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Differential Equations of Stiffened Panels and Fourier Series Expansion

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ABSTRACT

The LBR-5 software is an integrated package to perform cost and/or weight optimization of stiffened ship structures. This paper describes the theoretical background of the LBR-5 software about the linear elastic analysis procedure. This method is based on an analytical solution of the stiffened panels governing equations using Fourier series expansion.

KEY WORDS: Stiffened panel; governing differential equation; analytical solution; Fourier series expansion; LBR-5 software, linear elastic analysis.

1. Introduction

This paper presents the theory used to implement in the LBR-5 software an analytical solution of the stiffened panels governing equations. For that purpose, Fourier series expansions are used to solve the governing differential equations. In the present analysis, cylindrical shells are used as the reference panels. Stiffened plates are considered as a simplified case of the more general cylindrical shell. In the LBR-5 software, plates are analyzed as being cylindrical shells having a very large radius ($q = 10^{10}$ m). The present method has been developed for fast and accurate linear elastic analysis of stiffened structures, particularly in regard to structural optimization.

Applications of the LBR-5 software to ship structures including its associated background about the scantling optimization procedure have been presented in various papers and conferences, Rigo (2001a,b,c, 2003), Rigo and Fleury (2001). The presented developments were initiated in Rigo (1989a) and the general methodology presented in Rigo (1989b, 1992a,b, 2004). After 15 years the LBR5 software is now well-established and patented; it is therefore relevant to publish the extensive theoretical background of this method.

2. Differential Equations of Cylindrical Stiffened Shells

Fig. 1 shows the coordinate system oxφ with z = 0 at mid plate thickness. The relation between the φ coordinate (used for shell) and the y coordinate (used for plate) is: $y = q\,\varphi$, with q the radius.

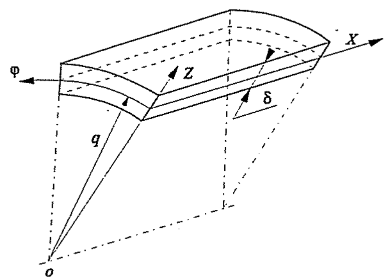

Fig. 1: Panel Coordinate System

Fig. 2 presents the stresses acting on a small volume element [dx, dz, (q+z)dφ]. In this study, the thin shell (plate) theory is used, i.e. τ_{xz}, $\tau_{\varphi z}$ and σ_z are not con-

sidered ($\varepsilon_z = \gamma_{xz} = \gamma_{\varphi z} = 0$).

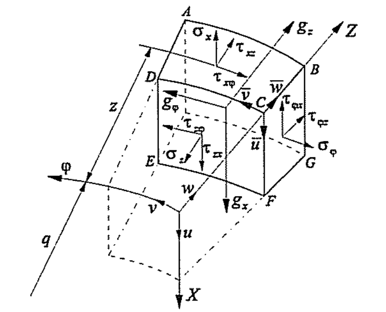

Fig. 2: Stresses Acting on a Small Volume Element

The governing differential equations, known as the D.K.J. differential equations (Donnell, von Karman and Jenkins), are based on the Love-Kirchoff hypotheses:

1. Thin shell theory, i.e. δ/q <<< 1. For LBR5, we impose that δ/q < 1/100.
2. Small deformation and linear analysis.
3. The points that are on a perpendicular line to the mid plate surface (z = 0) before deformation remain on the same perpendicular after deformation, thus γ_{xz} and $\gamma_{\varphi z} = 0$.
4. σ_z and its effects are negligible.
5. No deformation along oz ($\varepsilon_z = 0$).

Let us denote partial derivatives as follows:

$$f' = \frac{\partial f}{\partial x} \quad \text{and} \quad f^{o} = \frac{\partial f}{\partial y} = \frac{1}{q}\frac{\partial f}{\partial \varphi} \qquad (1)$$

Then the linear 'deformation-displacement' relations for a shell are:

$$\varepsilon_x = u' - zw''$$
$$\varepsilon_\varphi = v^{o} + \frac{w}{q} - zw^{oo} \qquad (2)$$
$$\gamma_{x\varphi} = u^{o} + v' - 2zw^{o\prime}$$

and the 'stress-displacement' relations are:

$$\sigma_x = \frac{E}{1-\upsilon^2}\left[u' + \upsilon\left(v^{o} + \frac{w}{q}\right) - z\left(w'' + \upsilon w^{oo}\right)\right]$$
$$\sigma_\varphi = \frac{E}{1-\upsilon^2}\left[\left(v^{o} + \frac{w}{q}\right) + \upsilon u' - z\left(w^{oo} + \upsilon w''\right)\right] \qquad (3)$$
$$\tau_{x\varphi} = G\left(u^{o} + v' - 2zw^{o\prime}\right)$$

E is the Young modulus, υ the Poisson coefficient, $G = E/[2(1+\upsilon)]$ the shear modulus. The special case of a plate is derived by simply setting w/q = 0 in Eqs. 2~3.

Fig. 3 shows the internal resultant forces N_x, Q_x, $N_{x\varphi}$, N_φ, Q_φ, $N_{\varphi x}$, and moments M_x, $M_{x\varphi}$, M_φ, $M_{\varphi x}$ that are applied on an elementary cylindrical shell (plate), hereafter called resultants. This element is included between the upper surface (z = δ/2) and the lower surface (z = -δ/2) and has a surface dimension of $dx \cdot q d\varphi$ (or $dx \cdot dy$). With reference to the thin shell element (Fig. 3), we can establish the 'resultant-stress' relationships (Eqs. 4). These resultant forces and moments are referenced to the plate neutral axis (z = 0).

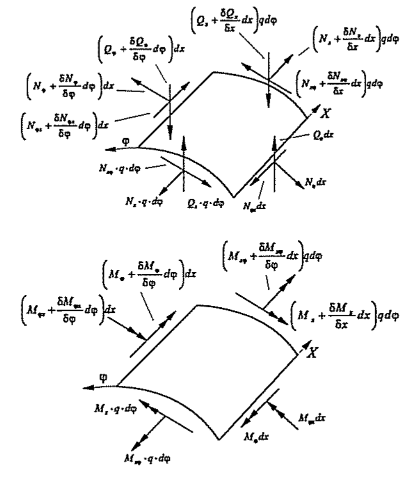

Fig. 3: Resultant Forces and Moments

$$N_\varphi = \int_{-\delta/2}^{+\delta/2} \sigma_\varphi \, dz \qquad N_x = \int_{-\delta/2}^{+\delta/2} \sigma_x \left(1+\frac{z}{q}\right) dz$$
$$M_\varphi = \int_{-\delta/2}^{+\delta/2} \sigma_\varphi \, z\, dz \qquad M_x = \int_{-\delta/2}^{+\delta/2} \sigma_x \left(1+\frac{z}{q}\right) z\, dz$$
$$N_{\varphi x} = \int_{-\delta/2}^{+\delta/2} \tau_{\varphi x} \, dz \qquad N_{x\varphi} = \int_{-\delta/2}^{+\delta/2} \tau_{x\varphi} \left(1+\frac{z}{q}\right) dz \qquad (4)$$
$$M_{\varphi x} = \int_{-\delta/2}^{+\delta/2} \tau_{\varphi x} \, z\, dz \qquad M_{x\varphi} = \int_{-\delta/2}^{+\delta/2} \tau_{x\varphi} \left(1+\frac{z}{q}\right) z\, dz$$

Q_x and Q_φ (transverse shear resultant) cannot be calculated by integration of the τ_{xz} and $\tau_{\varphi z}$ stresses as these shear stresses are assumed to be equal to 0 (thin plate assumption). Nevertheless, Q_x and Q_φ can be evaluated using the 4th and the 5th equilibrium equations (Eqs. 9).

If we replace the 'stress-displacement' relationships (Eqs. 3) within the 'resultant-stress' relationships

(Eqs. 4), we obtain the 'resultant-displacement' relationships:

$$N_\varphi = D(v^o + \frac{w}{q} + \upsilon u')$$
$$N_x = D(u' + \upsilon v^o + \upsilon \frac{w}{q})$$
$$N_{\varphi x} = N_{x\varphi} = D\frac{1-v}{2}(v' + u^o) \qquad (5)$$
$$M_\varphi = K(w^{oo} + \upsilon w'')$$
$$M_x = K(w'' + \upsilon w^{oo})$$
$$M_{\varphi x} = M_{x\varphi} = K(1-\upsilon)w^{o\prime}$$

with $D = E\delta / (1 - \upsilon^2)$ and $K = E\delta^3 / [12 (1 - \upsilon^2)]$

Fig. 4 shows the additional resultant forces and moments acting on the shell (plate) coming from a stiffener oriented along ox ($N_{x,Conc}$, $M_{x,Conc}$, $N_{xy,Conc}$, $Q_{x,Conc}$ and $M_{xy,Conc}$). For a transverse member (frame) oriented along oφ the additional resultant forces and moments are ($N_{y,Conc}$, $M_{y,Conc}$, $N_{yx,Conc}$, $Q_{y,Conc}$ and $M_{yx,Conc}$).

If we consider a stiffened thin shell element, the 'resultant-stress' relationships including the plate and the stiffener components become:

$$N_\varphi = \int_{-\delta/2}^{+\delta/2} \sigma_\varphi \, dz + f(x) \int_{\omega_\varphi} \sigma_\varphi \frac{e_\varphi}{d_\varphi} dz$$
$$N_x = \int_{-\delta/2}^{+\delta/2} \sigma_x (1 + \frac{z}{q}) dz + f(\varphi) \int_{\omega_x} \sigma_x \frac{e_x}{d_x} dz \qquad (6)$$
$$M_\varphi = \int_{-\delta/2}^{+\delta/2} \sigma_\varphi z \, dz + f(x) \int_{\omega_\varphi} \sigma_\varphi z \frac{e_\varphi}{d_\varphi} dz$$
$$M_x = \int_{-\delta/2}^{+\delta/2} \sigma_x z (1 + \frac{z}{q}) dz + f(\varphi) \int_{\omega_x} \sigma_x z \frac{e_x}{d_x} dz$$

with ω_φ and ω_x the cross section of, respectively, the frames and the stiffeners. e_x, e_φ, d_x and d_φ follow from Fig. 5. The second term of each equation in Eqs. 6 corresponds to the stiffeners $f(\varphi)$ [frames $f(x)$].

Eqs. 4 (unstiffened element) are a simplified form of Eqs. 6 (stiffened element). Their first terms are identical. For a stiffened element, Eqs. 6 include Heaviside functions [$f(x)$ and $f(\varphi)$] that are equal to zero except at the stiffener locations, $b - d/2 < x < b + d/2$, where they are equal to 1 (Fig. 5).

$$f(x) = H[x - (b - d/2)] - H[x - (b + d/2)] \qquad (7)$$

with $H(x) = 0$ if $x < 0$ and $H(x) = 1$ if $x > 0$

If we replace the 'stress-displacement' relationships (Eqs. 3) in the 'resultant-stress' relationships (Eqs. 6), we obtain the 'resultant-displacement' relationships for stiffened panels (including stiffeners and frames) (Eqs. 8).

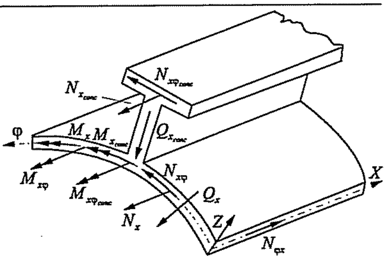

Fig. 4: Resultant Forces and Moments Acting on the Shell (Plate) Due to a Stiffener

$$N_\varphi = D(v^o + \frac{w}{q} + \upsilon u') + f(x)\frac{E}{d_\varphi}\left[(v^o + \frac{w}{q})\omega_\varphi - w^{oo} h_\varphi\right]$$
$$M_\varphi = K(w^{oo} + \upsilon w'') + f(x)\frac{E}{d_\varphi}\left[(v^o + \frac{w}{q})h_\varphi - w^{oo} I_\varphi\right]$$
$$M_{\varphi x} = K(1-\upsilon)w^{o\prime} + f(x)\frac{G}{d_\varphi}\left[K_\varphi w^{o\prime} + \lambda_\varphi \Omega_\varphi'(v' + u^o)\right]$$
$$N_x = D(u' + \upsilon v^o + \upsilon\frac{w}{q}) + f(\varphi)\frac{E}{d_x}(u'\omega_x - w'' h_x) \qquad (8)$$
$$M_x = K(w'' + \upsilon w^{oo}) + f(\varphi)\frac{E}{d_x}(u' h_x - w'' I_x)$$
$$M_{x\varphi} = K(1-\upsilon)w^{o\prime} + f(\varphi)\frac{G}{d_x}\left[K_x w^{o\prime} + \lambda_x \Omega_x'(v' + u^o)\right]$$
$$N_{\varphi x} = N_{x\varphi} = D(\frac{1-\upsilon}{2})(v' + u^o) + f(x)\frac{G}{d_\varphi}\Omega_\varphi'(v' + u^o) + f(\varphi)\frac{G}{d_x}\Omega_x'(v' + u^o)$$

with ω_x, ω_φ transversal section of a stiffener (frame) without plating

h_x, h_φ 1st sectional moment of ω_x (ω_φ) relative to the plate neutral axis $z = 0$

I_x, I_φ 2nd sectional moment of ω_x (ω_φ) relative to the plate neutral axis $z = 0$

K_x, K_φ torsional rigidity of a stiffener (frame)

Ω_x', Ω_φ' reduced flange section (for flange in-plane shear contribution)

λ_x, λ_φ flange eccentricity to the plate neutral axis $z = 0$

d_x, d_φ effective strip width of longitudinal stiffeners (frames)

A major uncertainty related to the validity of these equations concerns the flange contribution to the in-plane shear effects. In the following developments these contributions are not considered. Ω_x', Ω_φ' are thus set to zero in Eqs. 8. This is a conservative assumption.

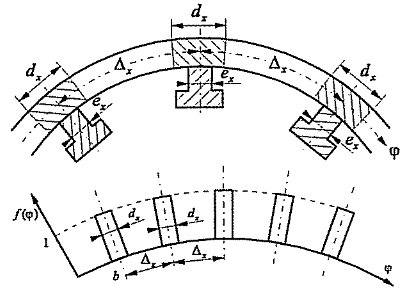

Fig. 5: Stiffener Spacing and Heaviside Function

To summarize, the D.K.J. governing differential equations of a cylindrical shell (plate) are obtained by using:

- The six equilibrium equations (Fig. 3):

$$N_x' + N_{\varphi x}^o + X = 0$$
$$N_\varphi^o + N_{x\varphi}' - \frac{Q_\varphi}{q} + Y = 0$$
$$\frac{N_\varphi}{q} + Q_\varphi^o + Q_x' - Z = 0 \qquad (9)$$
$$M_\varphi^o + M_{x\varphi}' - Q_\varphi = 0$$
$$M_x' + M_{\varphi x}^o - Q_x = 0$$
$$N_{x\varphi} - N_{\varphi x} + \frac{M_{\varphi x}}{q} = 0$$

The special case of a plate follows by omitting all terms containing q in Eqs. 9.

- The five hypotheses of linear thin shell theory (see above)
- The 'stress-displacement' relationships (Eqs. 3) and the 'resultant-stress' relationships (Eqs. 6) to establish the 'resultant-displacement' relationships (Eqs. 8).

The problem is composed of 13 unknowns:

- u, v, w
- N_x, M_x, Q_x, $N_{x\varphi}$, $M_{x\varphi}$
- N_φ, M_φ, Q_φ, $N_{\varphi x}$, $M_{\varphi x}$

and there are 13 available equations :

- 4 'resultant-displacement' relations corresponding to N_x, M_x, $N_{x\varphi}$, $M_{x\varphi}$ (Eqs. 8) (there is no available equation for Q_x)
- 4 'resultant-displacement' relations corresponding to N_φ, M_φ, $N_{\varphi x}$, $M_{\varphi x}$ (Eqs. 8) (there is no available equation for Q_φ)
- 5 equilibrium equations (Eqs. 9) (the last equilibrium equation has already been used and cannot be considered).

Replacing the 'resultant-displacement' relations (Eqs. 8) in the 5 first equilibrium equations (Eqs. 9) and replacing Q_x and Q_φ (4th and 5th equations) in the 3rd equilibrium equation, we obtain the three governing differential equations in u, v and w. This is a system of three differential equations.

$$D(u'' + \upsilon v^{o\prime} + \frac{\upsilon w'}{q}) + D\left(\frac{1-\upsilon}{2}\right)(u^{oo} + v^{o\prime}) + f(\varphi)\left[\Omega_x u'' - H_x w'''\right] + X = 0$$
$$D(v^{oo} + \frac{w^o}{q} + \upsilon u^{o\prime}) + D\left(\frac{1-\upsilon}{2}\right)(u^{o\prime} + v'') + f(x)\left[\Omega_\varphi(v^{oo} + w^o) - H_\varphi w^{ooo}\right] + Y = 0 \qquad (10)$$
$$\frac{D}{q}(v^o + \frac{w}{q} + \upsilon u') + K w^{oooo} + 2K w^{oo\prime\prime} + K w'''' + f(x)\left[\frac{\Omega_\varphi}{q}(v^o + \frac{w}{q}) - H_\varphi(\frac{2w^{oo}}{q} + v^{ooo}) + R_\varphi w^{oooo} + T_\varphi w^{oo\prime\prime}\right] + f(\varphi)\left[-H_x u''' + R_x w'''' + T_x w^{oo\prime\prime}\right] + f'(x)\left[T_\varphi w^{oo\prime}\right] + f^o(\varphi)\left[T_x w^{o\prime\prime}\right] - Z = 0$$

with:

$$\Omega_\varphi = \frac{E\omega_\varphi}{d_\varphi};\ \Omega_x = \frac{E\omega_x}{d_x};\ H_\varphi = \frac{Eh_\varphi}{d_\varphi};\ H_x = \frac{Eh_x}{d_x}$$
$$R_\varphi = \frac{EI_\varphi}{d_\varphi};\ R_x = \frac{EI_x}{d_x};\ T_\varphi = \frac{GK_\varphi}{d_\varphi};\ T_x = \frac{GK_x}{d_x}$$

3. Stiffened Cylindrical Shell (Plate) Elements

Fig. 6 shows a stiffened cylindrical shell element composed of a plating (δ) and two layers of stiffeners. Shells are the generic elements and plates are analyzed as a particular case. Typically the layer along oφ corresponds to the transverse frames and along ox to the longitudinal stiffeners. We assume that all stiffeners and frames belonging to the same panel are identical and have the same spacing.

Note: In the LBR-5 software, the typical stiffened cylindrical shell element is composed of a third layer along ox corresponding to the girders (larger than the stiffeners). Contrary to the stiffeners, they can differ and their spacing is free. Details about girders are available in Rigo (2004).

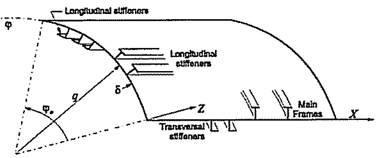

Fig. 6: Stiffened Cylindrical Shell Element

3.1 Resultant Forces and Moments of Two-Layered Stiffened Panels

Eqs. 11 give the resultant forces and moments of two-layered stiffened panel:

$$N_\varphi = (D+\Omega_\varphi)\left(v^o+\frac{w}{q}\right)+D\upsilon u' - H_\varphi w^{oo}$$

$$N_x = (D+\Omega_x)u' + D\upsilon\left(v^o+\frac{w}{q}\right) - H_x w''$$

$$M_\varphi = (K+R_\varphi)w^{oo} + K\upsilon w'' - H_\varphi\left(\frac{w}{q}+v^o\right)$$

$$M_x = (K+R_x)w'' + K\upsilon w^{oo} - H_x u'$$

$$M_{\varphi x} = [K(1-\upsilon)+T_\varphi]w^{o\prime}$$

$$M_{x\varphi} = [K(1-\upsilon)+T_x]w^{o\prime} \quad (11)$$

$$N_{\varphi x} = D\left(\frac{1-\upsilon}{2}\right)(v'+u^o)$$

$$N_{x\varphi} = N_{\varphi x}$$

$$Q_\varphi = (K+T_x)w^{o\prime\prime} + (K+R_\varphi)w^{ooo} - H_\varphi\left(\frac{w^o}{q}+v^{oo}\right)$$

$$Q_x = (K+T_\varphi)w^{oo\prime} + (K+R_x)w''' - H_x u''$$

with

$$\Omega_\varphi = \frac{E\omega_\varphi}{\Delta_\varphi};\ \Omega_x = \frac{E\omega_x}{\Delta_x};\ H_\varphi = \frac{Eh_\varphi}{\Delta_\varphi};\ H_x = \frac{Eh_x}{\Delta_x}$$

$$R_\varphi = \frac{EI_\varphi}{\Delta_\varphi};\ R_x = \frac{EI_x}{\Delta_x};\ T_\varphi = \frac{GK_\varphi}{\Delta_\varphi};\ T_x = \frac{GK_x}{\Delta_x}$$

and Δ_x, Δ_φ the spacing between longitudinal stiffeners resp. frames.

Eqs. 11 include different components:

- Plate components (D, K),
- Stiffener components (Ω_x, R_x, S_x, H_x, T_x, L_x) and
- Frame components (Ω_φ, R_φ, S_φ, H_φ, T_φ, L_φ).

For the stiffeners, the f(φ) Heaviside function is replaced by d_x/Δ_x. For the frames, f(x) is replaced by d_φ/Δ_φ (Fig. 7). This standardization does not mean that the stiffeners (frames) are smeared and replaced by an equivalent plate thickness but it means that each individual characteristic (cross section, first sectional moment, inertia moment, torsional rigidity,...) is standardized on the entire plate. Globally, the stiffened panel behaviour is accurately modelled but it is locally simplified. This simplification is only valid if the spacing between stiffeners (frames) is constant and remains small (compared to their span).

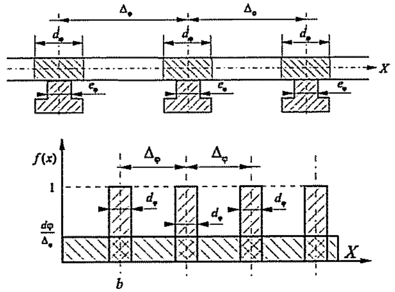

Fig. 7: Uniformly Distributed Frames:

$$f(x) = \frac{d_\varphi}{\Delta_\varphi} = \text{const.}$$

Based on the equilibrium equations (Eqs. 9) and the 'resultant-displacement' relationships (Eqs. 11), three governing differential equations are obtained:

$$(D+\Omega_x)u'' + D\left(\frac{1-\upsilon}{2}\right)u^{oo} + D\left(\frac{1+\upsilon}{2}\right)v^{o\prime} - H_x w''' + \frac{D\upsilon}{q}w' + X = 0$$

$$(D+\Omega_\varphi)v^{oo} + D\left(\frac{1+\upsilon}{2}\right)u^{o\prime} + D\left(\frac{1-\upsilon}{2}\right)v'' - H_\varphi w^{ooo} + \frac{1}{q}(D+\Omega_\varphi)w^o + Y = 0 \quad (12)$$

$$-H_x u''' + \frac{D\upsilon}{q}u' + \frac{1}{q}(D+\Omega_\varphi)v^o - H_\varphi v^{ooo} + \frac{1}{q^2}(D+\Omega_\varphi)w + (K+R_\varphi)w^{oooo} + (2K+T_\varphi+T_x)w^{oo\prime\prime} + (K+R_x)w'''' - \frac{2H_\varphi}{q}w^{oo} - Z = 0$$

4. Analytical Solution for the Governing Equations of Stiffened Panels

Only for the unstiffened plate u and v (in-plane displacements) are not coupled with w (transversal displacements) within linear thin plate theory. In all other cases, u, v and w are coupled and the three equations for them (Eqs. 10 or 12) have to be solved simultaneously. The principle to solve any of these three governing differential equations is the same. They can be written as:

$$a_1u + b_1v + c_1w = +X(x,\varphi)$$
$$a_2u + b_2v + c_2w = +Y(x,\varphi) \quad (13)$$
$$a_3u + b_3v + c_3w = -Z(x,\varphi)$$

with

- u(x,φ), v(x,φ) and w(x,φ) the displacements.
- x and φ are the coordinates of a point on the mid-plane of the cylindrical shell (plate). The z coordinate does not appear as we only look for the displacements (u,v,w) at the mid-plate thickness where z = 0 (linear thin shell theory).
- X, Y and Z(x,φ) are the surface loads.
- a_1, b_1, ..., c_3 are the derivative operators. E.g. for the system of Eqs. 10 we have:

$$a_1 = D\frac{\partial}{\partial x} + D\left(\frac{1-\upsilon}{2}\right)\frac{\partial}{\partial y}$$

4.1 Homogeneous Solution (or Complementary Solution)

The homogeneous solution of the governing differential equations (Eqs. 13) yields:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0 \quad \text{or} \quad (14)$$

$$a_1(b_2c_3 + b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1) = 0$$

If we apply this operator (Eq. 14) to the w(x,φ) displacement, we obtain:

$$Aw_{80} + Bw_{60} + Cw_{62} + Dw_{40} + Ew_{42} + \ldots + Jw_{26} + Kw_{08} = 0 \quad (15)$$

This is an 8^{th} order differential equation with two coupled variables x and φ. w_{ij} denotes the i^{th} order derivative of w by x and j^{th} order derivative by y (y = qφ). E.g. $w_{13} = w'^{ooo}$.

4.2 Fourier Series Expansions

To solve this 8^{th} order differential equation we have to make an assumption on the shape of the displacements u, v, w to obtain an 8^{th} order differential equation with two separate variables:

$$w(x,\varphi) = w_1(\varphi) \cdot w_2(x) \quad (16)$$

We use the Fourier series expansion theory and assume:

$$u(x,\varphi) = u(\varphi) \cdot \cos(\lambda x)$$
$$v(x,\varphi) = v(\varphi) \cdot \sin(\lambda x) \quad (17)$$
$$w(x,\varphi) = w(\varphi) \cdot \sin(\lambda x)$$

with $\lambda = n\pi / L$, n the term number of the Fourier series expansion, and L the span of the structure (and panels) along ox. L is the same for each panel.

The shape of the assumed displacements imposes some limitations on the boundary conditions. The two edges x = 0 and x = L must behave as simply supported edges, i.e. $w = v = M_x = N_x = 0$ (Fig. 8).

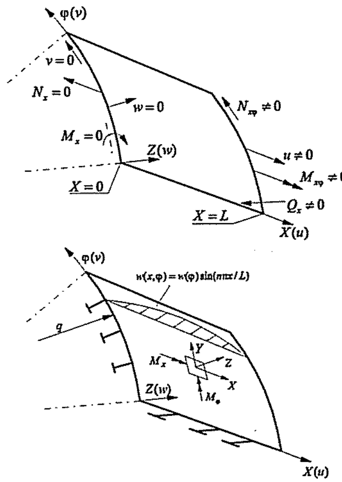

Fig. 8: Fourier Series Expansion and Boundary Conditions

Inserting Eqs. 17 in the one of the considered governing differential equations (Eqs. 10 or 12) yields an 8^{th} order polynomial differential equation with now only one variable φ.

4.3 Loads' Fourier Series Expansions

Having decided to expand the displacements using Fourier series (Section 4.2) to solve the governing differential equations means that the Z(x,φ) loads (Eqs.13) have to also satisfy the Fourier series expansion's shapes:

$$Z(x,\varphi) = Z^*(\varphi) \cdot Q(x) = Z^*(\varphi) \cdot \Sigma\, a \sin(\lambda x) \quad (18)$$

The way to implement the actual loads in the analytical procedure is explained in Section 4.5.

Presented here are the sine and cosine Fourier series expansions of a Q(x) generic load, which consists of a uniform load Q_0 between x_1 and x_2, and zero elsewhere.

For a sine expansion

$$Q(x) = \sum_{n=1}^{\infty}\left[\frac{4Q}{n\pi}\sin\left[\frac{n\pi}{2L}(x_1+x_2)\right] \cdot \sin\left[\frac{n\pi}{2L}(x_2-x_1)\right]\right] \cdot \sin\frac{n\pi x}{L} \quad (19)$$

For a cosine expansion

$$Q(x)=\sum_{n=1}^{\infty}\left[\begin{array}{c}\frac{4Q}{n\pi}\cos\left[\frac{n\pi}{2L}(x_1+x_2)\right]\\ \sin\left[\frac{n\pi}{2L}(x_2-x_1)\right]\end{array}\right].\cos\frac{n\pi x}{L} \quad (20)$$

Hydrostatic pressure is usually uniformly distributed along ox and varies linearly along oφ. The variation along oφ is considered in Section 4.5. The expansion along ox for such a symmetric load uses only the odd terms in a sine series:

$$Q(x)=\sum_{n=1}^{\infty}\frac{4Q}{(2n-1)\pi}.\sin\frac{(2n-1)\pi x}{L} \quad (21)$$

In practice, the first three terms of the series are enough to model such loads with sufficient accuracy.

Cargo loads and weight distribution can be approximated by step functions. In such cases, 7 to 13 terms usually suffice to model the loads with sufficient accuracy.

To model the primary bending moment, it is necessary to apply axial longitudinal loads at the both ends of each panel. As concentrated loads cannot be expanded with the Fourier series, these end loads are applied on a small zone on each side. The width of these zones is taken in LBR-5 as 1/20 of the span L. This is a compromise between computational effort and accuracy. For such expansions, cosine Fourier series are used truncated typically after 7 to 13 terms.

4.4 Homogeneous Solution of Differential Equations

From the solution of the 8th order polynomial differential equation with a single variable φ (Eq. 15) and keeping in mind that w(x,φ) = w(φ) . sin (λx) (Eqs. 17) we obtain:

$$w(x,\varphi)=\left[\begin{array}{l}e^{\alpha_i q\varphi}(A_i\cos\beta_i q\varphi+B_i\sin\beta_i q\varphi)\\ +e^{\alpha_i q(\varphi_0-\varphi)}.(C_i\cos\beta_i q(2\pi-\varphi)\\ \qquad +D_i\sin\beta_i q(2\pi-\varphi))\\ +\ldots\end{array}\right]\cdot\sin\lambda x \quad (22)$$

Table 1: Values of index 'i'

If	then i	i.e.
β_1 & $\beta_2 \neq 0$	1 to 2	$(\alpha_1, \pm\beta_1)$, $(\alpha_2, \pm\beta_2)$ 2 complex solutions
$\beta_1 \neq 0$ & $\beta_2 = 0$	1 to 3	$(\alpha_1, \pm\beta_1)$, $(\alpha_2, 0)$, $(\alpha_3, 0)$ 1 complex, 2 real sol.
$\beta_1 = 0$ & $\beta_2 \neq 0$	1 to 3	$(\alpha_1, \pm\beta_1)$, $(\alpha_2, 0)$, $(\alpha_3, 0)$ 1 complex, 2 real sol.
$\beta_1 = \beta_2 = 0$	1 to 4	$(\alpha_1,0)$,$(\alpha_2,0)$,$(\alpha_3,0)$,$(\alpha_4,0)$ 4 real solutions

A_i, B_i, C_i, D_i are the eight integration constants included in Eq. 22. These constants are determined through the boundary conditions (Section 4.6).

For u(φ) and v(φ) similar equations can be written.

The u(φ) and v(φ) equations contain other integration constants that depend directly on the eight integration constants of w (A_i, B_i, C_i, D_i). This means that once these eight constants are fixed for w, the equations for u and v are also completely defined. In addition, using the 'resultant-displacement' relationships (like Eqs. 11) the resultant and displacement derivatives (e.g. w° = slope) are also known. These will be required later (Section 4.6) to find the boundary forces to apply along the panel boundary edges (φ = 0 and φ = φ_0).

Eq. 22 for w(φ) and those for u(φ) and v(φ) are the 'homogeneous solution' of the differential equations. The 'homogeneous solution' is our basic solution to determine the solution of the actual panel.

4.5 Superposition Principle

At this stage it is valuable to resume briefly the general philosophy to solve analytically the governing differential equations of structures composed of cylindrical stiffened shells (plates):

1. We decompose (mesh modelling) the global structure in a series of stiffened cylindrical shells and stiffened plates (Fig. 9).

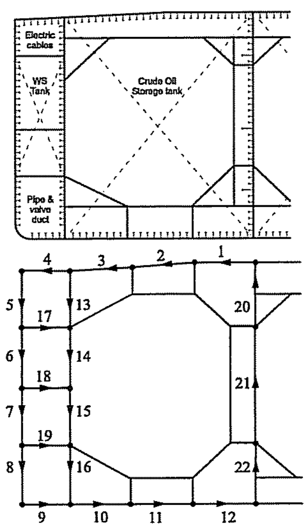

Fig. 9: Modelling of the Structure with Stiffened Panels

2. Using the displacement shape of the Fourier series expansion, we solve for each panel the governing differential equations without second member (homogeneous solution). For each panel, Eq. 22 gives the homogeneous solution, which includes the eight unknown integration constants. This procedure is

repeated for each term of the Fourier series expansion. At the end, the superposition principle is applied by summing all the solutions (one per term) to get the actual solution. The number of terms to use depends on the problem's complexity in terms of the load patterns. Usually 3 to 13 terms are required. Each panel (cylindrical shell) is considered as a complete 360° cylinder (i.e. the shell opening angle is 360°). The actual opening angle φ_0 will be considered later.

3. Definition of the four 'basic unitary load lines': X_u, Y_u, Z_u, M_u. The principle is to find the eight integration constants for the four 'basic unitary load lines' applied on the complete cylinder. Four sets of integration constants are determined (one per unitary load line). The superposition principle allows then to find the solution (u,v,w) for the actual stiffened panels (actual opening angle φ_0 and loads) that compose the structure. The four 'basic unitary load lines' applied on the complete cylinder are:

$$\begin{array}{lll} X_u = 10000\cos(\lambda x) & \text{(N/m)} & \\ Y_u = 10000\sin(\lambda x) & \text{(N/m)} & (23)\\ Z_u = 10000\sin(\lambda x) & \text{(N/m)} & \\ M_u = 10000\sin(\lambda x) & \text{(Nm/m)} & \end{array}$$

Their forms are compatible with the Fourier series expansions of the actual loads. These unitary load lines are applied at φ = 0 (and φ = 360°).

For each of these 'unitary load lines', eight integration constants are obtained through the boundary conditions at φ = 0 and φ = 360°. To satisfy the boundary conditions, we can define four equations (equilibrium and/or compatibility). In addition, the symmetry or the anti-symmetry of the resultants and displacements induced by the load line provides four other equations. E.g. for the Z_u load case, the conditions are, Fig. 10:

$$\begin{array}{llll} v & = 0.0 & \sin(\lambda x) & \text{in } \varphi = 0 \text{ (per symmetry)}\\ N_{\varphi x} & = 0.0 & \cos(\lambda x) & \text{in } \varphi = 0 \text{ (per symmetry)}\\ w^{\circ} & = 0.0 & \sin(\lambda x) & \text{in } \varphi = 0 \text{ (per symmetry)}\\ R_{\varphi} & = -5000 & \sin(\lambda x) & \text{in } \varphi = +\varepsilon \end{array}$$

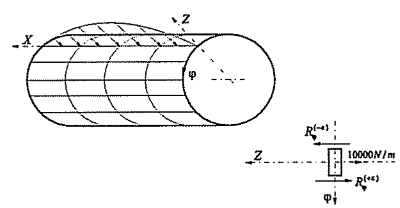

Fig. 10: Z_u = 10000 sin (λx) (N/m) Basic Unitary Load Line

Lateral pressure (varying along oφ), the deadweight, and the longitudinal axial compression (induced by the primary bending moment) can also be considered using the basic unitary load lines. The unitary load lines are assumed to be applied on a small surface (L.dy or L.qdφ) at z = 0. Integrating the solutions obtained for the basic load lines according to the actual load distribution, we get the solutions (u, v and w) for a complete cylinder under the real load conditions.

4.6 Actual Panels

In order to get the solution of the real panel (for the actual shell opening angle φ_0) we have to consider the actual boundary conditions imposed along the two longitudinal edges (φ = 0 and φ = φ_0). To satisfy these boundary conditions, we apply along each edge a set of four basic load lines (X_u, Y_u, Z_u and M_u). The problem is to find the amplitude of these load lines. For each panel, the unknowns are the 'edge amplification factors' of these load lines. Conditions to determine these 'edge amplification factors' are:

- For a free edge: $M_\varphi = N_\varphi = N_{x\varphi} = R_\varphi = 0$
- For a clamped edge: w = v = u = dw/dy = 0
- For a simply supported edge: $w = u = M_\varphi = N_\varphi = 0$
- For an edge (node) corresponding to the junction between two panels, we impose four compatibility conditions between the displacements of the two panels and four equilibrium equations.
- For an edge (node) corresponding to the junction between three panels, we impose eight compatibility conditions between the displacements of the three panels and four equilibrium equations.

The 'edge amplification factors' for all panels are determined at the final stage (Section 4.7). For a structure with N panels, there are 8N unknowns corresponding to the eight 'edge amplification factors' per panel. They are determined by solving a system of 8N linear equations.

The equations (compatibility or equilibrium) at the panel edges require the displacements (u, v, w, w°) and the resultants (M_φ, N_φ, $N_{x\varphi}$, R_φ) acting along the edge φ = 0 and the edge φ = φ_0. These are determined for the nine 'standard loading cases', Fig. 11:

- The actual external loads:
  - o pressures (quasi-static): Z type
  - o gravity loads (deadweight, cargo, ...) having component along oφ and along oz: Y and Z types
  - o axial compression (induced by the primary bending moment) : X type
- The four basic unitary load lines (X_u, Y_u, Z_u and M_u) acting at φ = 0
- The four basic unitary load lines (X_u, Y_u, Z_u and M_u) acting at φ = φ_0

All these displacements and forces are calculated from the solutions of the homogeneous differential equations for the four basic load lines applied on the 360° cylinder.

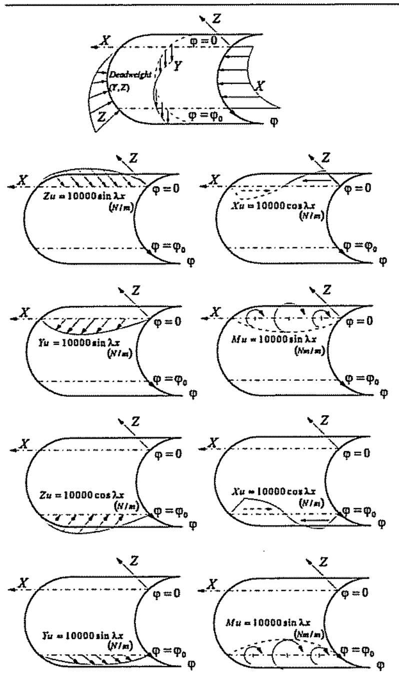

Fig. 11: The Nine 'Standard Loading Cases' (Applied on the 360° Cylinder)

4.7 Final Solution

At the final stage, each panel is a 360° cylindrical shell including stiffener and frame contributions.

For these panels we know the displacements (u, v, w, w°) and the resultants (M_φ, N_φ, $N_{x\varphi}$, R_φ) along their two boundary edges ($\varphi = 0$ and $\varphi = \varphi_0$) for the nine 'standard loading cases'. To satisfy the actual boundary conditions of each panel, we determine the 'amplification factors' of the four 'unitary load lines' applied at $\varphi = 0$ and the four 'unitary load lines' applied at $\varphi = \varphi_0$. This is done through the compatibility and the equilibrium equations between panels (Section 4.6). By solving the global system including all these equations (8 per panel) we get the 'amplification factors'.

Then, the final solutions (u,v,w) of a panel of the structure is obtain by adding nine different solutions of the same 360° cylindrical panel (including stiffeners and frames):

- the 360° cylindrical panel under actual external loads,
- at $\varphi = 0$, the 360° cylindrical panel under the X_u, Y_u, Z_u and M_u 'unitary load lines' multiplied by their respective 'amplification factor',
- at $\varphi = \varphi_0$, the 360° cylindrical panel under the X_u, Y_u, Z_u and M_u 'unitary load lines' multiplied by their respective 'amplification factor'.

5. Conclusions

An analytical method to analyze stiffened structures was presented in this paper. It is based on the resolution of the stiffened panel's differential equations by using Fourier series expansions. This method has been developed for fast and accurate linear elastic analysis, particularly in regard to structural optimization. This is precisely the reason it was implemented in the LBR-5 optimization software.

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