A NEW SEPARABLE APPROXIMATION SCHEME FOR TOPOLOGICAL PROBLEMS AND OPTIMIZATION PROBLEMS CHARACTERIZED BY A LARGE NUMBER OF DESIGN VARIABLES

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ABSTRACT

This paper is dedicated to a new convex separable approximation for solving optimization problems characterized by a very large number of design variables as in topology design. For such problems, the convergence speed can be accelerated if one uses high quality approximation schemes for structural responses. To achieve this task, we propose, here, a new approximation procedure that belongs to the GMMA family. The originality of this new scheme is to rely on an automatic selection procedure of the asymptotes, using only first and zero order information accumulated during the previous iterations. This is possible owing to a diagonal Quasi-Newton update technique. Firstly, the new approximation procedure is validated on classical "benchmarks" of structural optimization. Then, it is compared to other schemes on a typical topology optimization problem.

KEYWORDS

Convex approximation, MMA, Diagonal Quasi-Newton update, Topology optimization.

INTRODUCTION AND DEFINITION OF THE PROBLEM OF INTEREST

The present work is concerned with the solution aspects of large scale optimization problems. Such problems have particularly received great attention since the emergence of topology optimization. Determination of the topology of structural component can be achieved through a material distribution problem over a design domain. The main difficulty inherent to that formulation is the very large number of design variables introduced by the material distribution. The earliest resolutions of the topology problem such in Bendsøe and Kikuchi (1988) used optimality criteria. Even if this technique was suited to problems with a single constraint and well-defined characteristics, the generalization of the topological problem under various design conditions can be facilitated by the use of mathematical programming techniques. One major feature of this procedure is the replacement of the real implicit optimization problem by a sequence of approximate subproblems. Since the dual space has a smaller dimension, the dual methods are very well adapted to solve the convex, explicit subproblems, if the involved approximations are separable. The quality of the approximations of the structural constraints has a great influence on the speed of convergence of candidate optimum designs. Because of the high cost of topology problems, the need for good approximations is very important. As it can be seen in numerous applications, first order schemes such as CONLIN (Fleury and Braibant, 1986) and MMA (Svanberg, 1987) exhibit a good convergence rate during the first iterations but slow down during the final convergence phase. On another hand, the second order approximations (Fleury, 1989a) have good convergence properties in the final convergence, but they seem to be more sensitive to local optima and necessitate a second order sensitivity analysis which can be very expensive for a large number of design variables. Here, we propose a new approximation procedure whose aim is to combine the advantages of the first and second order schemes. The new approximation scheme belongs to the MMA family, but the main originality of the procedure is to update the moving asymptotes on the basis of the first order information. To the authors' knowledge, besides update procedures of the asymptotes based on zero order or second order information, there was no update that relied on the gradients. This paper brings a satisfactory answer to this problem. The asymptotes are selected automatically with a scheme similar to the second order strategy of Smaoui *et al.* (1988), but the curvatures are estimated by a Quasi-Newton update preserving the diagonal structure of the Hessian estimates. As it is limited to vector manipulation, it can be achieved with a reasonable cost on large problems. Since one is able to choose a particular set of asymptotes for each structural constraints, the new approximation scheme belongs to what some people calls the GMMA (Generalized Method of Moving Asymptotes) family.

A CONVEX APPROXIMATION OF THE GMMA FAMILY

Structural optimization by mathematical programming methods relies on the concept of approximation of mechanical responses. Various first order approximations are generated by making a first order Taylor expansion of the constraint after a suitable change of design variables. Since pioneer works, the key role of the reciprocal variables is well known to reduce the non-linearity of the mechanical responses, and thus to yield a first order approximation scheme of high quality. When every first derivatives are negative, the approximation have positive curvatures that fits the structural behaviour much better than a direct linear expansion. Unfortunately, when sensitivities have mixed signs, the approximation is no longer convex. CONLIN (Fleury and Braibant, 1986) is a mixed approximation where the direct expansion is selected if the first derivative is positive while the reciprocal variables are chosen when the related first derivative is negative. Thus, CONLIN is the first unconditionally convex approximation scheme. Nevertheless, besides the numerous successes in sizing and shape optimization, CONLIN failures can be attributed to an inflexible choice of the convexity of the approximation which is not well adapted to some problems. The method of moving asymptotes (MMA) (Svanberg, 1987) brings a possible remedy to these problems. The MMA allows to modify additional parameters to adjust the approximation curvature to the problem's one. Nonetheless, the selection of the moving asymptotes is still a question which is only partially solved. Svanberg (1987) proposed an heuristic selection strategy of the asymptotes based on the design variables oscillation. Zhang and Fleury (1994) gave an alternative strategy by fitting the approximation to the previous function value. If the second order sensitivity is available, the asymptotes can be automatically selected to match the true curvatures (Smaoui *et al.*,1988). Latter, Fleury (1989a) showed that such second order schemes were well adapted to structural problems despite the fact that the computation of the second order sensitivity was onerous.

We propose here to use a new structural approximation which derives from the family of the method of moving asymptotes. To generate approximations of high quality, the scheme uses an automatic selection strategy for the moving asymptotes which is similar to the one described by Smaoui *et al.* (1988). The structural response $g(x)$ at the design point x^0 is replaced by its convex approximation $\tilde{g}(x)$. The parameters are adjusted to match the value of the function, its first derivatives as well as its diagonal curvatures at the current design point.

$$\tilde{g}(x) \simeq c_0 + \sum_{i=1}^{n} \frac{a_i}{x_i - b_i} \tag{1}$$

$$a_i = -(x_i^0 - b_i)^2 \frac{\partial g}{\partial x_i}(x^0) \qquad b_i = x_i^0 + 2\frac{\partial g}{\partial x_i} / \max(\epsilon, \frac{\partial^2 g}{\partial x_i^2}) \qquad (0 < \epsilon \ll 1)$$

Of course, the linear and the reciprocal expansions, the CONLIN or the pure MMA approximations can be recovered by setting the asymptotes to particular values (e.g. 0, +∞, -∞), or choosing special curvatures for the approximation. But, the main difficulty of this strategy is to estimate the coefficient of curvature $\partial^2 g / \partial x_i^2$. The second order sensitivity information is often unavailable or too prohibitive, especially for a large number of design variables. That's why, despite their higher accuracy, the emergence of the second order approximations, like the Smaoui's scheme, is slowed down. We propose here to avoid the direct second order derivatives evaluation by the use of an inexpensive, special purpose, Quasi-Newton update which generates only diagonal estimates.

DIAGONAL QUASI-NEWTON UPDATES

To avoid the second order sensitivity analysis, the idea is to use the available first order information and to build an approximation of the Hessian with a Quasi-Newton update procedure. Nevertheless, the "full" Quasi-Newton becomes also expensive when the number of design variables increases. On another hand, only diagonal terms are useful since separable approximations are used for problems with a large number of design variables. Thus, we present here a modified BFGS update scheme able to generate a sequence of diagonal Hessian estimates. The algorithm is the adaptation (Haubruge and Nguyen, 1994), to diagonal matrices, of more general results for Quasi-Newton updates preserving the sparse structure of the Hessian estimates (Thapa, 1981).

Let $\boldsymbol{B}$ be a diagonal approximation of the Hessian matrix of a given structural response at the current design point $\boldsymbol{x}$. If the new design $\boldsymbol{x}^+$ doesn't satisfy convergence criteria, one seeks to enrich the estimation of the Hessian with a Quasi-Newton update procedure. The update formula are all based on the Quasi-Newton equation.

$$\boldsymbol{B}^+ \boldsymbol{s} = \boldsymbol{y} \quad \textit{where} \quad \boldsymbol{s} = \boldsymbol{x}^+ - \boldsymbol{x} \quad \textit{and} \quad \boldsymbol{y} = \nabla g(\boldsymbol{x}^+) - \nabla g(\boldsymbol{x}) \tag{2}$$

One of the most famous update formula is the Broyden-Fletcher-Goldfard-Shanno (BFGS) one :

$$\boldsymbol{B}^+ = \boldsymbol{B} + \boldsymbol{U}_{BFGS} \quad \textit{with} \quad \boldsymbol{U}_{BFGS} = \frac{\boldsymbol{y}\boldsymbol{y}^T}{\boldsymbol{s}^T\boldsymbol{y}} - \frac{\boldsymbol{B}\boldsymbol{s}\boldsymbol{s}^T\boldsymbol{B}}{\boldsymbol{s}^T\boldsymbol{B}\boldsymbol{s}} \tag{3}$$

Even if this powerful update satisfies simultaneously the symmetry and the positive definiteness of the update as well as the Quasi-Newton condition, it doesn't preserve sparse or diagonal structure of the previous estimate.

In the following, we note by $\boldsymbol{B}_D^+$ and $\boldsymbol{B}_{ND}^+$ the matrices which are formed respectively with the diagonal and the off-diagonal terms of the "full" update while $\hat{\boldsymbol{B}}^+$ is the diagonal Quasi-Newton update we look for. According to the theorems of Thapa (1981), the diagonal update can be found by defining a correction matrix $\boldsymbol{E}$, so that the diagonal updated matrix $\hat{\boldsymbol{B}}^{+} = \boldsymbol{B}_{D}^{+} + \boldsymbol{E}$ be the closest, in the Frobenius norm, by the classic updated matrix $\boldsymbol{B}_{D}^{+}$ and still satisfies the Quasi-Newton condition $\hat{\boldsymbol{B}}^{+} \boldsymbol{s} = \boldsymbol{y}$. The correction matrix $\boldsymbol{E}$ that we look for is the solution of the minimum problem:

$$\begin{aligned} &\min \ \|\boldsymbol{E}\|_F \\ &\text{s.t. } \boldsymbol{E}\,\boldsymbol{s} = \boldsymbol{B}_{ND}^{+}\,\boldsymbol{s} \\ &\qquad E_{ij} = 0 \quad (\, i \neq j\,) \end{aligned} \tag{4}$$

The solution writes $\boldsymbol{E} = \operatorname{diag}\{2\,\lambda_i\, s_i\}$ where the vector $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$, itself, is solution of the linear system :

$$\boldsymbol{Q}\boldsymbol{\lambda} = \boldsymbol{B}_{ND}^{+}\boldsymbol{s} = \boldsymbol{y} - \boldsymbol{B}\boldsymbol{s} - \boldsymbol{U}_{D}\,\boldsymbol{s} \qquad \textit{with} \qquad \boldsymbol{Q} = \operatorname{diag}\{s_i^2\} \tag{5}$$

The computation of the diagonal BFGS update requires solving this last diagonal system. Nonetheless, it is not necessary to compute the off-diagonal terms of the classical update correction, since only the diagonal terms are used. So, this diagonal update is very inexpensive and requires a much lower storage capacity than a "full" BFGS update since the diagonal update needs only some vector manipulations.

ADAPTING THE DIAGONAL BFGS UPDATE TO STRUCTURAL OPTIMIZATION

The diagonal BFGS update has to be adapted to structural problems to yield approximations of high quality. Since the optimum has to be localized in few iterations, the sequence of the estimations of curvatures must be quickly convergent despite the large number of variables. This can be practically achieved if the update sequence is begun with a first diagonal Hessian estimate of good quality and if the non-linearity of the response is low. Then, we developed an efficient procedure based on the key role of reciprocal variables in structural design. Indeed, it is well known that working in the space of reciprocal variables is favourable to reduce the curvature of structural constraints. So, it is better to update the Hessian matrix in the reciprocal design space. The choice of the initial Hessian estimate is very important too. In the reciprocal design space, a diagonal of small positive numbers gives good results:

$$\boldsymbol{B}^{0}(y_i = 1/x_i) \simeq \mathit{diag}\{0 < \kappa \leq 1\} \tag{6}$$

If the update is performed in the direct design space, the choice of the curvatures introduced by CONLIN (Fleury and Braibant, 1986) or pure MMA (Svanberg, 1987) approximations appears usually very relevant. But a more conservative choice might be :

$$B_{ii}^{0}(x_i^0) \simeq \frac{2\,|g_i'|}{x_i^0} \qquad \textit{with} \qquad g_i' = \frac{\partial g}{\partial x_i} \tag{7}$$

Finally, to our experience, it doesn't worth forcing positive definite character of the updates which can slow down the convergence of the Hessian update and over-convexify the approximations.

SECOND ORDER CORRECTION PROCEDURE (SOC)

Quasi-Newton updates exploit first order information accumulated during the previous iterations to enrich the curvature estimate. The second order correction (SOC) that we propose, tries also to improve the quality of the approximation by using the function value information. As this has already been suggested by Zhang and Fleury (1994), the convexity of the approximation can be adjusted by fitting to the function value at the previous design point. The convexity of the approximation is uniformly modified through a dilation of the available estimations of the asymptotes $\hat{\boldsymbol{b}}^k$, around the point $\boldsymbol{x}^k$:

$$b_i := x_i^k - s\,(x_i^k - \hat{b}_i^k) \tag{8}$$

The curvature of the approximation is changed by a factor 1/s. The approximation is a one-dimensional, monotonic, decreasing and convex function of the parameter "s" . The optimal parameter "s*" is found by solving numerically the equation :

$$\tilde{g}(x^{k-1},s) - g(x^{k-1}) = 0 \tag{9}$$

The parameter "s" is nevertheless searched in the interval $0 < s_{\min} \le s \le s_{\max}$. The lower and the upper bounds avoid too large modifications of the convexity of the approximation. The lower bound $s_{\min}$ keeps the approximation being degenerated.

NUMERICAL APPLICATIONS

Eight Bar Truss Problem

The new approximation with an automatic selection of the asymptotes based on a diagonal Quasi-Newton update (QNMMA) can be validated on classical "bench-marks" of structural optimization. The eight bar truss problem, reported by Svanberg (1987), is one of the most severe ones. The design problem consists in minimizing the mass while keeping the stresses under 100 N/mm² under the given load case. In this problem, the degree of redundancy is very high and the behaviour is highly non-convex. The initial cross sections are x_i^0=400 mm² and they must remain over $\underline{x}_i^0$=100 mm². As suggested by Fleury (1989a), one often gets an optimum solution of equal mass, but with different configurations. The solution we get in all our experiences is x_1=859.6, x_2=743.5, x_3=241.1, x_4=537.4, x_5=100.0, x_6=100.0, x_7=100.0, x_8=100.0 (mm²). Table 1. gives a comparison of the QNMMA scheme with other different well-known approximation schemes. Convergence of CONLIN is disastrous in that example. Surprisingly, the Smaoui's second order MMA scheme is quite slow (17 iterations). The QNMMA takes only 9 iterations which is as quick as the best performance of the pure MMA of Svanberg (1987). Nonetheless, with the QNMMA, the asymptotes are selected automatically. In this example, the diagonal Quasi-Newton update is performed in the space of the direct design variables. The adopted initial curvature is the most conservative choice presented in equation 7. One can note also that our stopping criterion is very severe (4 digits on the objective function), so that optimisation is continued until iteration 9, despite the fact that the mass has nearly converged after 8 iterations. When combined with the SOC procedure, the number of iterations is reduced to 7. Indeed, the 8 bar truss is a non-convex problem, and the SOC correction reduces the convexity of the approximation at each iteration. Nevertheless, the benefits of the SOC procedure cannot be generalized, because, in some problems, it leads sometimes to unfeasible designs that finally slow down the convergence.

Table 1. : Comparison of the results for the 8 bar truss

It.	CONLIN	best MMA (Svanberg, 1987)	second order MMA	QNMMA	QNMMA + SOC
1	13.05	13.05	13.05	13.05	13.05
2	12.05	12.10	10.84	12.27	12.27
3	11.66	11.67	11.51	11.67	11.65
4	11.64	11.65	11.58	11.64	11.64
5	11.63	11.61	11.55	11.42	11.25
6	11.61	11.52	11.51	11.30	11.23
7	11.60	11.42	11.47	11.26	11.23
8	11.59	11.28	11.44	11.23	-
9	11.57	11.23	11.40	11.23	
10	11.55	-	11.37	-	
11	11.54		11.34		
...	...		...		
17	11.46		11.23		
...	...		-		
50	11.22				

Topological Optimization of a Short Cantilever Beam

Finally, we illustrate the new QNMMA approximation combining the diagonal Quasi-Newton update and the automatic choice of the moving asymptotes on a topology problem. The benchmark we have selected is the well-known short cantilever beam problem (fig. 1).

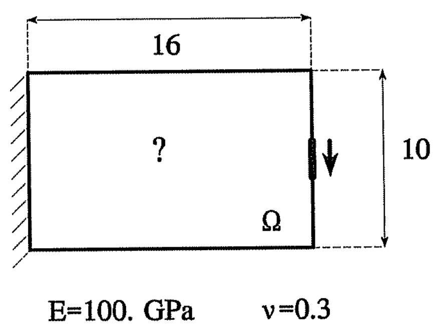

Fig. 1. Short cantilever beam problem

The material law is simply given by a cubic relation between the rigidity and the relative density (Bendsøe, 1989) : $E = \mu^3 E^0$ and $\rho = \mu \rho^0$. The Poisson's coefficient and the Young modulus of the solid are : ν^0=0.3 and E^0=100 GPa. The compliance under the given load case is minimized while the volume is bounded to 37.5% of the volume of the design domain. The problem is discretized by a regular mesh of 1040 finite elements of degree 2 and is solved with three different approximations for compliance (the volume is linearized). Since all the first derivatives of compliance are negative, CONLIN and the reciprocal variables expansion are the same. As this direct relaxation (Bendsøe, 1989) of the topology problem is similar to a plate problem, we try also to expand compliance in terms of the same power of the reciprocal variables. Then, we use the new approximation QNMMA. The optimal distribution of the material is given in fig. 2.

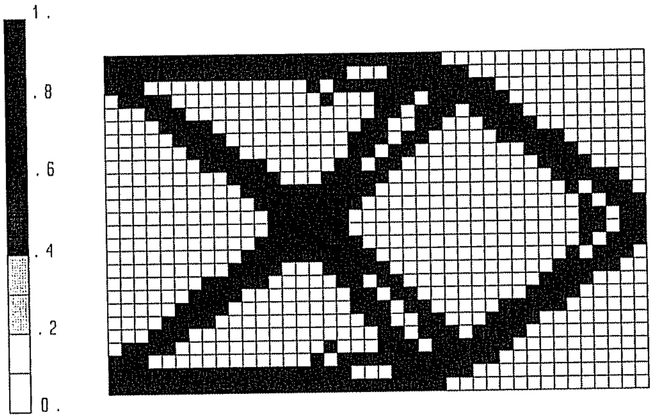

Fig. 2. Optimal distribution of material

The finite element analysis and the sensitivity computation are realized with the SAMCEF package. The resolution of the convex optimization subproblems is let to the CONLIN optimizer (Fleury, 1989b). The comparison of the three approaches is given at fig. 3. (compliance history) and fig. 4 (evolution of the root mean square of the modification of the design variables between two iterations).

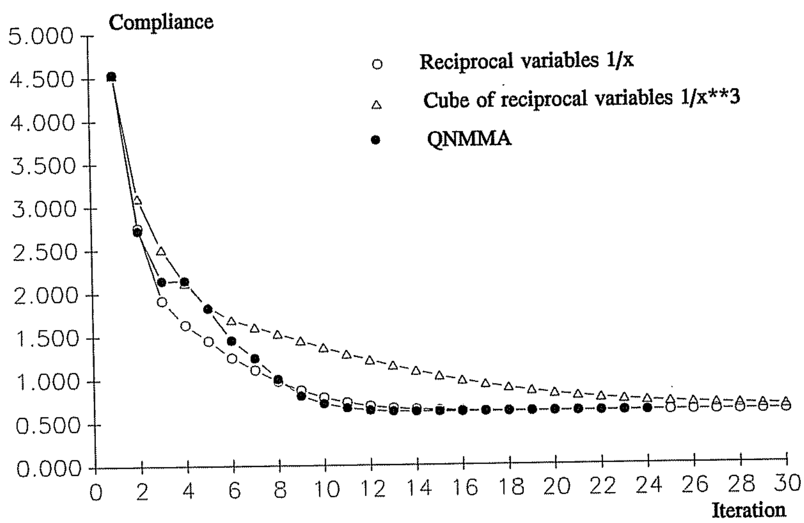

Fig. 3. Convergence histories

With the QNMMA approximation, the objective function is completely stationary after 26 iterations. At this moment, no design modifications can be observed, whereas the convergence of the other first order schemes is not achieved yet. CONLIN's final convergence is much slower. Expansion in terms of the cube of reciprocal variables is too conservative; the convergence is too slow and is far from being achieved.

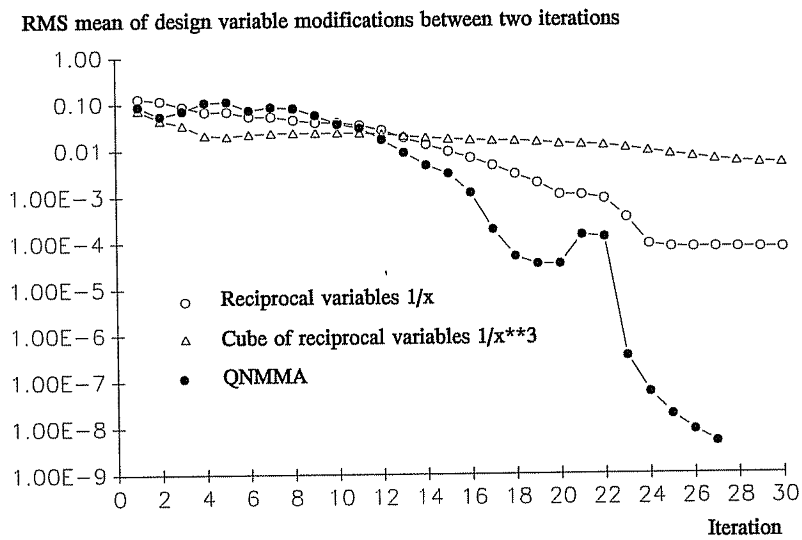

Fig. 4. Design variable modification history

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A new separable approximation scheme for topological problems and optimization problems characterized by a large number of design variables

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1. Introduction and definition of the problem of interest

The present work is concerned with the solution aspects of large scale optimization problems. Such problems have particularly received great attention since the emergence of topological optimization. Determination of the topology of structural component can be achieved through a material distribution problem over a design domain. The main difficulty inherent to that formulation is the very large number of design variables introduced by the material distribution. The earliest works such as Bendsøe and Kikuchi (Ref.1) used optimality criteria. Even if this technique was suited to problems with a single constraint and well-defined characteristics, the generalization of the topological problem under various design conditions can be facilitated by the use of mathematical programming techniques. One major feature of this procedure is the replacement of the real implicit optimization problems by an approximate optimization subproblem. For the resolution of these convex, explicit and separable subproblems, it is clear that dual methods are very well adapted, since they work in the dual space of smaller dimension. The other important aspect is concerned with the formulation of the involved explicit convex approximations of the structural constraints. The quality of the approximations has a great influence on the speed of convergence of candidate optimum designs. Because of the high cost of topological problems, the need for good approximations is very important. As it can be seen in numerous applications, first order schemes such as CONLIN[2] and MMA[3] exhibit a good convergence rate during the first iterations but slow down during the final convergence phase, while second order approximations[4] seem to be more sensitive to local optima and necessitate a very expensive second order sensitivity computation for a large number of design variables.

2. Convex approximations : a generalized version of the Method of Moving Asymptotes

We propose here to use a new structural approximation which is derived from the family of the method of moving asymptotes[3] (MMA). This new version is particularly well adapted to the solution of the optimization of material distribution and is based on an automatic selection algorithm of the moving asymptotes similar to the one described by Smaoui and Fleury[5] in the second order MMA scheme. The structural response $g(x)$ at the design point x^0 is replaced by its convex approximation $\tilde{g}(x)$. The parameters are adjusted to match the value of the function, its gradients as well as its diagonal curvatures.

$$\tilde{g}(x) = c_0 + \sum_{i=1}^{n} \frac{a_i}{x_i - b_i} \quad with \quad a_i = -(x_i^0 - b_i)^2 \frac{\partial g}{\partial x_i}(x^0) \quad and \quad b_i = x_i^0 + 2\frac{\partial g}{\partial x_i} / \max(\epsilon, \frac{\partial^2 g}{\partial x_i^2}) \quad (0<\epsilon<1) \tag{1}$$

Of course, the linear approximation, the reciprocal expansion, the CONLIN[2] or the pure MMA[3] approximations can be recovered by setting the b_i's to particular values (e.g. 0, $+\infty$, $-\infty$), or choosing special curvatures for the approximation. Nevertheless, despite their higher accuracy, the emergence of the second order approximations like Smaoui's scheme is slowed down because of the difficulty to have the second order sensitivity information. Particularly for a large number of design variables, the computation cost becomes very prohibitive. We propose here to avoid the direct second order derivatives evaluation by the use of a special Quasi-Newton procedure.

3. Diagonal Quasi-Newton updates

To avoid the second order sensitivity analysis, the idea is to use the available first order information and to build an approximation of the Hessian with a Quasi-Newton update procedure. Nevertheless, the full "Quasi-Newton" becomes also expensive when the number of design variables increases. On another hand, only diagonal terms are useful since one uses separable approximations for problems with a large number of design variables. Thus, we present here a modified BFGS update scheme able to generate a sequence of diagonal Hessian matrices. The algorithm is the adaptation[6] to diagonal matrices of more general results for Quasi-Newton updates[7] preserving sparse structure of the Hessian estimates. Let be B a diagonal approximation of the Hessian matrix of a given structural response at the current design point x. If the new design x^+ doesn't satisfy convergence criteria, one seeks to enrich the estimation of the Hessian with a Quasi-Newton update technique preserving the previous diagonal structure. The classic Broyden-Fletcher-Goldfard-Shanno (BFGS) update

$$\hat{B}^+ = B + U_{BFGS} \quad with \quad U_{BFGS} = \frac{yy^T}{s^T y} - \frac{Bss^TB}{s^TBs} \quad where \quad s = x^+ - x \quad and \quad y = \nabla g(x^+) - \nabla g(x) \tag{2}$$

satisfies simultaneously the symmetry and positive definiteness of the update and the Quasi-Newton condition $\hat{B}^+ s = y$, but it doesn't preserve sparse or diagonal structure of the previous estimate. Such an update can be found by adding a correction matrix E so that the diagonal updated matrix $B^+ = \hat{B}_D^+ + E$ be the closest, in the Frobenius norm, by the classic updated matrix $\hat{B}_D^+$ and still satisfies the Quasi-Newton condition $B^+ s = y$ too. If we note $\hat{B}_D^+$ and $\hat{B}_{ND}^+$ the matrices which are formed respectively with the diagonal and off-diagonal terms of $\hat{B}^+$, the correction matrix E that we look for is the solution of the minimum problem:

$$\begin{aligned} &\min \|E\|_F \\ &\text{s.t. } Es = \hat{B}_{ND}^+ s \\ &\quad E_{ij} = 0 \quad (i \neq j) \end{aligned} \quad \rightarrow \quad E = \text{diag}\{2\lambda_i s_i\} \tag{3}$$

where the vector $\bar{\lambda} = (\lambda_1, \ldots, \lambda_n)$ itself is solution of the linear system :

$$Q\lambda = \hat{B}^{+}_{ND}s = y - Bs - U_D s \qquad with \qquad Q = \mathrm{diag}\{s_i^2\} \tag{4}$$

The computation of the diagonal BFGS update requires solving this last diagonal system. Nonetheless, it is not necessary to compute off-diagonal terms of the classical update correction U_{ND}; only the diagonal terms U_D are necessary. This diagonal update is very inexpensive and requires much lower storage capacity than a "full" BFGS update since it needs only some vector manipulations.

4. Adapting the diagonal BFGS update to structural optimization

The diagonal BFGS update has to be adapted to structural problems to yield approximation schemes of high quality. Since the optimum has to be localized in few iterations, the sequence of estimations of curvatures must be quickly convergent despite the large number of variables. The efficiency of the procedure is based on the key role of reciprocal variables in structural design. It is well known that working in the space of reciprocal variables is favourable to reduce the curvature of structural constraints. So, it is better to update the Hessian matrix in the reciprocal design space. The choice of the initial Hessian estimate is very important too. The choice of curvatures of CONLIN[2] or pure MMA[3] approximations appears usually very relevant. Finally, to our experience, it doesn't worth forcing positive definite character of the updates which can slow down the convergence of the Hessian update and over-convexify the approximations.

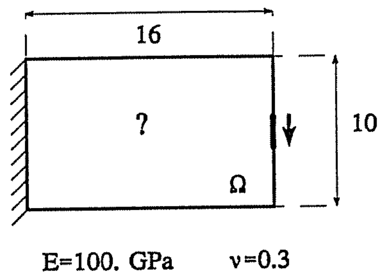

Fig.1 : Short cantilever beam

5. Applications : topological optimization of a short cantilever beam

Finally, we want to illustrate the new QNMMA approximation combining the diagonal Quasi-Newton update and the automatic choice of the moving asymptotes. The bench-mark we propose is a typical example of topological optimization: the well-known short cantilever beam problem (figures 1 and 2). The material law is simply given by a cubic relation between rigidity and density[8]. The problem is discretized by a regular mesh of 1040 finite elements of degree 2 and is solved with three different approximations for compliance (the volume is linearized). Since all the gradients of compliance are negative, CONLIN and the reciprocal variables expansion are the same. As this direct relaxation[8] of the topology problem is similar to a plate problem, we try also to expand compliance in terms of the same power of the reciprocal variables. Then, we use the new approximation QNMMA. The solution of the optimization problem is let to the CONLIN optimizer[9]. The comparison of the three approaches is given at figure 3 (compliance history) and figure 4 (evolution of the root mean square of the modification of the design variables between two iterations). With the QNMMA approximation, the objective function is completely stationary after 26 iterations. At this moment, no design modifications can be observed, whereas the convergence of the other first order schemes is not achieved yet. CONLIN's final convergence is much slower. Expansion in terms of the cube of reciprocal variables is too conservative; the convergence is too slow and is far from being achieved.

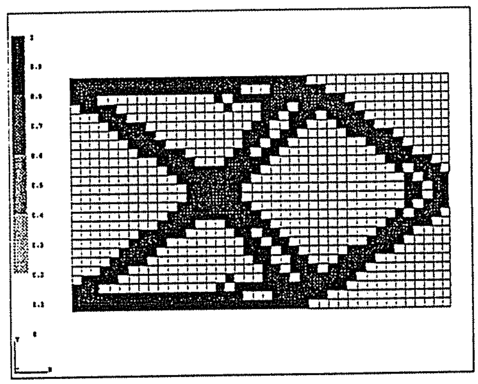

Fig.2 : Optimal distribution of material

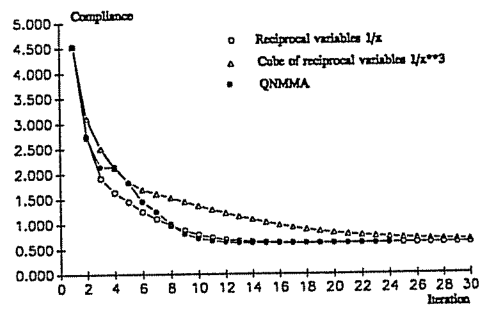

Fig. 3 : Convergence histories

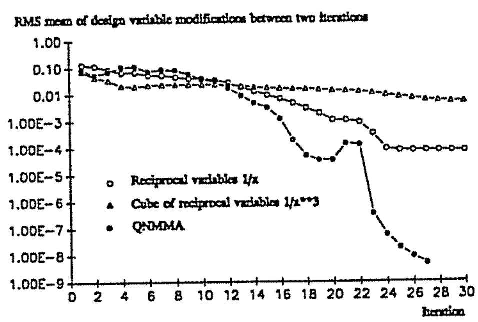

Fig. 4 : Design variable modification history

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