Report OA-28

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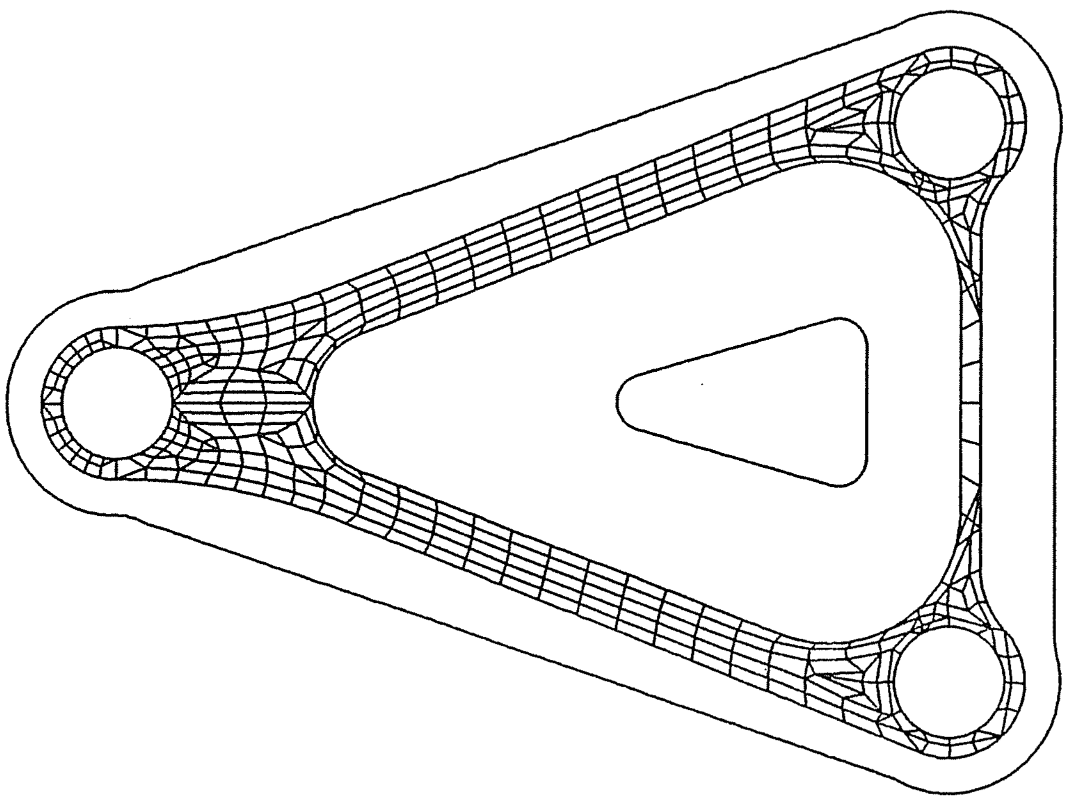

March 1993

*Paper to be presented at the 3rd International Conference OPTI93 in Zaragoza, Spain*

# A PARAMETRIC DESIGN BASED APPROACH TO STRUCTURAL SHAPE OPTIMIZATION

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ABSTRACT

In recent years, the integration of structural optimization methods with the modern CAD technology has made remarkable progress to achieve automatic engineering designs. The objective of this paper is to introduce the practical design methodology in structural shape optimization. A general parametric design based approach is developed and fully discussed here. Based on the CAD geometric model of the structure, shape design variables can be either manually or automatically identified with the help of equality constraints. Because the chosen design variables are reduced to a minimum number and are independent of each other, the original shape optimization problem is simplified into a compact form without equality constraints. Finally, the problem can be suitably solved by using the convex programming algorithm (CONLIN). Furthermore, it will be illustrated that the FEM based sensitivity analysis becomes easy to handle in this formulation.

Discussions and comparisons of the numerical results are given, which show that this approach is especially adapted to problems with many geometric design constraints, i.e., to structures composed of straight line segments and circular arcs, with at least $C^1$ order of continuity required along the boundary.

INTRODUCTION

During the last decade, the development of CAD oriented shape optimization techniques is one of the most important subjects and has been widely studied. Their successful applications have been recognized especially in aerospace and automobile industries. However, as reported by Haftka and Grandhi [1], one of the basic existing problems is concerned with the appropriate formulation of shape design variables, because this formulation has the great importance to the definition of the problem and the final design result of the structure.

In principle, one needs to design the boundary shape with several curve segments when the considered 2D structure has a complicated geometry. The reason is that this description has the advantage to make local design modifications. Here, the rising problem is to maintain the smoothing and the regularity of the whole boundary during the entire design process. For this reason, geometric design constraints are needed, which form, in fact, the nonlinear equality constraints in shape optimization problems. Generally, shape optimization problems appear more difficult to deal with than sizing optimization ones. The main facts are: (i). sizing design variables are naturally assigned according to the structural pattern. e.g., section areas for truss and thickness for plate but shape design variables are implicitly coupled by equality constraints and not directly available; (ii). general convex mathematical programming methods fail due to the nonlinearity of the equality constraints; (iii). equality constraints have to be respected during the perturbation step if sensitivity analysis is carried out by means of the semi-analytical method.

Until now, the usual approach is to represent the design boundary with one flexible mathematical CAD curve such as cubic splines, Bézier or B-spline curves where shape design variables are assigned by the coordinates of moving control points as utilized by Luchi, Braibant and Fleury [2,3]. Note that the flexibility is a key factor here which implies that continuity requirements can be implicitly satisfied by the curve's formulation whereas shape design variables are always independent. For example, cubic splines insure a continuity of $C^2$. Unfortunately, the produced optimum shape by this approach is sometimes rather mathematical than practical and thus lacks its direct adaptation to industrial design contexts. Moreover, it was found that the curves' formulation has also a great influence upon the quality of shape design variables.

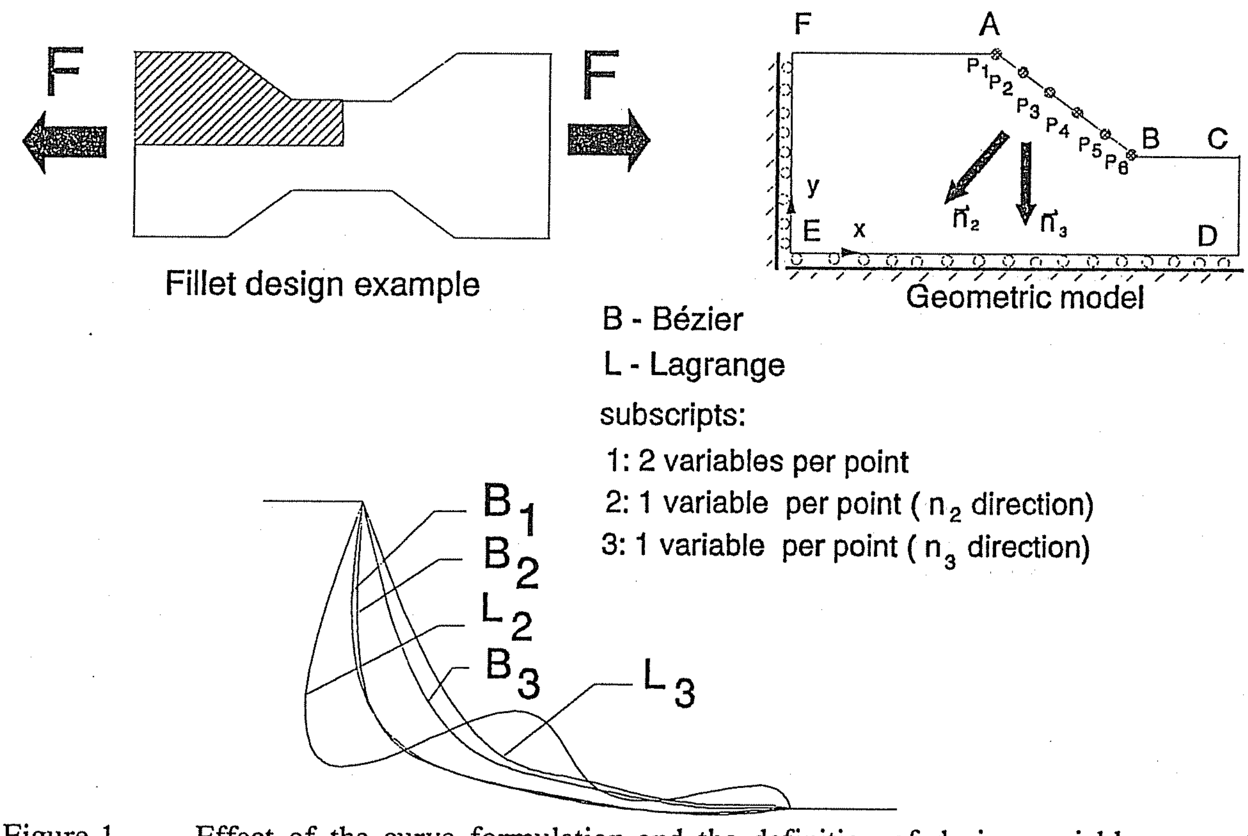



Figure 1 Effect of the curve formulation and the definition of design variables upon the optimum solution

As shown in figure 1, the minimum weight design of the fillet problem initially described by Seong and Choi [4] is solved here. The results demonstrate that Bézier curves exhibit a more superior reliability than Lagrangian curves which result in an oscillating solution still verifying stress constraints. Hence, a mathematically accepted solution is not surely an admissible engineering design.

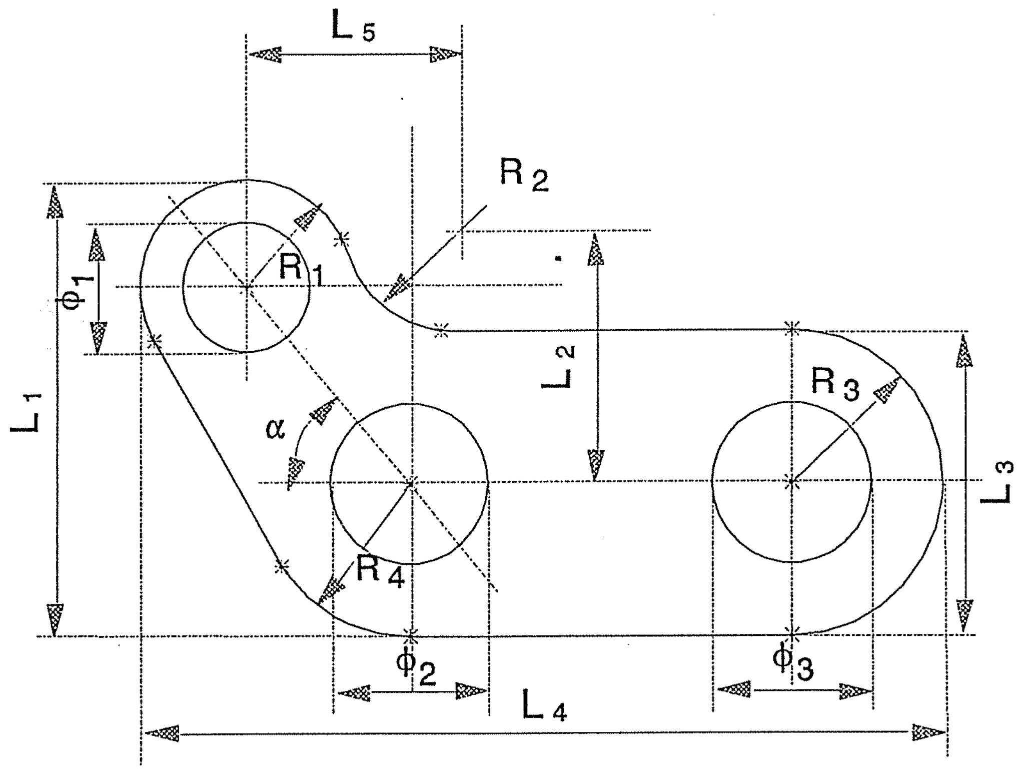



Figure 2 Parametric design annotation of mechanical drawings

In this work, efforts are devoted to shape optimization of 2D structures having a composed boundary of several portions of straight line segments and circular arcs typically like that in figure 2. The presentation of the paper is arranged as follows: at first, the popular parametric design methodology employed to annotate mechanical drawings is reviewed in order to interpret the physical meanings of shape design variables. It is shown that independence is their basic property. From this conclusion, a new general approach is developed to choose automatically independent shape design variables. Secondly, from the mathematical formulation of shape optimization problems, sequential convex subproblems are derived by introducing independent design variables as well as the convex linearization approximation. After, semi-analytical sensitivity analysis method is presented, emphasis is focused upon its efficient implementation in optimization codes. This is realized by analyzing the evaluation scheme of the velocity field quantifying the finite element mesh deformation. Discussions are also made about the relation between the velocity field and mesh generators. Finally, numerical examples are treated and the results are illustrated.

## REVIEW OF THE PARAMETRIC DESIGN METHODOLOGY

As is well known, mechanical design is a cycle process of several steps: description, analysis, interpretation of the results and modification. The first one is to create mechanical drawings, which includes two parts: (i). specification of the structural shape by means of one or more views, usually as plane parallel projections where hidden parts are removed or dotted and visible ones are represented by solid lines. (ii). quantitative annotation of views for which reference points and reference lines are previously specified and fixed to prevent rigid body translation and rotation. A dimensioning scheme is then imposed to parameterize the object with a set of scalar values, called dimension parameters, as length, angles and radius (see figure 2). The criterion is that the object is neither over- nor under-defined whatever the dimensioning scheme is.

The parametric design methodology also called variational geometry theory was originally established to generate variants of the current design as used by Light, Gossard and Aldefeld [5,6] in engineering production. It consists in choosing independent dimension parameters as shape design variables. For example, in figure 2, the exterior boundary having a $C^1$ order of continuity is described with nine high-level independent parameters ($L_1$, $L_2$, $L_3$, $L_4$, $L_5$, $R_1$, $R_2$, $R_3$, $R_4$) which are normally invariant to coordinate systems. In practice, this approach provides an efficient way to incorporate numerical optimization methods into the modern CAD dimension-driven systems. But the difficulty is how to construct a generalized dimensioning scheme because it depends much on how the structure will be fabricated, measured, toleranced and assembled as pointed out by Hillyard and Braid [7]. Even now, it is impossible to identify automatically independent parametric design variables in general cases, they are obtained either manually by observing the geometric features of the considered problem or through interactive dialogues with CAD design systems.

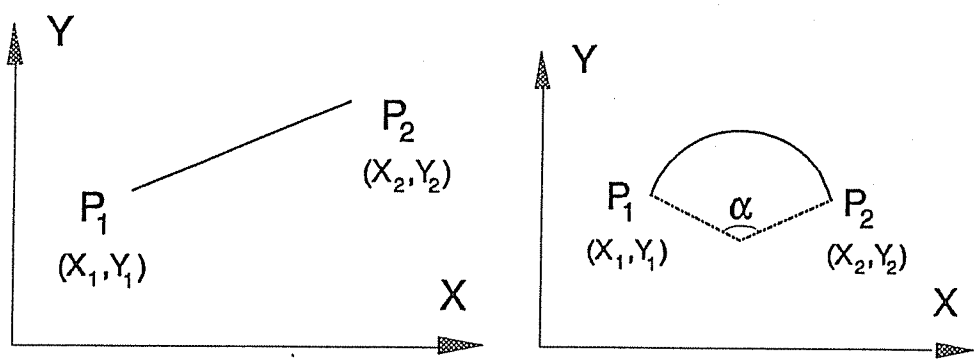



Figure 3 Description of straight line segments and circular arcs

It is known that each kind of geometric primitives can be characterized by the corresponding controlling points. For instance, as shown in figure 3, a planar straight line segment is defined by four independent variables: the coordinates of the starting point and the ending point; a circular arc is defined by five independent variables: the coordinates of the two extremity points and the opening angle. Certainly, the description can be made in other possible ways, but the proposed definition can insure automatically a $C^0$ continuity along the composed curve. Here, all these variables will be called low-level variables to distinguish with the high-level dimension parameters defined above.

In fact, the validity of independent parametric design variables can be numerically checked by the set of geometric equality constraints, which specify the relationship between independent parametric design variables with low-level geometric variables.

As illustrated in [5], geometric equality constraints can be generally expressed as the following implicit nonlinear system:

$$h_k(X,D) = 0 \qquad k = 1, L \tag{1}$$

where $X=(X_1,X_2,...,X_R)$ denotes the R-dimensioned vector of all low-level variables excluding the coordinates of fixed reference points. $D=(D_1,D_2,...,D_N)$ denotes the N-dimensioned vector of parametric design variables. The following cases may occur: if $R \neq L$, the dimensioning scheme is then invalid because low-level variables are not all constrained for a given design vector D. Precisely, if $R < L$, the structure can not be modified since the geometry is over-defined; if $R > L$, the number of parametric design variables is insufficient so that the geometry is under-defined. Hence, the valid dimensioning scheme for the design vector $D=(D_1,D_2,...,D_N)$ is achieved only if $R = L$. In this condition, the differentiation of (1) leads to a linearized system as follows:

$$T\delta X = -B\delta D \tag{2}$$

in which the two Jacobian matrix are defined respectively as:

$$T_{L\times L} = \{t_{ki}\} = \{\frac{\partial h_k}{\partial X_i}\} \qquad B_{L\times N} = \{b_{ki}\} = \{\frac{\partial h_k}{\partial D_i}\}$$

Physically, system (2) signifies that the geometric variation $\delta X$ of the current design is obtainable for a given $\delta D$ modification of design variables only if the Jacobian matrix $T$ is not singular. The solution is

$$\delta X = -T^{-1}B\delta D \tag{3}$$

If non singular square matrix $T$ can be found from (1),this means that the design problem is badly defined or eventually the system (2) is not an independent one.

In fact, one can also define independent shape design variables differently. Here, an automatic approach initially developed by Zhang and Beckers [8] is presented, which consists in choosing low-level variables as independent shape design variables. In this case, the original system (1) is reduced to that containing (L-N) equations.

$$h_k(X) = 0 \qquad k = 1, L\text{-}N \tag{4}$$

and the corresponding linear system is:

$$A\,\delta X = 0 \qquad \text{with } A_{(L-N)\times L} = \{a_{ki}\} = \{\frac{\partial h_k}{\partial X_i}\} \tag{5}$$

By splitting the vector $X = [Y,Z]$, the system (5) becomes

$$A_1\,\delta Y + A_2\,\delta Z = 0 \tag{6}$$

to which the solution is

$$\delta Y = -A_1^{-1} A_2\,\delta Z = -\tilde{A}\,\delta Z \qquad (\tilde{A} = A_1^{-1} A_2)$$
$$\text{only if } \det(A_1) \neq 0 \tag{7}$$

where $Y$ and $Z$ have the dimension of (L-N) and N respectively. By definition, Z is the vector of independent design variables and Y is the vector of dependent variables because the former is able to be expressed as a function of the latter. Numerically, to get a non singular and well-conditioned square matrix $A_1$, the automatic way adopted here is to use the Gauss-Jordan elimination method with pivoting manipulation.

In reality, two types of independent design variables can be transformed from one to another. Their relationship can be derived by decomposing the system (2) into:

$$T_1\,\delta Y + T_2\,\delta Z + B\,\delta D = 0 \tag{8}$$

Finally, to meet some practical requirements, a semi-automatic mixed approach can be achieved to identify independent design variables as a combination of a number of prescribed high-level dimension parameters with a number of low-level variables which will be automatically split out from (2). In this case, the number of equations in (2) should be less than that of low-level variables (L < R).

MATHEMATICAL FORMULATION OF THE PROBLEM

Definition of the original problem

Shape optimization problems can be generally stated as follows:

$$Min \quad f(X) \tag{9.1}$$

$$c_j(X) \leq \bar{c}_j \quad j = 1,M \tag{9.2}$$

$$h_k(X) = 0 \quad k = 1,L \tag{9.3}$$

$$X_i^- \leq X_i \leq X_i^+ \quad i = 1,N \tag{9.4}$$

Where (9.1) is the objective function to be minimized, usually defined by the structural weight or the stress concentration; (9.2) specifies constraints to structural responses which are commonly the displacements, the stress...etc. (9.3) corresponds to the geometric equality constraints discussed above and finally (9.4) represents technical constraints related to manufacturing requirements and the limitation of the working geometric domain.

Sequential subproblems with convex approximation

By retaining the independent design variables $Z$ from relation (9.3), the problem (9.1-9.4) is simplified into a reduced problem without equality constraints.

$$\begin{aligned} & Min \quad f(Z) \\ & c_j(Z) \leq \bar{c}_j \qquad j = 1,M \\ & Z_i^- \leq Z_i \leq Z_i^+ \qquad i = 1,L-N \end{aligned} \tag{10}$$

This problem can be suitably solved by sequential convex mathematical programming method (CONLIN), which was elaborated by Fleury and Braibant [9]. In this method, the unique condition is that all the design variables have to take positive values. This can be verified either by translating design variables or by changing their signs. If the convex linearization approximation is effected with respect to the current design $Z^0$, the sequential subproblem corresponds to:

$$Min \quad f(Z^0) + \sum_i^{+} \frac{\partial f(Z^0)}{\partial Z_i}(Z_i - Z_i^0) - \sum_i^{-} \frac{\partial f(Z^0)}{\partial Z_i}(Z_i^0)^2(\frac{1}{Z_i} - \frac{1}{Z_i^0})$$

$$c_j(Z^0) + \sum_i^{+} \frac{\partial c_j(Z^0)}{\partial Z_i}(Z_i - Z_i^0) - \sum_i^{-} \frac{\partial c_j(Z^0)}{\partial Z_i}(Z_i^0)^2(\frac{1}{Z_i} - \frac{1}{Z_i^0}) \le \bar{c}_j \qquad j = 1,M \tag{11}$$

$$Z_i^- \le Z_i \le Z_i^+ \qquad i = 1,N-L$$

where $\overset{+}{\underset{i}{\Sigma}}$ and $\overset{-}{\underset{i}{\Sigma}}$ designate the sum for which the corresponding first order partial derivatives are positive and negative respectively.

Each time when the optimal solution of the subproblem, noted $Z^*$, is obtained, dependent variables $Y$ should be correspondingly updated by solving (9.3) written as $h_k(Y,Z^*)=0 \quad k=1,L$. In our work, the solution is obtained by using the Newton-Raphson iteration method.

Sensitivity Analysis

Sensitivities of the objective function and the constraints are evaluated with the semi-analytical method whose original F.E. formulation is:

$$\frac{\partial q}{\partial Z_i} = K^{-1}(\frac{\partial g}{\partial Z_i} - \frac{\partial K}{\partial Z_i}q) \tag{12}$$

in which the derivative of the structural stiffness matrix $K$ and that of the load vector $g$ are approximated by finite difference calculation. That is,

$$\frac{\partial g}{\partial Z_i} \approx \frac{g(x_j^*) - g(x_j^0)}{\delta Z_i}$$

$$\frac{\partial K}{\partial Z_i} \approx \frac{K(x_j^*) - K(x_j^0)}{\partial Z_i} \tag{13}$$

where $\delta Z_i$, the perturbed step size of the $i^{th}$ design variable, has to be properly chosen, here we use $\delta Z_i = 10^{-5} Z_i$; $x_j^0$ and $x_j^*$ represent nodal coordinates related to the initial mesh and the perturbed mesh respectively. The latter is determined through the formula

$$x_j^* = x_j^0 + V\delta Z_i \qquad ( V=\frac{\partial x_j}{\partial Z_i} ) \tag{14}$$

$V$ is the velocity field representing how the mesh is deformed with respect to the perturbation of design variables. In this process, the deformation of the mesh does not change its original topology, only the nodal positions are slightly modified. In the work by Zhang and Beckers [10,11], different numerical approaches to evaluate $\cdot V$, such as transfinite mapping method, the physical approach, Laplacian smoothing method and the boundary node approach were compared and investigated, it was shown that the results are very similar as long as the approximation of the velocity field is global enough. Here, the physical approach is employed in which the velocity field is simulated as an elastic displacement field whereas the perturbation of the moving boundary is regarded as a prescribed displacement field on the boundary. In this way, the velocity field is the solution of the corresponding F.E. equations. Due to its generality, the physical approach is easily coupled with free mesh generators.

To implement the semi-analytical method (12), an efficient way is to treat the step size $\delta Z_i$ outside the parenthesis. Equation (12) is thus converted into:

$$\frac{\partial q}{\partial Z_i} = K^{-1}(\delta g - \delta K q)\frac{1}{\delta Z_i} \tag{15}$$

Two cases may happen here: firstly, when the computation of the velocity field is highly located to a small region of F.E. mesh near the moving boundary, for example, if the concept of design element patches is used, one can retain only the active finite elements having a non-zero velocity field. Then, equation (15) becomes:

$$\frac{\partial q}{\partial Z_i} = K^{-1} \left(\sum_e \delta g_e - \sum_e \delta k_e\, q\right)\frac{1}{\delta Z_i} \tag{16}$$

Secondly, when the velocity field has a global distribution, covers nearly the entire structural domain. For example, if it is calculated by the physical approach or the Laplacian smoothing scheme, then one can transform (15) into:

$$\frac{\partial q}{\partial Z_i} = K^{-1}(\delta g - \delta K q)\frac{1}{\delta Z_i} = K^{-1}(g^* - g - K^* q + Kq)\frac{1}{\delta Z_i} \tag{17}$$

by introducing the finite element equation $Kq = g$ into (17), one gets

$$\frac{\partial q}{\partial Z_i} = K^{-1}(g^* - K^* q)\frac{1}{\delta Z_i} \tag{18}$$

In this expression, finite difference calculations disappeared, only the perturbed global load vector and the perturbed structural stiffness matrix are needed. In the case when the load vector is independent of design variables, the load vector is unchanged equal to the initial one ($g^* = g$).

## NUMERICAL EXAMPLES

We demonstrate now the performance of the proposed design method. Three examples are treated which are all chosen from the literature with available numerical solutions. The structural weight is defined as the objective function to be minimized. Constraints are defined as the limitation of the average von-mises stress in each finite element. In order to test the developed design method, the boundary shape of each structure is redefined here by a set of straight line segments and circular arcs.

### Plate with a hole

Optimization of the hole shape in a square plate with a given biaxial planar forces is an academic problem. In figure 4, only a quart of the whole plate is represented due to its symmetry.

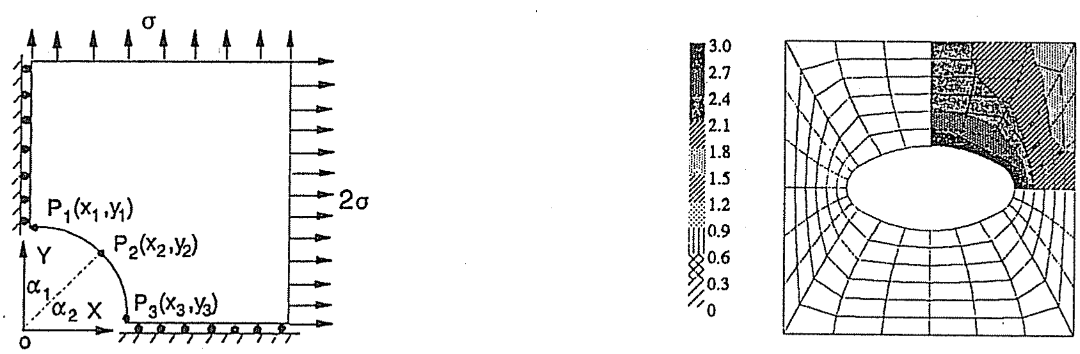



Figure 4 Optimization of the hole shape in a square plate

The hole segment is described by two circular arcs which yield eight low-level variables ( $x_1$, $y_1$, $\alpha_1$, $x_2$, $y_2$, $\alpha_2$, $x_3$, $y_3$ ). Five geometric equality constraints exist: the tangent condition between two circular arcs and symmetric conditions keeping the centre, the point $P_1$ of the first circular arc on the Y-axis ( $x_1 = x_{c1} = 0$ ) and the centre, the point $P_3$ of the second circular arc on the X-axis ( $y_3 = y_{c2} = 0$ ). As a result, three (8-5=3) independent design variables are automatically identified by the Gauss-Jordan pivoting scheme, which are ( $y_1$, $y_2$, $x_3$ ). Alternatively, if a mixed selection is made, one may adopt ( $y_1$, $R_1$, $R_2$ ) as an other solution.

The optimum shape together with the final distribution of the von-mises stress is shown in figure 4. One can see that the optimal shape is very close to the analytical solution of the ellipse.

Shape design of a torque arm

The problem initially studied by Bennett and Botkin [13] is shown now in figure 5-(a), where the outside and inside contours need to be modified. In the right extremity, the structure is simultaneously loaded with the traction force and mainly the bending force. Fixations are imposed along the circular hole in the left. Related data are given below:

$\mu = 0.3$, t = 0.3 cm E = 20.74 $10^6$ N/cm$^2$ $\rho = 7.81\ 10^{-3}$ kg/cm$^3$
$F_x = 2789$ N $F_y = 5066$ N $\sigma_{admi} = 8\ 10^4$ N/cm$^2$
$d_{min.}$ = *1 cm (minimal separation between boundaries)*

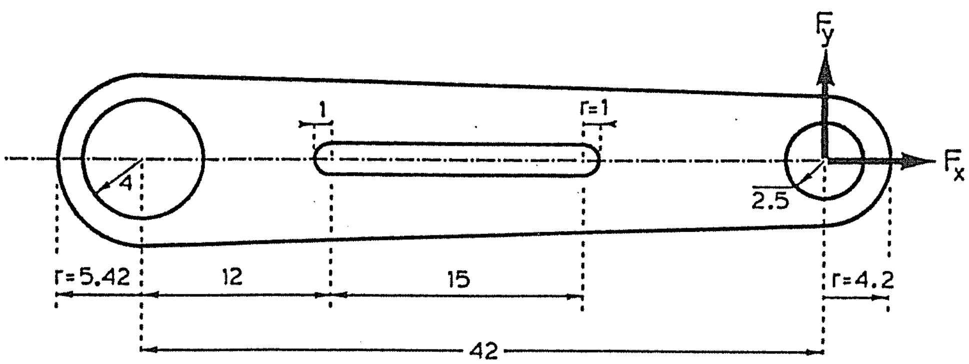



Figure 5-(a) Description of the problem (in cm)

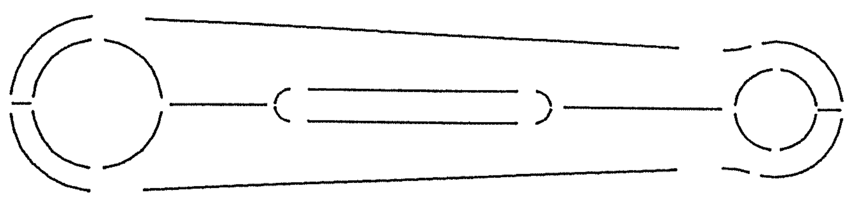


Figure 5-(b) Geometric model defined with straight line segments and circular arcs

In figure 5-(b) is shown our used geometric model. At the beginning, although it contains twenty six low-level parameters for the moving boundaries, only eight among them are retained as independent design variables in the optimization model.

Here, it is necessary to mention the problem related to the finite element mesh distortion

during the design process. This is a common issue in shape optimization. Ideally, a complete design tool would integrate adaptive finite element techniques for mesh generation and error control, as suggested by Kikuchi et al. [13]. In this example, the free mesh generator is used to remesh the modified geometric domain after each design iteration. Geometrically, the created mesh has an acceptable reliability provided the free mesh generator is robust enough. It was found that remeshing is reasonable and efficient when large modifications are made at each iteration step. In contrast, if each iteration leads to relatively a small geometric modification, one can utilise the simple relocation procedure to create the new mesh, i.e., by using the Laplacian smoothing scheme. The solution is reached after eight iterations with a weight reduction of about 50%. As shown in figure 6, The final design is feasible, the maximum von-mises stress increases from 4.21 $10^4$ N/cm$^2$ to 7.88 $10^4$ N/cm$^2$ under the allowance of the limit. The iteration history is shown in figure 7.

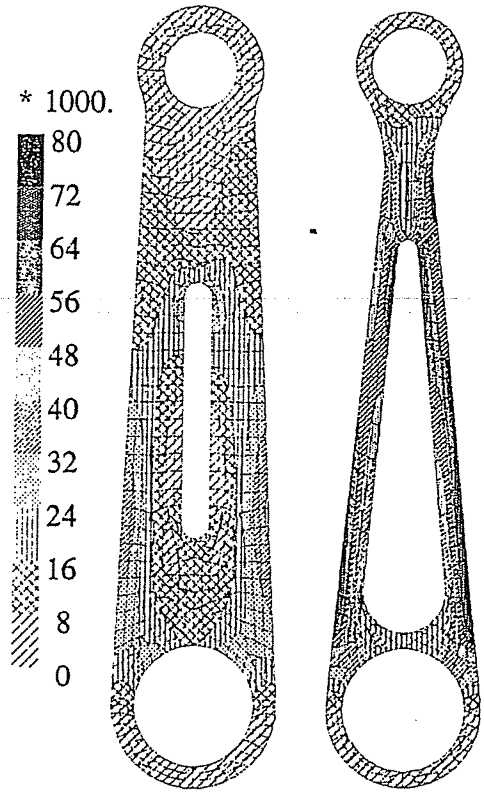



Figure 6 Isovalues of von-mises stress for the initial and final design

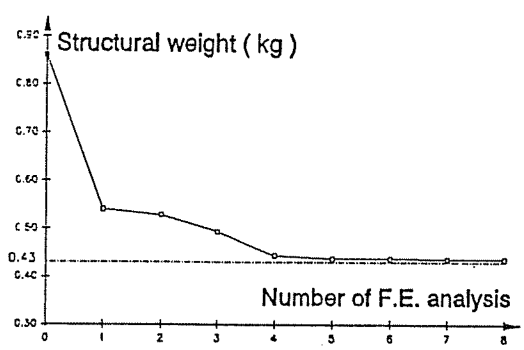



Figure 7 Iteration history

Shape design of a bracket

It is known that numerical shape optimization is typically established in such a way that the geometric model is constructed only one time when the design process begins. One has to determine how many pieces of curve segments are used to represent the real boundary, their types and how they will be orderly connected among them. After, all these specifications will be kept unchanged until the ending of the design history. In this example, one can understand that geometric modelling plays an important role in shape optimization. If a refined design model is established with a suitable number of independent design variables, considerable improvements can be achieved to the final design.

The initial design of the structure with its boundary conditions and the loading are given in figure 8. Here, two design models are considered and described in figure 9. The first one is supposed to be the same as the original model. It holds seven independent design variables. The final design is obtained after five iterations, as shown in figure 10. The weight decreases to 0.14 kg. The second model is more flexible with an addition of one circular arc in the left part. Finally, this model acquires ten independent design variables, the optimum solution is reached after six iterations which gives a greater reduction of weight from 0.36 to 0.11 kg. Here, both two cases produce feasible solutions. In figure 11, these solutions are compared with the reference one.

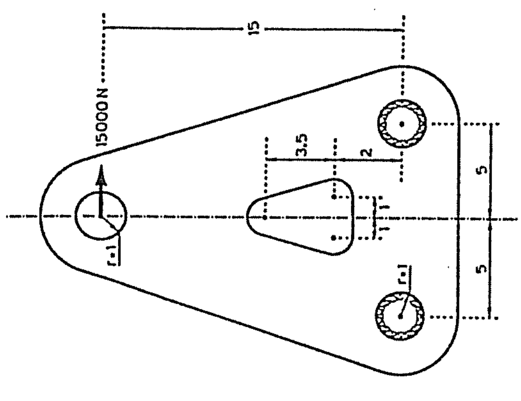
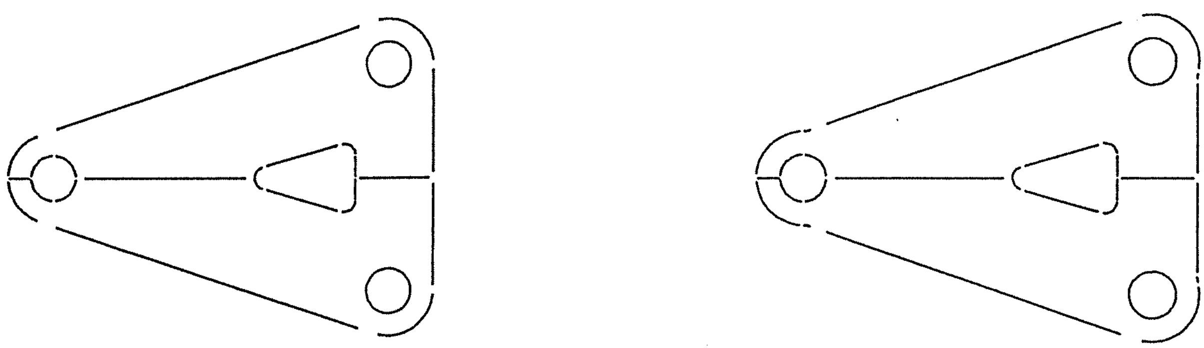



Figure 8 The dimensions and loading of the bracket



model 1 model 2

Figure 9 Two different geometric models for the problem

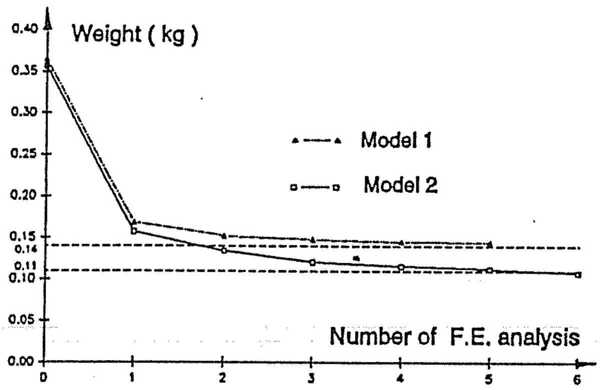
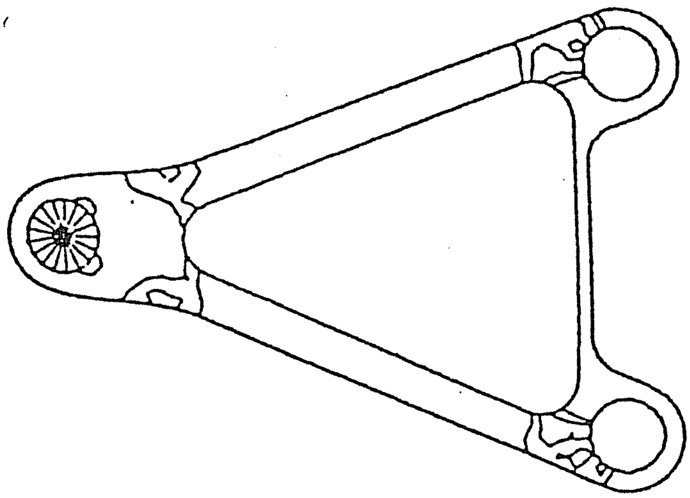
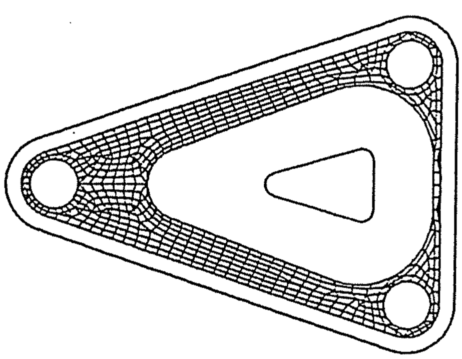


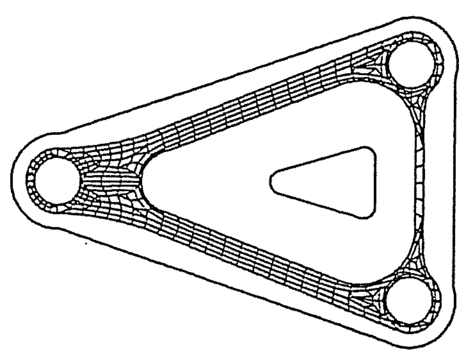

Figure 10 Iteration history



reference solution by Bennett and Botkin [12]



solution obtained from model 1



solution obtained from model 2

Figure 11 Comparison of different solutions

## CONCLUSIONS

A parametric design based approach is established to shape optimization of 2D structures defined with pieces of curve segments, typically straight line segments and circular arcs. Based on the CAD model of the problem, this approach consists in choosing automatically low-level geometric parameters from the set of geometric equality constraints to constitute independent shape design variables. The advantage is that a minimum number of design variables are retained, equality constraints are eliminated from the optimization design model and they will be respected through a post process. Finally, the reduced problem can be easily solved by utilizing sequential convex programming methods (CONLIN).

An new scheme is given to make an efficient implementation of the semi-analytical method to sensitivity analysis. The goal is to reduce the cost of the finite difference evaluation of the stiffness matrix and the load vector, which depends on how the velocity field is distributed locally or globally.

Numerical examples have shown that the integration of the parametric scheme to define shape design variables with the semi-analytical method to sensitivity analysis and the physical approach to evaluate the velocity and the free mesh generator forms a reliable numerical tool for practical design applications.

## REFERENCES


[1]. Haftka, R.T. and Grandhi, R.V. 'Structural Shape Optimization - A Survey' *Comp. Meth. Appl. Mech. Eng.*, Vol.57, pp.91-106, 1986

[2]. Luchi, M.L. et al, 'An Interactive Optimization Procedure Applied to the Design of Gas Turbine Discs' *Comp. Struct.*, Vol.11, pp.629-637, 1980

[3]. Braibant, V. and Fleury, C. 'Shape Optimal Design Using B-splines' *Comp. Meth. Appl. Mech. Eng.*, Vol.44, pp.247-267, 1984

[4]. Seong, H.G. and Choi, K.K. 'Boundary-Layer Approach to Shape Design Sensitivity Analysis' *Mech. Struct. & Mach.*, Vol.15, pp.241-263, 1987

[5]. Light, R. and Gossard, D. 'Modification of Geometric Models through Variational Geometry' *CAD* Vol.14, No.4, pp.209-214. 1982

[6]. Aldefeld, B. 'Variation of Geometries Based on a Geometric-Reasoning Method' *CAD* Vol.20, No.3, pp.117-126. 1988

[7]. Hillyard, RC. and Braid, IC. 'Analysis of Dimensions and Tolerances in Computer-Aided Mechanical Design' *CAD* Vol.10, No.3, pp.161-166, 1978

[8]. Zhang, W.H. and Beckers, P. 'A Systematic Strategy to Select Design Variables for Shape Optimization of Structures,' pp.348-351, *Proceedings of the 2th Belgium National Conf. on Theoritical and Applied Mechanics* (Ed. Dick, E and Hogge. M), Bruxelles, Belgium, 1990.

[9]. Fleury, C. and Braibant, V. 'Structural Optimization - A New Dual Method Using Mixed Variables' *Int. J. Num. Meth. Eng.* Vol.23, pp.409-428, 1986

[10]. Zhang, W.H. and Beckers, P. 'Comparison of Different Sensitivity Analyses Approaches for Structural Shape Optimization,' in Opti89 (Ed. Brebbia, C.A. and Hernandez, S.), pp.347-356, *Proceedings of the 1st Int. Conf. on Computer Aided Optimum Design of Structures: Recent Advances,* Southampton, U.K. 1989.

[11]. Zhang, W.H. 'Sensitivty Analysis and Structural Shape Optimization by Finite Element Method,' *Ph.D Thesis,* University of Liège, 1991

[12]. Bennett, J.A. and Botkin, M.E. 'Structural Shape Optimization with Geometric Description and Adaptive Mesh Refinement' *AIAA J.* Vol.23, No.3, pp.458-464, 1984

[13]. Kikuchi, N et al. 'Adaptive Finite Element Methods for Shape Optimization of Linearly Elastic Structures' *Comp. Meth. Appl. Mech. Eng.*, Vol.57, pp.67-89, 1986