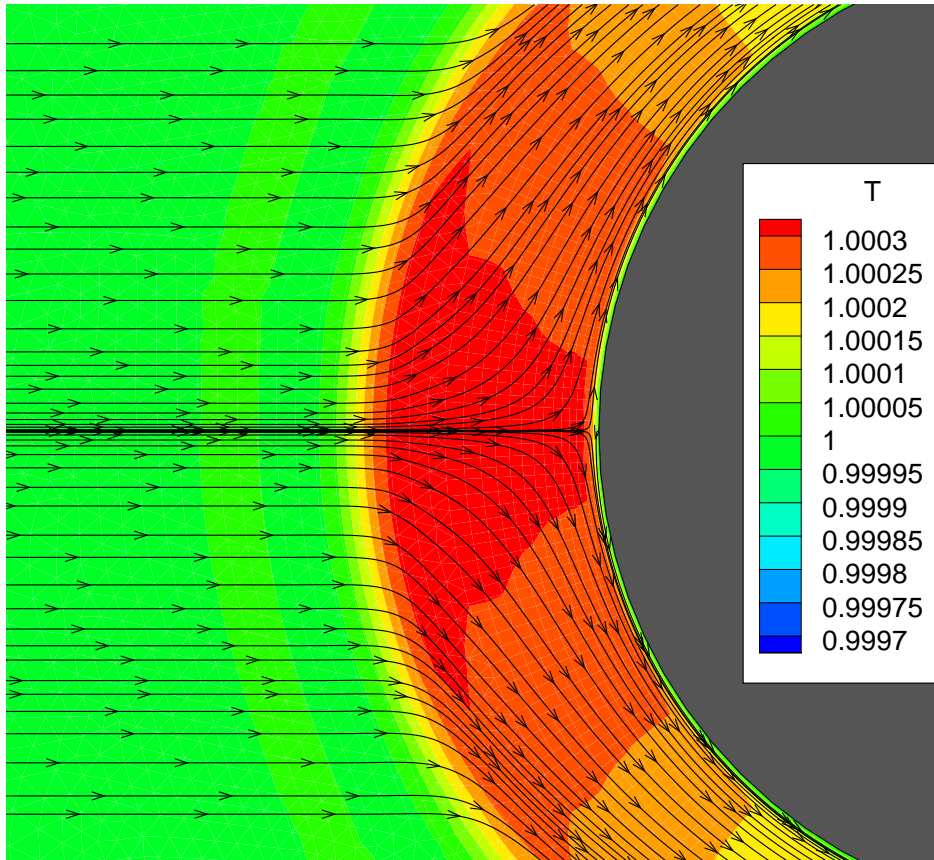
Steady Prec. $\rightarrow f_c = \frac{\sqrt{(1-M_*^2)^2 \overline{M}^2 + 4 M_*^2}}{(1+M_*^2)}$

AUSM+up scheme

Impulsively starting cylinder at $M_0 = 10^{-3}$ and $CFL_C \sim 1$



AUSM+up scheme



$$f_c = 1$$

$$f_c = \frac{\sqrt{(1-M_*^2)^2 \overline{M}^2 + 4 M_*^2}}{(1+M_*^2)}$$

AUSM+up scheme

Modification of the function f_c

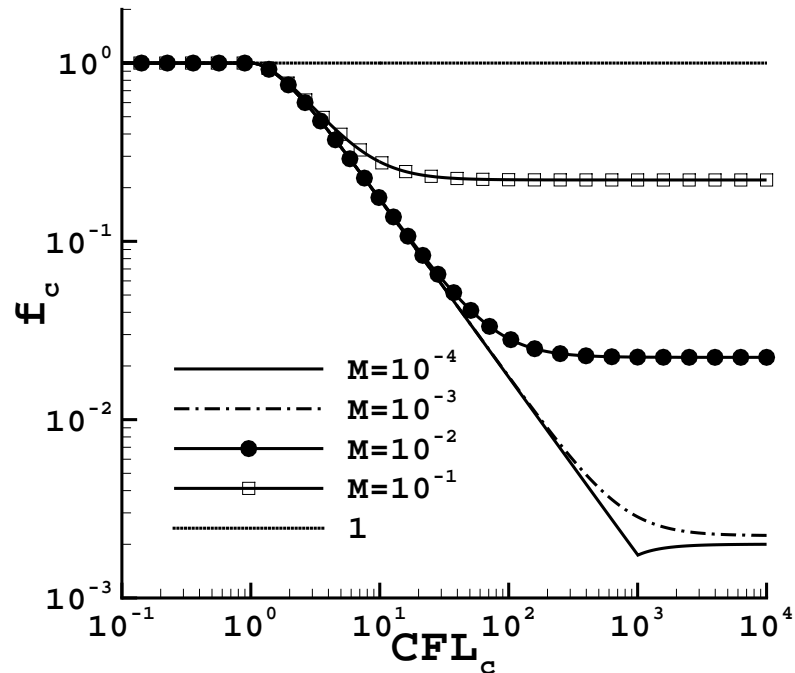
$$\lambda_{4,5} = \theta u_k (1 - i CFL_u^{-1}) T_{\pm}$$

$$T_{\pm} = \frac{1}{2} (1 + M_*^2) \pm \frac{1}{2} \sqrt{(1 - M_*^2)^2 - \frac{4 M_*^2}{(i M + CFL_c^{-1})^2}}$$

If we only look to the waves propagation speed

$$\begin{aligned} \operatorname{Re}(\lambda_{4,5}) &= \theta u_k [\operatorname{Re}(T_{\pm}) + CFL_u^{-1} \operatorname{Im}(T_{\pm})] \\ &= \theta u_k \frac{1}{2} (1 + M_*^2) \left[1 \pm \frac{1}{M} \frac{M \operatorname{Re}(\sqrt{*}) + CFL_c^{-1} \operatorname{Im}(\sqrt{*})}{(1 + M_*^2)} \right] \\ &= \theta u_k \frac{1}{2} (1 + M_*^2) \left[1 \pm \frac{1}{M} f_c \right] \end{aligned}$$

AUSM+up scheme



No Prec.

$$f_c = 1$$

Steady Prec.

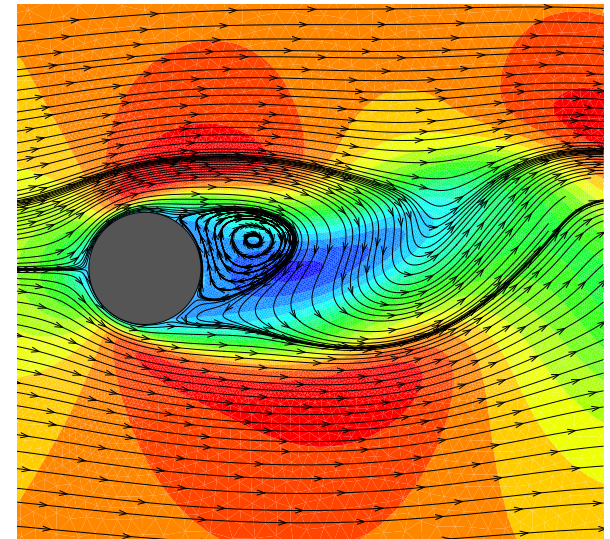
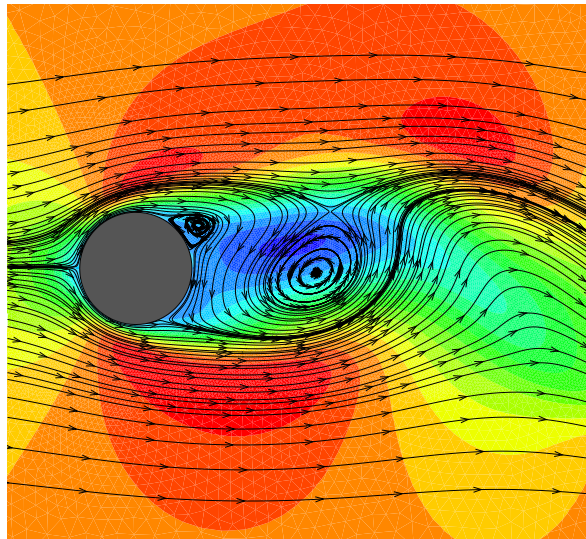
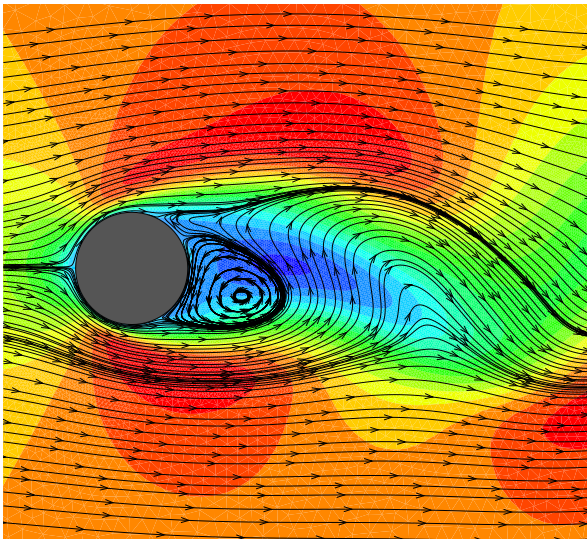
$$f_c = \frac{\sqrt{(1-M_*^2)^2 \bar{M}^2 + 4 M_*^2}}{(1+M_*^2)}$$

Unsteady Prec.

$$f_c = \frac{M \operatorname{Re}(\sqrt{*}) + CFL_c^{-1} \operatorname{Im}(\sqrt{*})}{(1+M_*^2)}$$

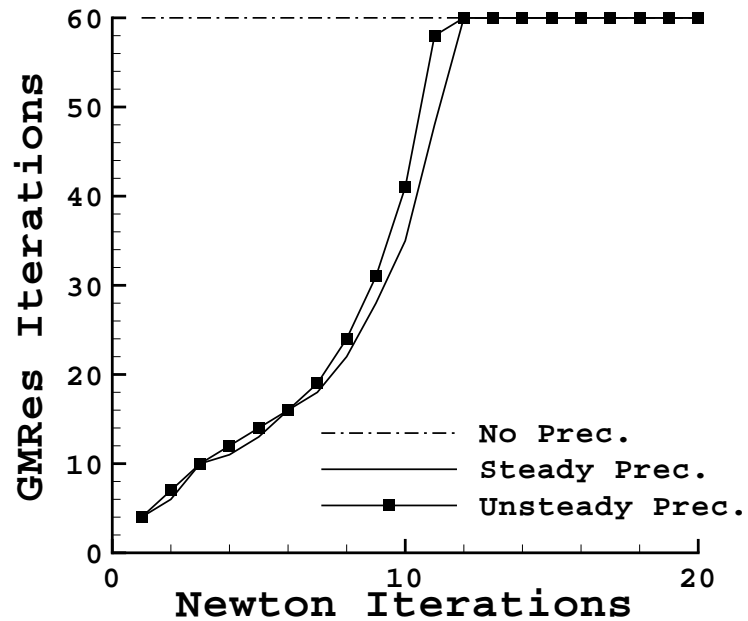
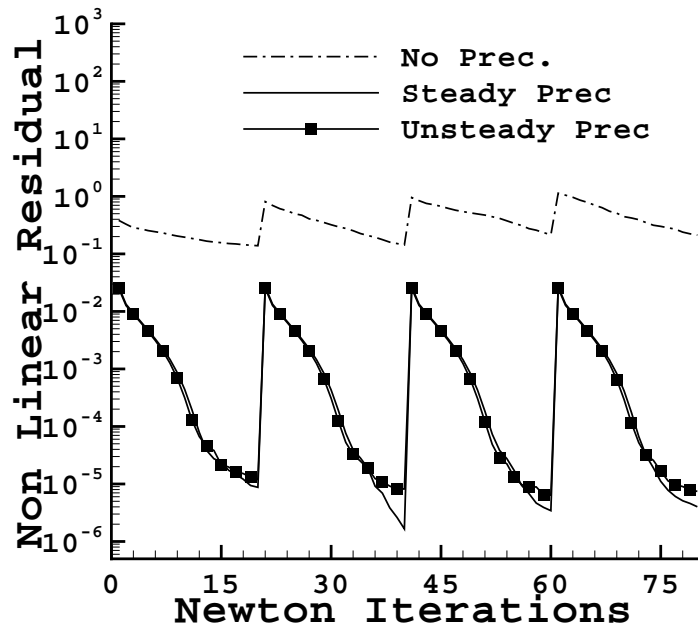
Periodic laminar flow past a cylinder

$$Re = 100, M_0 = 10^{-3} \text{ and } CFL_c = 10^3, CFL_u = 1$$



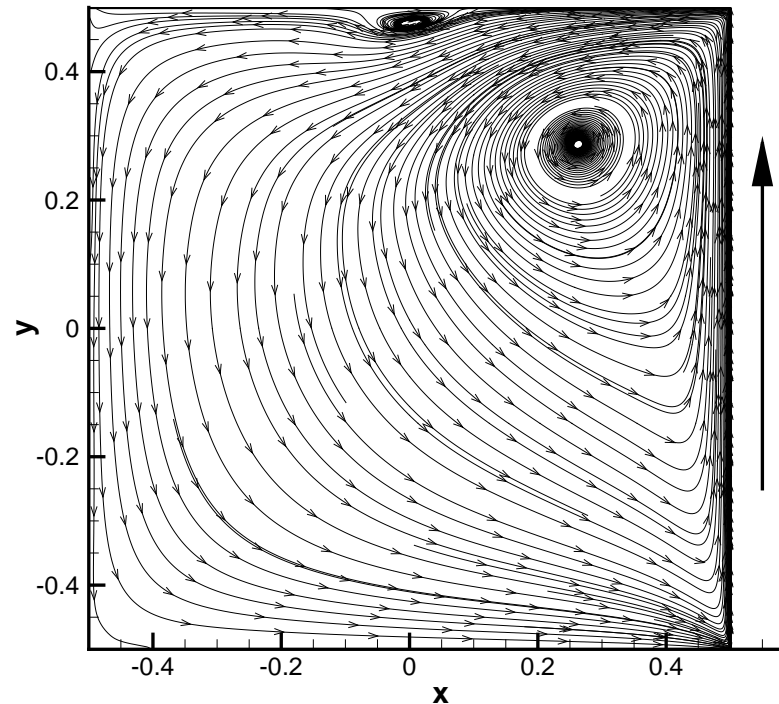
$$\overline{C_D} = 1.32 \quad \Delta C_D = 0.0085 \quad \Delta C_L = 0.315 \quad Str_{vs} = 0.166$$

Periodic laminar flow past a cylinder



3D laminar lid driven cavity flow

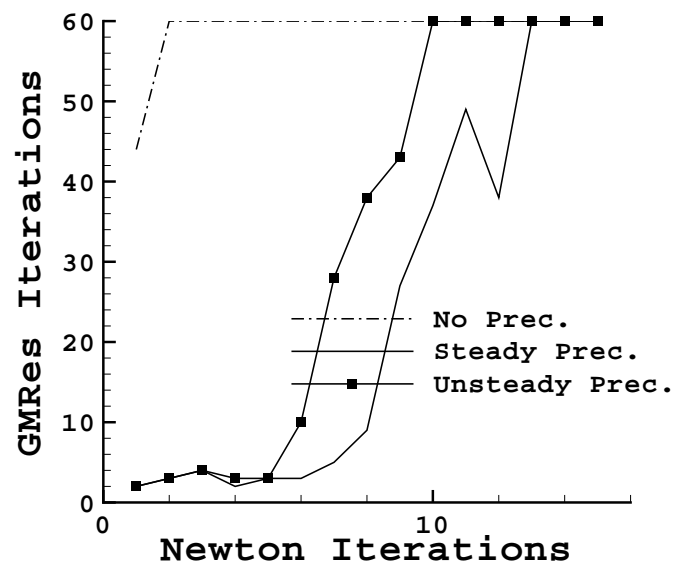
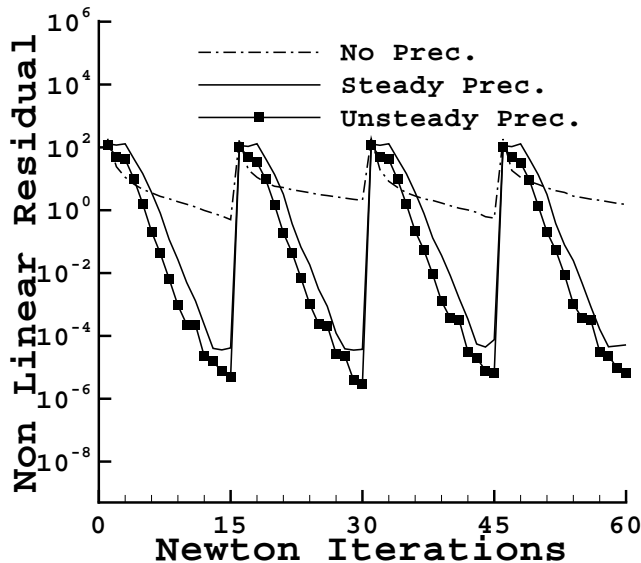
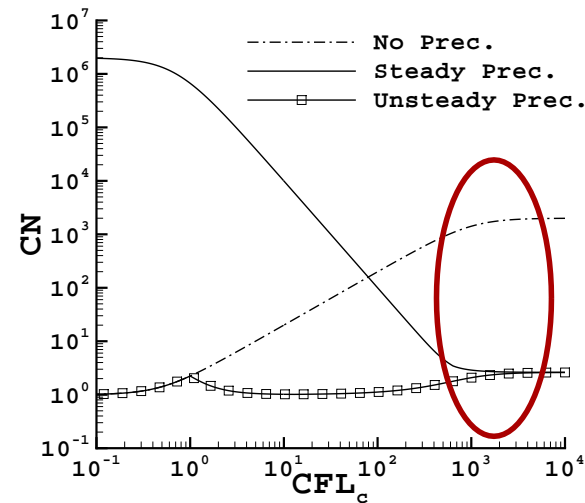
$$Re = 1000 \text{ and } M_0 = 10^{-3}$$



Convergence study for $CFL_c = 1, 10^2, 10^3 = M_0^{-1}$

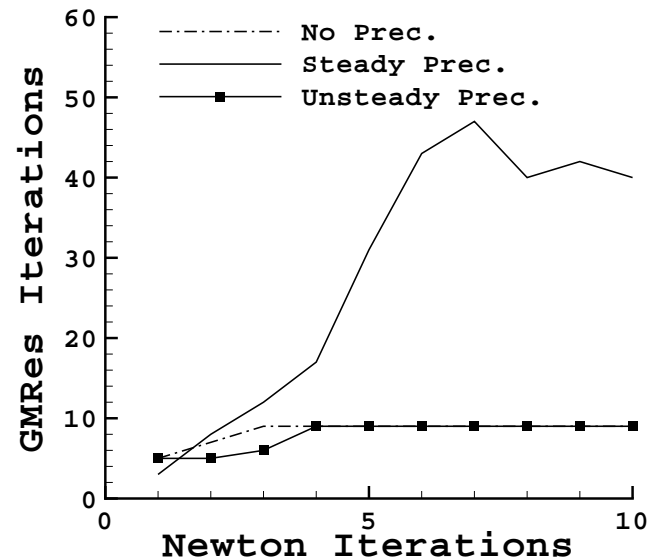
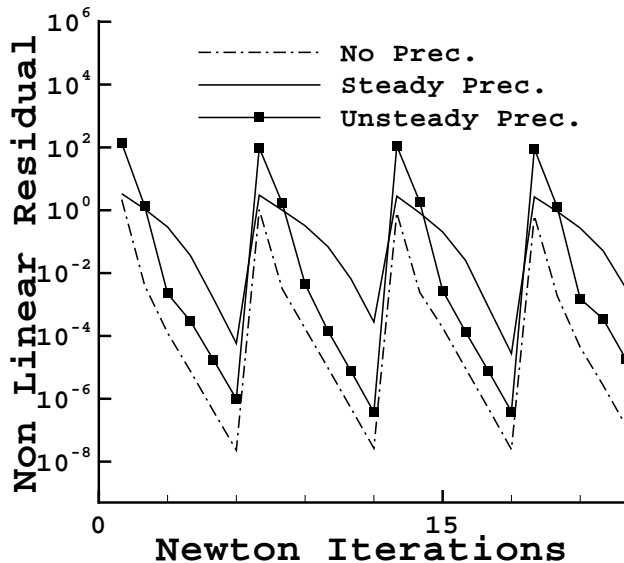
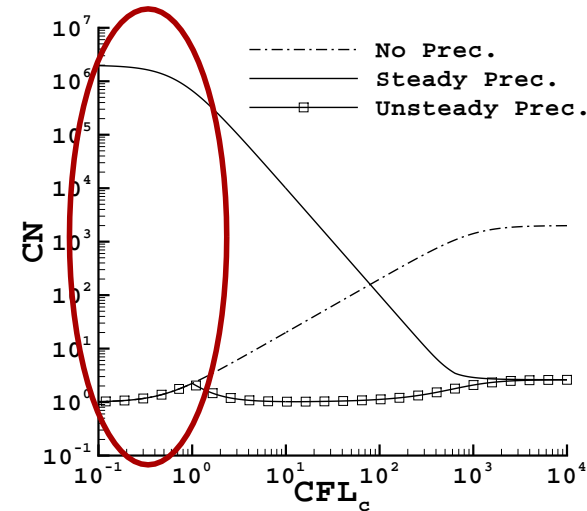
3D laminar lid driven cavity flow

- $CFL_c = 10^3, CFL_u = 1$
- Steady and Unsteady Prec. converge
- Steady and Unsteady Prec. are equivalent
- No Prec. does not converge



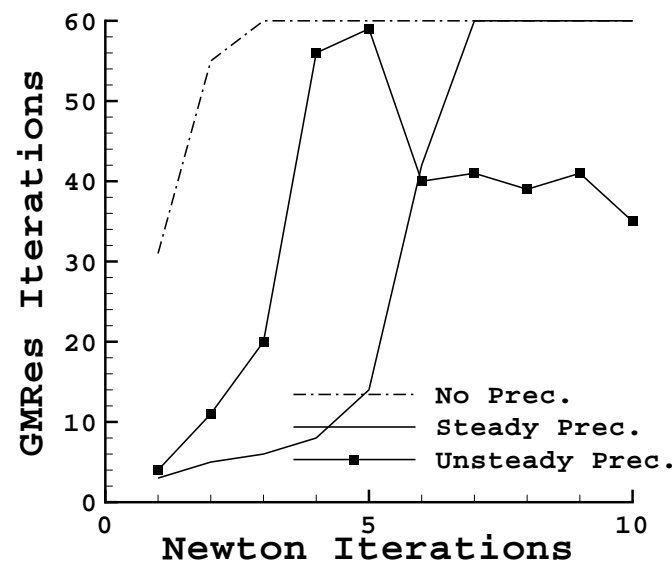
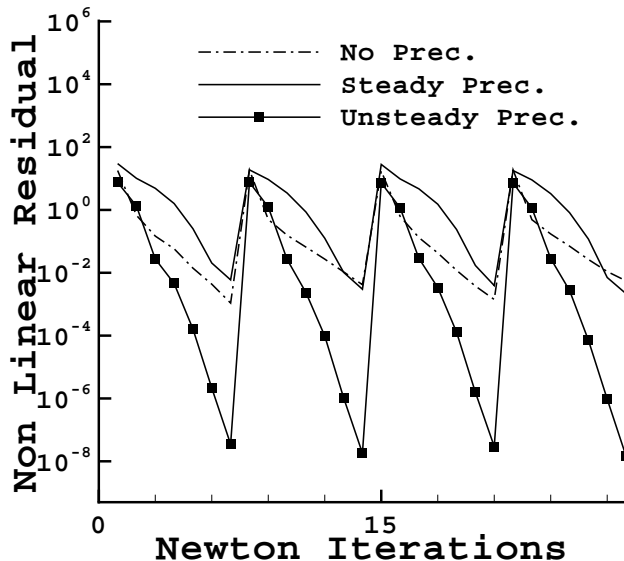
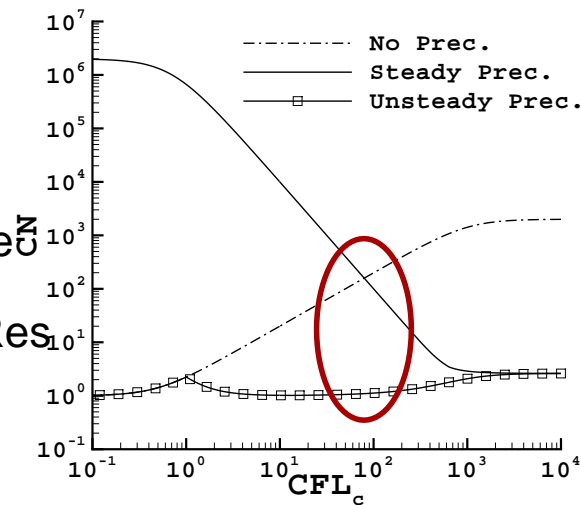
3D laminar lid driven cavity flow

- $CFL_c = 1, CFL_u = 10^{-3}$
- Unsteady and No Prec. converge
- Unsteady and No Prec. are equivalent
- Steady Prec. converges but lots of GMRes iterations are needed



3D laminar lid driven cavity flow

- $CFL_c = 10^2, CFL_u = 10^{-1}$
- All methods seem to converge but Unsteady Prec. improves convergence
- Unsteady Prec. yields more efficient GMRes
- Effect of f_c can be seen for low residuals



Conclusions

- Condition number depends on CFL_c and Mach numbers
- New unsteady preconditioning improves convergence
- AUSM+up scheme has been modified for solving low Mach number flows with any CFL_c numbers
- What about the diffusion terms ?
- A better study of space and time accuracy should be done