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SHIP TECHNOLOGY RESEARCH

SCHIFFSTECHNIK

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Differential Equations of Stiffened Panels of Ship Structures and Fourier Series Expansions

by Philippe Rigo

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Differential Equations of Stiffened Panels of Ship Structures and Fourier Series Expansions

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1 Introduction

This text presents the theory used to implement in the LBR-5 software an analytical solution of the stiffened panels governing equations. For that purpose, Fourier series expansions are used to solve the governing differential equations. In the present analysis, cylindrical shells are used as the reference panels. Stiffened plates are considered as a simplified case of the more general cylindrical shell. In the LBR-5 software, plates are analyzed as being cylindrical shells having a very large radius ($q = 10^{10}$ m). The present method has been developed for fast and accurate linear elastic analysis of stiffened structures, particularly in regard to structural optimization.

Applications of the LBR-5 software to ship structures including its associated background about the scantling optimization procedure have been presented in various papers and conferences, *Rigo (2001a,b,c,2003), Rigo and Fleury (2001)*. The presented developments were initiated in *Rigo (1989a)* and the general methodology presented in *Rigo (1989b,1992a,b)*. After 15 years the LBR-5 software is now well-established and patented; it is therefore relevant to publish the extensive theoretical background of this method.

2 Differential equations of the cylindrical stiffened shell

Fig.1 shows the coordinate system $ox\varphi$ with $z = 0$ at mid plate thickness. The relation between the φ coordinate (used for shell) and the y coordinate is: $y = q\varphi$, with q the radius, δ the plate thickness, L the panel length (along x), and φ_0 the opening angle (radian).

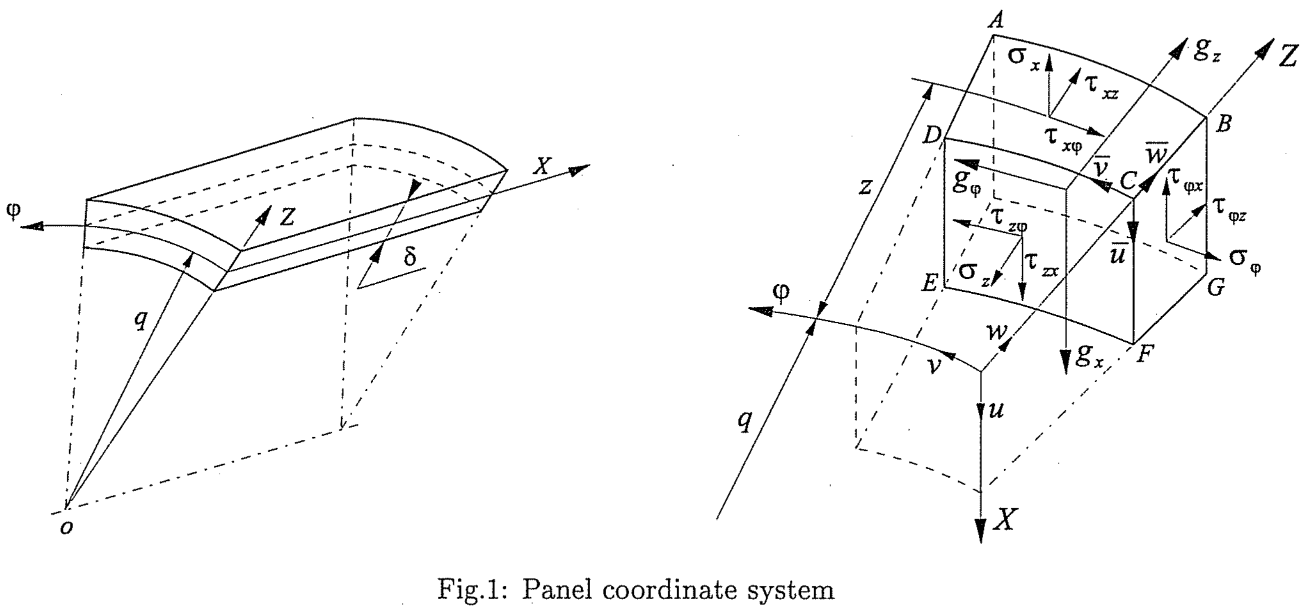

Fig.1: Panel coordinate system

Fig.1 presents the stresses acting on a small volume element $[dx, dz, (q+z)d\varphi]$. In this study, the thin shell (plate) theory is used, i.e. τ_{xz}, $\tau_{\varphi z}$, and σ_z are not considered ($\varepsilon_z = \gamma_{xz} = \gamma_{\varphi z} = 0$).

The governing differential equations, known as the D.K.J. differential equations (Donnell, von Karman and Jenkins), are based on the Love-Kirchoff hypotheses:

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1. Thin shell theory, i.e. $\delta/q \ll 1$. For LBR-5, we impose that $\delta/q < 1/100$.
2. Small deformation and linear analysis.
3. The points that are on a perpendicular line to the mid plate surface ($z = 0$) before deformation remain on the same perpendicular after deformation, thus γ_{xz} and $\gamma_{\varphi z} = 0$.
4. σ_z and its effects are negligible.
5. No deformation along oz ($\varepsilon_z = 0$).

Let us denote partial derivatives as follows:

$$f' = \frac{\partial f}{\partial x} \qquad f^\circ = \frac{\partial f}{\partial y} = \frac{1}{q}\frac{\partial f}{\partial \varphi} \tag{1}$$

Then the linear 'deformation-displacement' relations for a shell are:

$$\begin{aligned}
\varepsilon_x &= u' - zw'' \\
\varepsilon_\varphi &= v^\circ + \frac{w}{q} - zw^{\circ\circ} \\
\gamma_{x\varphi} &= u^\circ + v' - 2zw^{\circ\prime}
\end{aligned} \tag{2}$$

and the 'stress-displacement' relations are:

$$\begin{aligned}
\sigma_x &= \frac{E}{1-\nu^2}\left[u' + \nu \cdot \left(v^\circ + \frac{w}{q}\right) - z(w'' + \nu \cdot w^{\circ\circ})\right] \\
\sigma_\varphi &= \frac{E}{1-\nu^2}\left[\left(v^\circ + \frac{w}{q}\right) + \nu \cdot u' - z(w^{\circ\circ} + \nu \cdot w'')\right] \\
\tau_{x\varphi} &= G(u^\circ + v' - 2zw^{\circ\prime})
\end{aligned} \tag{3}$$

E is the Young's modulus, ν the Poisson coefficient, $G = E/[2(1+\nu)]$ the shear modulus. The special case of a plate is derived by simply setting $w/q = 0$ in Eqs.(2) and (3)

Fig.2 shows the internal resultant forces N_x, Q_x, $N_{x\varphi}$, N_φ, Q_φ, $N_{\varphi x}$, and moments M_x, $M_{x\varphi}$, M_φ, $M_{\varphi x}$ that are applied on an elementary cylindrical shell (plate), hereafter called resultants. This element is included between the upper surface ($z = \delta/2$) and the lower surface ($z = -\delta/2$) and has a surface dimension of $dx \cdot q\ d\varphi$ (or $dx \cdot dy$). With reference to the thin shell element, Fig.2, we can establish the 'resultant-stress' relationships, Eqs.(4). These resultant forces and moments are referenced to the plate neutral axis ($z = 0$).

$$\begin{aligned}
N_\varphi &= \int_{-\delta/2}^{\delta/2} \sigma_\varphi\, dz & N_x &= \int_{-\delta/2}^{\delta/2} \sigma_x(1 + \frac{z}{q})\, dz && \text{inplane axial resultant} \\
M_\varphi &= \int_{-\delta/2}^{\delta/2} \sigma_\varphi z\, dz & M_x &= \int_{-\delta/2}^{\delta/2} \sigma_x(1 + \frac{z}{q}) z\, dz && \text{bending moments} \\
N_{\varphi x} &= \int_{-\delta/2}^{\delta/2} \tau_{\varphi x}\, dz & N_{x\varphi} &= \int_{-\delta/2}^{\delta/2} \tau_{x\varphi}(1 + \frac{z}{q})\, dz && \text{inplane shear resultant} \\
M_{\varphi x} &= \int_{-\delta/2}^{\delta/2} \tau_{\varphi x} z\, dz & M_{x\varphi} &= \int_{-\delta/2}^{\delta/2} \tau_{x\varphi}(1 + \frac{z}{q}) z\, dz && \text{torsion moments}
\end{aligned} \tag{4}$$

Q_x and Q_φ (transverse shear resultant) cannot be calculated by integration of the τ_{xz} and $\tau_{\varphi z}$ stresses as these shear stresses are assumed to be equal to 0 (thin plate assumption). Nevertheless, Q_x and Q_φ can be evaluated using the 4^{th} and the 5^{th} equilibrium equations, Eqs.(9).

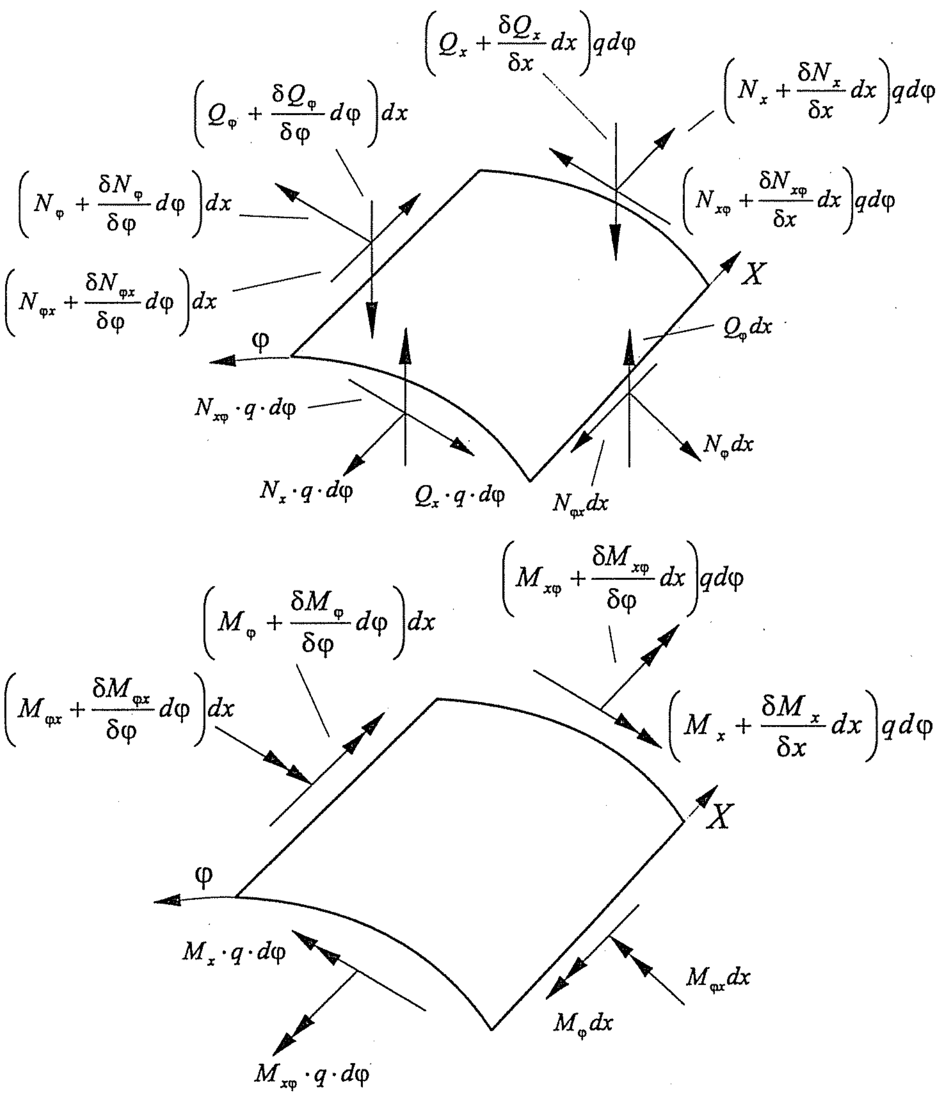

Fig.2: Resultant forces and moments

If we replace the 'stress-displacement' relationships, Eqs.(3), within the 'resultant-stress' relationships Eqs.(4), we obtain the 'resultant-displacement' relationships:

$$
\begin{aligned}
N_\varphi &= D(v^\circ + \frac{w}{q} + \nu u') \\
N_x &= D(u' + \nu v^\circ + \nu \frac{w}{q}) \\
N_{\varphi x} &= N_{x\varphi} = D\frac{1-\nu}{2}(v' + u^\circ) \\
M_\varphi &= K(w^{\circ\circ} + \nu w'') \\
M_x &= K(w'' + \nu w^{\circ\circ}) \\
M_{x\varphi} &= M_{\varphi x} = K(1-\nu) w^{\circ\prime}
\end{aligned}
\qquad (5)
$$

with $D = E\delta/(1-\nu^2)$ and $K = E\delta^3/[12(1-\nu^2)]$.

Fig.3 shows the additional resultant forces and moments acting on the shell (plate) coming from a stiffener oriented along ox ($N_{x,Conc}$, $M_{x,Conc}$, $N_{xy,Conc}$, $Q_{x,Conc}$, and $M_{xy,Conc}$). For a transverse member (frame) oriented along $o\varphi$ the additional resultant forces and moments are ($N_{y,Conc}$, $M_{y,Conc}$, $N_{yx,Conc}$, $Q_{y,Conc}$, and $M_{yx,Conc}$).

If we consider a stiffened thin shell element, the 'resultant-stress' relationships including the plate and the stiffener components become:

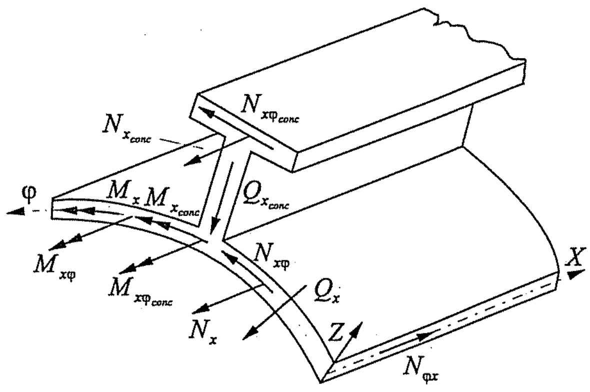

Fig.3: Resultant forces and moments acting on the shell (plate) due to a stiffener.

$$
\begin{aligned}
N_\varphi &= \int_{-\delta/2}^{\delta/2} \sigma_\varphi \, dz + f(x) \int_{\omega_\varphi} \sigma_\varphi \frac{e_\varphi}{d\varphi} \, dz && \text{inplane axial resultant} \\
N_x &= \int_{-\delta/2}^{\delta/2} \sigma_x (1 + \frac{z}{q}) \, dz + f(\varphi) \int_{\omega_x} \sigma_x \frac{e_x}{d_x} \, dz && \\
M_\varphi &= \int_{-\delta/2}^{\delta/2} \sigma_\varphi z \, dz + f(x) \int_{\omega_\varphi} \sigma_\varphi z \frac{e_\varphi}{d\varphi} \, dz && \text{bending moments} \\
M_x &= \int_{-\delta/2}^{\delta/2} \sigma_x z (1 + \frac{z}{q}) \, dz + f(\varphi) \int_{\omega_x} \sigma_x z \frac{e_x}{d_x} \, dz &&
\end{aligned}
\qquad (6)
$$

with ω_φ and ω_x the cross-sections of, respectively, the frames and the stiffeners. e_x, e_φ, dx, and $d\varphi$ follow from Fig.4. The second term of each equation in Eqs.(6) corresponds to the stiffeners $f(\varphi)$ (frames ($f(x)$)).

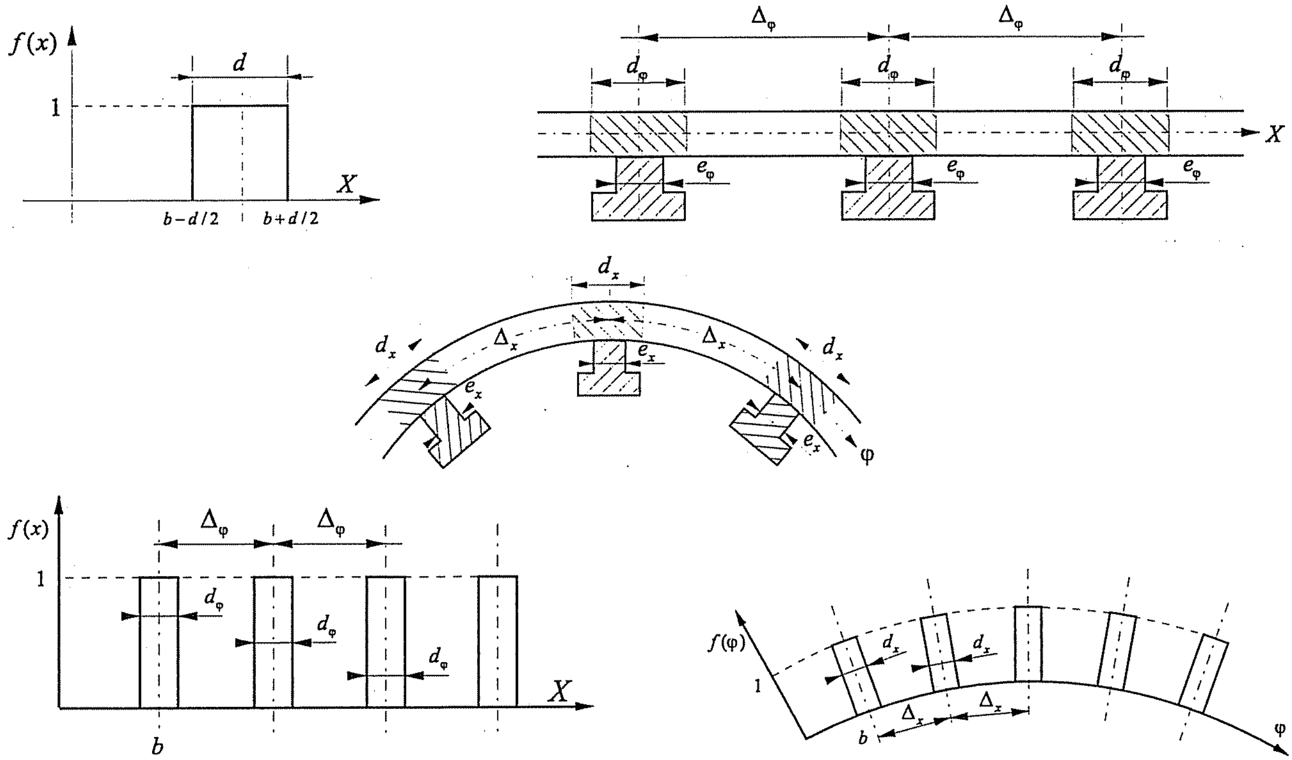

Fig.4: Heaviside function (top left) and stiffener/frame spacings

Eqs.(4) (unstiffened element) are a simplified form of Eqs.(6) (stiffened element). Their first terms are identical. For a stiffened element, Eqs.(6) include Heaviside functions [$f(x)$ and $f(\varphi)$] that are equal to zero except at the stiffener locations, $b - d/2 < x < b + d/2$, where they are equal to 1.

$$F(x) = H[x - (b - d/2)] - H[x - (b + d/2)] \quad \text{with } H(x) = \begin{cases} H(x) = 0 & \text{if} \quad x < 0 \\ H(x) = 1 & \text{if} \quad x > 0 \end{cases} \tag{7}$$

with $H(x)$ the Heaviside function. If we replace the 'stress-displacement' relationships (3) in the 'resultant-stress' relationships (6), we obtain the 'resultant-displacement' relationships for stiffened panels (including stiffeners and frames):

$$\begin{array}{lll}
N_\varphi = D(v^\circ + \frac{w}{q} + \nu u') & +f(x)\frac{E}{d_\varphi}\left((v^\circ + \frac{w}{q})\omega_\varphi - w^{\circ\circ}h_\varphi\right) & \text{incl. frame contr.} \\
M_\varphi = K(w^{\circ\circ} + \nu w'') & +f(x)\frac{E}{d_\varphi}\left((v^\circ + \frac{w}{q})h_\varphi - w^{\circ\circ}I_\varphi\right) & \\
M_{\varphi x} = K(1-\nu)w^{\circ\prime} & +f(x)\frac{G}{d_\varphi}(K_\varphi w^{\circ\prime} + \lambda_\varphi \Omega'_\varphi(v' + u^\circ)) & \\
N_x = D(u' + \nu(v^\circ + \frac{w}{q})) & +f(\varphi)\frac{E}{d_x}(u'\omega_x - w''h_x) & \text{incl. stiffener contr.} \\
M_x = K(w'' + \nu w^{\circ\circ}) & +f(\varphi)\frac{E}{d_x}(u'h_x - w''I_x) & \\
M_{x\varphi} = K(1-\nu)w^{\circ\prime} & +f(\varphi)\frac{G}{d_x}(K_x w^{\circ\prime} + \lambda_x \Omega'_x(v' + u^\circ)) & \\
N_{x\varphi} = D\left(\frac{1-\nu}{2}\right)(v' + u^\circ) & +f(x)\frac{G}{d_\varphi}\Omega'_\varphi(v' + u^\circ) & \text{frame contribution} \\
 & +f(\varphi)\frac{G}{d_x}\Omega'_x(v' + u^\circ) & \text{stiffener contribution} \\
N_{\varphi x} = N_{x\varphi} & &
\end{array} \tag{8}$$

with
- ω_x, ω_φ — transversal section of a stiffener (frame) without plating
- h_x, h_φ — 1^{st} sectional moment of ω_x (ω_φ) relative to the plate neutral axis $z = 0$
- I_x, I_φ — 2^{nd} sectional moment of ω_x (ω_φ) relative to the plate neutral axis $z = 0$
- K_x, K_φ — torsional rigidity of a stiffener (frame)
- Ω'_x, Ω'_φ — reduced flange section (for flange in-plane shear contribution)
- λ_x, λ_φ — flange eccentricity to the plate neutral axis $z = 0$
- d_x, d_φ — effective strip width of longitudinal stiffeners (frames)

A major uncertainty related to the validity of these equations concerns the flange contribution to the in-plane shear effects, Fig.5. If these contributions are not considered, set Ω'_φ and Ω'_x to zero in Eqs.(8). For the special case of a flat plate, set $w/q = 0$ in Eqs.(8)

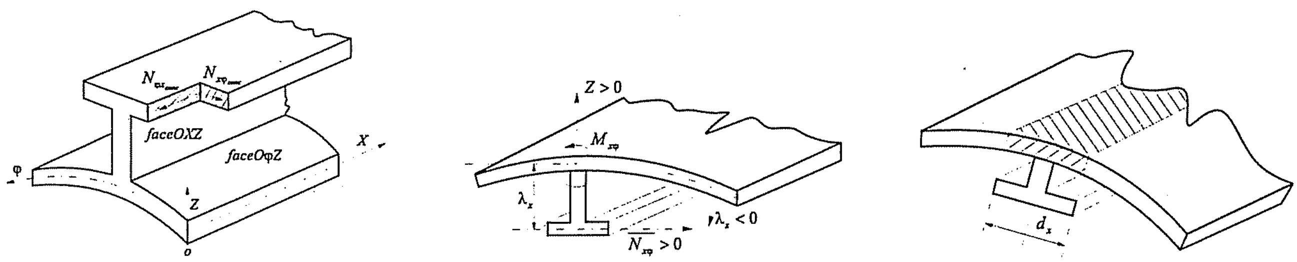

Fig.5: Flange contributions to the in-plane shear resultant ($N_{x\varphi}$) and torsional moment ($M_{x\varphi}$)

To summarize, the D.K.J. governing differential equations of a cylindrical shell (plate) are obtained by:

- Using the 6 equilibrium equations, Fig.2:

$$
\begin{array}{ll}
N'_x + N^\circ_{\varphi x} + X & = 0 \\
N^\circ_\varphi + N'_{x\varphi} - (Q_\varphi/q) + Y & = 0 \\
(N_\varphi/q) + Q^\circ_\varphi + Q'_x - Z & = 0 \\
M^\circ_\varphi + M'_{x\varphi} - Q_\varphi & = 0 \\
M'_x + M^\circ_{\varphi x} - Q_x & = 0 \\
N_{x\varphi} - N_{\varphi x} + M_{\varphi x}/q & = 0
\end{array}
\qquad (9)
$$

The special case of a plate follows by omitting all terms containing q in Eq.(9).

- Using the five hypotheses of linear thin shell theory (see above),
- Using the 'stress-displacement' relationships (3) and the 'resultant-stress' relationships (6) to establish the 'resultant-displacement' relationships (8).

The problem is composed of 13 unknowns:

- u, v, w
- N_x, M_x, Q_x, $N_{x\varphi}$, $M_{x\varphi}$
- N_φ, M_φ, Q_φ, $N_{\varphi x}$, $M_{\varphi x}$

and there are 13 available equations:

- 4 'resultant-displacement' relations corresponding to N_x, M_x, $N_{x\varphi}$, $M_{x\varphi}$ (8)
  (there is no available equation for Q_x)
- 4 'resultant-displacement' relations corresponding to N_φ, M_φ, $N_{\varphi x}$, $M_{\varphi x}$ (8)
  (there is no available equation for Q_φ)
- 5 equilibrium equations (9)
  (the last equilibrium equation has already been used and cannot be considered).

Replacing the 'resultant-displacement' relations (8) in the 5 first equilibrium equations (9) and replacing Q_x and Q_φ (4^{th} and 5^{th} equations) in the 3^{rd} equilibrium equation, we obtain the 3 governing differential equations in u, v, and w. This is a system of three differential equations.

2.1 Shell with shear contribution of the flanges

Stiffened shell with stiffeners $[f(\varphi)]$ and frames $[f(x)]$

$$
\begin{aligned}
& D(u'' + \nu v^{\circ\prime} + \nu \frac{w'}{q}) + D\left(\frac{1-\nu}{2}\right)(u^{\circ\circ} + v^{\circ\prime}) + f(x)[S_\varphi(v^{\circ\prime} + u^{\circ\circ})] \\
& \quad + f(\varphi)[\Omega_x u'' - H_x w''' + S_x(v^{\circ\prime} + u^{\circ\circ})] + f^\circ(\varphi)[S_x(v' + u^\circ)] + X = 0 \\
& D(v^{\circ\circ} + w^\circ/q + \nu u^{\circ\prime}) + D\left(\frac{1-\nu}{2}\right)(u^{\circ\prime} + v'') \\
& \quad + f(x)[\Omega_\varphi(v^{\circ\circ} + w^\circ) - H_\varphi w^{\circ\circ\circ} + S_\varphi(v'' + u^{\circ\prime})] + f'(x)[S_\varphi(v' + u^\circ)] \\
& \quad + f(\varphi)[S_x(v'' + u^{\circ\prime})] + Y = 0 \\
& \frac{D}{q}(v^\circ + \frac{w}{q} + \nu u') + K w^{\circ\circ\circ\circ} + 2K w^{\circ\circ\prime\prime} + K w'''' \\
& \quad + f(x)\left[\frac{\Omega_\varphi}{q}(v^\circ + \frac{w}{q}) - H_\varphi(\frac{2w^{\circ\circ}}{q} + v^{\circ\circ\circ}) + R_\varphi w^{\circ\circ\circ\circ} + T_\varphi w^{\circ\circ\prime\prime} + L_\varphi(v^{\circ\prime\prime} + u^{\circ\circ\prime})\right] \\
& \quad + f'(x)[T_\varphi w^{\circ\circ\prime} + L_\varphi(v^{\circ\prime} + u^{\circ\circ})] + f(\varphi)[-H_x u''' + R_x w'''' + T_x w^{\circ\circ\prime\prime} + L_x(v^{\circ\prime\prime} + u^{\circ\circ\prime})] \\
& \quad + f^\circ(\varphi)[T_x w^{\circ\prime\prime} + L_x(v'' + u^{\circ\prime})] - Z = 0
\end{aligned}
\qquad (10)
$$

with

$$\Omega_\varphi = \frac{E\omega_\varphi}{d_\varphi} \quad \Omega_x = \frac{E\omega_x}{d_x} \quad H_\varphi = \frac{Eh_\varphi}{d_\varphi} \quad H_x = \frac{Eh_x}{d_x} \quad R_\varphi = \frac{EI_\varphi}{d_\varphi} \quad R_x = \frac{EI_x}{d_x}$$

$$S_\varphi = \frac{G\Omega'_y}{d_\varphi} \quad S_x = \frac{G\Omega'_x}{d_x} \quad T_\varphi = \frac{G}{d_\varphi}K_\varphi \quad T_x = \frac{G}{d_x}K_x \quad L_\varphi = \frac{\lambda_\varphi}{d_\varphi}G\Omega'_\varphi \quad L_x = \frac{\lambda_x}{d_x}G\Omega'_x$$

A major uncertainty related to the validity of these equations concerns the flange contribution to the in-plane shear effects. In the following developments these contributions will not be considered. This is a conservative assumption.

2.2 Shell without flange contribution

Stiffened shell with stiffeners $[f(\varphi)]$ and frames $[f(x)]$:

$$\begin{aligned}
&D(u'' + \nu v^{\circ\prime} + \nu\frac{w'}{q}) + D\left(\frac{1-\nu}{2}\right)(u^{\circ\circ} + v^{\circ\prime}) + f(\varphi)[\Omega_x u'' - H_x w'''] + X = 0 \\
&D(v^{\circ\circ} + \frac{w^\circ}{q} + \nu u^{\circ\prime}) + D\left(\frac{1-\nu}{2}\right)(u^{\circ\prime} + v'') + f(x)[\Omega_\varphi(v^{\circ\circ} + w^\circ) - H_\varphi w^{\circ\circ\circ}] + Y = 0 \\
&\frac{D}{q}(v^\circ + \frac{w}{q} + \nu u') + Kw^{\circ\circ\circ\circ} + 2Kw^{\circ\circ\prime\prime} + Kw'''' \\
&\quad + f(x)\left[\frac{\Omega_\varphi}{q}(v^\circ + \frac{w}{q}) - H_\varphi(\frac{2w^{\circ\circ}}{q} + v^{\circ\circ\circ}) + R_\varphi w^{\circ\circ\circ\circ} + T_\varphi w^{\circ\circ\prime\prime}\right] \\
&\quad + f'(x)[T_\varphi w^{\circ\circ\prime}] + f(\varphi)[-H_x u''' + R_x w'''' + T_x w^{\circ\circ\prime\prime}] + f^\circ(\varphi)[T_x w^{\circ\prime\prime}] - Z = 0
\end{aligned} \tag{11}$$

Unstiffened shell:

$$\begin{aligned}
&D(u'' + \nu v^{\circ\prime} + \nu\frac{w'}{q}) + D\left(\frac{1-\nu}{2}\right)(u^{\circ\circ} + v^{\circ\prime}) + X = 0 \\
&D(v^{\circ\circ} + \frac{w^\circ}{q} + \nu u^{\circ\prime}) + D\left(\frac{1-\nu}{2}\right)(u^{\circ\prime} + v'') + Y = 0 \\
&\frac{D}{q}(v^\circ + \frac{w}{q} + \nu u') + Kw^{\circ\circ\circ\circ} + 2Kw^{\circ\circ\prime\prime} + Kw'''' - Z = 0
\end{aligned} \tag{12}$$

The corresponding formulae for plates are again obtained by simply setting $1/q = 0$ in Eqs.(11) and (12). These equations are then no longer coupled. The bending equation (3^{rd} equation) can be solved independently of the in-plane forces (1^{st} and 2^{nd} equations).

3 Stiffened cylindrical shell (plate) elements

Fig.6 shows a typical LBR-5 stiffened cylindrical shell element composed of a plating (δ) and three layers of stiffeners. Shells are the generic elements and plates are analyzed as a particular case. There are two layers along the longitudinal (ox) direction and one layer along the ($o\varphi$) direction.

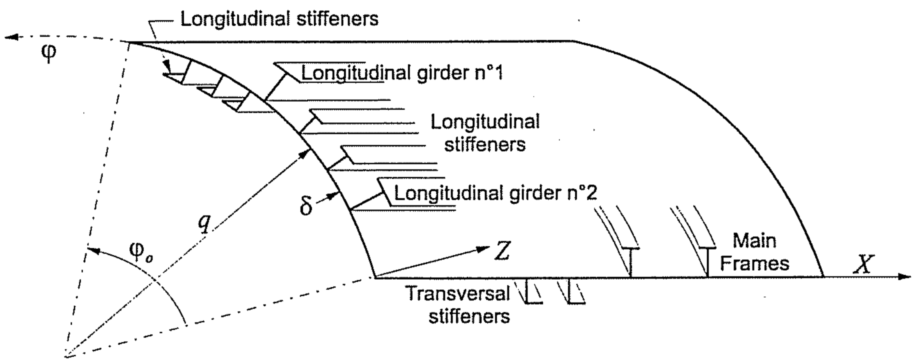

Fig.6: Stiffened cylindrical shell elements

Typically the layer along $o\varphi$ corresponds to the transverse frames. Longitudinally (ox), there are the stiffeners (smaller in size) and the girders (larger). We assume that:
– all stiffeners belonging to the same panel are identical and have the same spacing,
– all frames belonging to the same panel are identical and have the same spacing.
On the contrary, the girders can differ and their spacing is free.

3.1 Resultant forces and moments of the three-layered stiffened panels

Fig.7 shows the standardization procedure used to consider the stiffener and frame contributions. Eqs.(13) give the resultant forces and moments of the three-layered stiffened panels:

$$\begin{aligned}
N_\varphi &= (D+\Omega_\varphi)\left(v^\circ + \frac{w}{q}\right) + D\nu u' - H_\varphi w^{\circ\circ} \\
N_x &= (D+\Omega_x)u' + D\nu\left(v^\circ + \frac{w}{q}\right) - H_x w'' + f(\varphi)[\Omega_x^t u' - H_x^t w''] \\
M_\varphi &= (K+R_\varphi)w^{\circ\circ} + K\nu w'' - H_\varphi\left(\frac{w}{q} + v^\circ\right) \\
M_x &= (K+R_x)w'' + K\nu w^{\circ\circ} - H_x u' - f(\varphi)[H_x^t u' - R_x^t w''] \\
M_{\varphi x} &= [(K(1-\nu)+T_\varphi]w^{\circ\prime} + L_\varphi(v' + u^\circ) \\
M_{x\varphi} &= [(K(1-\nu)+T_x]w^{\circ\prime} + L_x(v' + u^\circ) + f(\varphi)[T_x^t w^{\circ\prime} - L_x^t(v' + u^\circ)] \\
N_{\varphi x} &= \left[D\left(\frac{1-\nu}{2}\right) + S_x + S_\varphi\right](v' + u^\circ) + f(\varphi)[S_x^t(v' + u^\circ)] \\
N_{x\varphi} &= N_{\varphi x} \\
Q_\varphi &= (K+T_x)w^{\circ\prime\prime} + (K+R_\varphi)w^{\circ\circ\circ} - H_\varphi\left(\frac{w^\circ}{q} + v^{\circ\circ}\right) + L_x(v'' + u^{\circ\prime}) \\
&\quad + f(\varphi)[T_x^t w^{\circ\prime\prime} + L_x^t(v'' + u^{\circ\prime})] \\
Q_x &= (K+T_\varphi)w^{\circ\circ\prime\prime} + (K+R_x)w''' - H_x u'' + L_\varphi(v^{\circ\prime} + u^{\circ\circ}) \\
&\quad - f(\varphi)(H_x^t u'' - R_x w''')
\end{aligned} \tag{13}$$

with (subscript x for stiffeners, subscript φ for frames, subscript xt for each individual girders):

$$\Omega_\varphi = \frac{E\omega_\varphi}{\Delta_\varphi} \quad \Omega_x = \frac{E\omega_x}{\Delta_x} \quad H_\varphi = \frac{Eh_\varphi}{\Delta_\varphi} \quad H_x = \frac{Eh_x}{\Delta_x} \quad R_\varphi = \frac{EI_\varphi}{\Delta_\varphi} \quad R_x = \frac{EI_x}{\Delta_x}$$

$$S_\varphi = \frac{G\Omega'_\varphi}{\Delta_\varphi} \quad S_x = \frac{G\Omega'_x}{\Delta_x} \quad T_\varphi = \frac{GK_\varphi}{\Delta_\varphi} \quad T_x = \frac{GK_x}{\Delta_x} \quad L_\varphi = \frac{G\lambda_\varphi}{\Delta_\varphi}\Omega'_\varphi \quad L_x = \frac{G\lambda_x}{\Delta_x}\Omega'_x$$

with Δ_x, Δ_φ the spacing between longitudinal stiffeners resp. frames,
and for each individual girder:

$$\Omega_x^t = \frac{E\omega_{xt}}{d_{xt}} \quad H_x^t = \frac{Eh_{xt}}{d_{xt}} \quad R_x^t = \frac{EI_{xt}}{d_{xt}} \quad T_x^t = \frac{GK_{xt}}{d_{xt}} \quad S_x^t = \frac{G\Omega'_{xt}}{d_{xt}} \quad L_x^t = S_x^t\lambda_{xt}$$

The symbols have the same meaning as above for the other stiffeners, the index xt indicates that they pertain now to a girder.

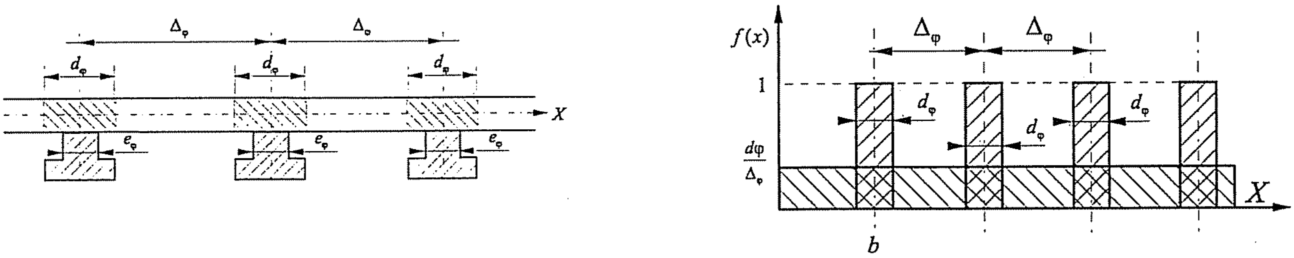

Fig.7: Uniformly distributed frames: $f(x) = \dfrac{d\varphi}{\Delta\varphi}$ =const.

Eqs.(13) include different components:
- Plate components (D, K),
- Stiffener components (Ω_x, R_x, S_x, H_x, T_x, L_x),
- Frame components (Ω_φ, R_φ, S_φ, H_φ, T_φ, L_φ) and
- Girder components (Ω_x^t, R_x^t, S_x^t, H_x^t, T_x^t, L_x^t; terms multiplied by $f(\varphi)$).

The $f(\varphi)$ Heaviside functions allow to model the non-uniformity of the girders (in size and in location). For the stiffeners, however, the $f(\varphi)$ function is replaced by $d_\varphi/\Delta\varphi$. For the frames $f(x)$ is replaced by $d_x/\Delta x$, Fig.7. This standardization does not mean that the stiffeners (frames) are smeared and replaced by an equivalent plate thickness but it means that each individual characteristic (cross section, first sectional moment, inertia moment, torsional rigidity,...) is standardized on the entire plate. Globally, the stiffened panel behavior is accurately modeled but it is locally simplified. This simplification is only valid if the spacing between stiffeners (frames) is constant and remains small (compared to their span).

Based on the equilibrium equations (9) and the 'resultant-displacement' relationships (13), three governing differential equations are obtained:

$$(D+\Omega_x)u'' + \left[D\left(\frac{1-\nu}{2}\right)+S_x+S_\varphi\right]u^{\circ\circ} + \left[D\left(\frac{1+\nu}{2}\right)+S_x+S_\varphi\right]v^{\circ\prime} - H_x w''' + \frac{D\nu}{q}w'$$

$$= \underbrace{-f(\varphi)[\Omega_x^t u'' - H_x^t w + S_x^t(v^{\circ\prime}+u^{\circ\circ})]}_{X_0} - \underbrace{f^\circ(\varphi)[S_x^t(v'+u^\circ)]}_{U_{0Z}} - X$$

$$(D+\Omega_\varphi)v^{\circ\circ} + \left[D\left(\frac{1+\nu}{2}\right)+S_x+S_\varphi\right]u^{\circ\prime} + \left[D\left(\frac{1-\nu}{2}\right)+S_x+S_\varphi\right]v'' - H_\varphi w^{\circ\circ\circ} + \frac{1}{q}(D+\Omega_\varphi)w^\circ$$

$$= -\underbrace{f(\varphi)[S_x^t(v''+u^{\circ\prime})]}_{Y_0} - Y \qquad (14)$$

$$-H_x u''' + (L_x+L_\varphi)(u^{\circ\circ\prime}+v^{\circ\prime\prime}) + \frac{D\nu}{q}u' + \frac{1}{q}(D+\Omega_\varphi)(v^\circ + \frac{w}{q}) - H_\varphi v^{\circ\circ\circ} + (K+R_\varphi)w^{\circ\circ\circ\circ}$$

$$+(2K+T_\varphi+T_x)w^{\circ\circ\prime\prime} + (K+R_x)w'''' - \frac{2H_\varphi}{q}w^{\circ\circ} =$$

$$-\underbrace{f(\varphi)[T_x^t w^{\circ\circ\prime\prime} - H_x^t u''' + R_x^t w'''' + L_x^t(v^{\circ\prime\prime}+u^{\circ\circ\prime})]}_{Z_0} - \underbrace{f^\circ(\varphi)[T_x^t w^{\circ\prime\prime} + L_x^t(v''+u^{\circ\prime})]}_{U_{0X}} + Z$$

In these equations we can distinguish the components related to the plating (D, K), stiffeners (S_x, Ω_x,...), frames (S_φ, Ω_φ,...) and girders ($f(\varphi)\cdot X_0$, $f(\varphi)\cdot Y_0$, $f(\varphi)\cdot Z_0$, $f^\circ(\varphi)\cdot U_{0X}$, $f^\circ(\varphi)\cdot U_{0Z}$). The X_0, Y_0, Z_0, U_{0X}, and U_{0Z} components can be considered as external loads for the girders. The X_0, Y_0, Z_0 loads are in fact 'force load lines', Fig.8, corresponding to the girder reactions on the panels. U_{0X} and U_{0Z} are 'moment load lines'. They correspond to the $f^\circ(\varphi)$ terms.

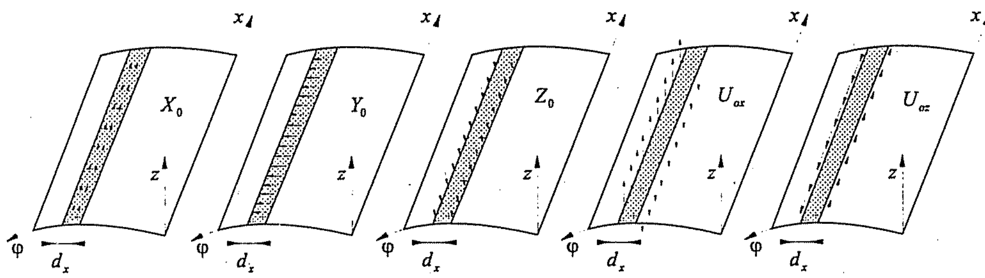

Fig.8: The five girder load lines (forces and moments)

4 Analytical solution for the governing equations for stiffened panel

Only for the unstiffened plate u and v (in-plane displacements) are not coupled with w (transversal displacements) within linear thin plate theory. In all other cases, u, v, and w are coupled and the three equations for them (Eqs.(10), (11), (12), or (14)) have to be solved simultaneously. The principle to solve any of these three governing differential equations is the same. They can be written as:

$$\begin{aligned} a_1 u + b_1 v + c_1 w &= +X(x, \varphi) \\ a_2 u + b_2 v + c_2 w &= +Y(x, \varphi) \\ a_3 u + b_3 v + c_3 w &= -Z(x, \varphi) \end{aligned} \tag{15}$$

with:

- $u(x, \varphi)$, $v(x, \varphi)$, and $w(x, \varphi)$ the displacements;
- x and φ are the coordinates of a point on the mid-plane of the cylindrical shell (plate). The z coordinate does not appear as we only look for the displacements (u, v, and w) at the mid-plate thickness where $z = 0$ (linear thin shell theory).
- X, Y and $Z(x, \varphi)$ are the surface loads, Fig.12.

a_1, b_1, ..., c_3 are the derivative operators. E.g. for the system of Eq.(11) we have:

$$a_1 = D\frac{\partial}{\partial x} + D\left(\frac{1-\nu}{2}\right)\frac{\partial}{\partial y}$$

Note: At this stage the $f(\varphi)$ concentrated terms of the differential equations (14) corresponding to the girders must be discarded. Their effects will be considered at a later stage (STEP 6).

4.1 STEP 1: Homogeneous solution (or Complementary solution)

The homogeneous solution of the governing differential equations (15) yields:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1) = 0 \tag{16}$$

If we apply this to the $w(x, \varphi)$ displacement, we obtain:

$$Aw_{80} + Bw_{60} + Cw_{62} + Dw_{40} + Ew_{42} + \ldots + Jw_{26} + Kw_{08} = 0 \tag{17}$$

This is an 8^{th} order differential equation with two coupled variables (x and φ). w_{ij} denotes the i^{th} order derivative of w by x and j^{th} order derivative by y ($y = q\varphi$). E.g. $w_{13} = w'^{\circ\circ\circ}$.

4.2 STEP 2: Fourier Series Expansions

To solve this 8^{th} order differential equation make an assumption on the shape of the displacements u, v, w to obtain an 8^{th} order differential equation with two separate variables:

$$w(x, \varphi) = w_1(\varphi) \cdot w_2(x) \tag{18}$$

We use the Fourier series expansion theory and assume:

$$\begin{aligned} u(x, \varphi) &= u(\varphi) \cdot \cos(\lambda x) \\ v(x, \varphi) &= v(\varphi) \cdot \sin(\lambda x) \\ w(x, \varphi) &= w(\varphi) \cdot \sin(\lambda x) \end{aligned} \tag{19}$$

with $\lambda = n\pi/L$, n the term number of the Fourier series expansion, and L the span of the structure (and panels) along ox. L is the same for each panel.

The shape of the assumed displacements imposes some limitations on the boundary conditions. The two edges $x = 0$ and $x = L$ must behave as simply supported edges, i.e. $w = v = M_x = N_x = 0$, Fig.9.

Inserting Eq.(19) in the one of the considered governing differential equations (10), (11), (12), or (14) yields an 8^{th} order polynomial differential equation with now only one variable (φ).

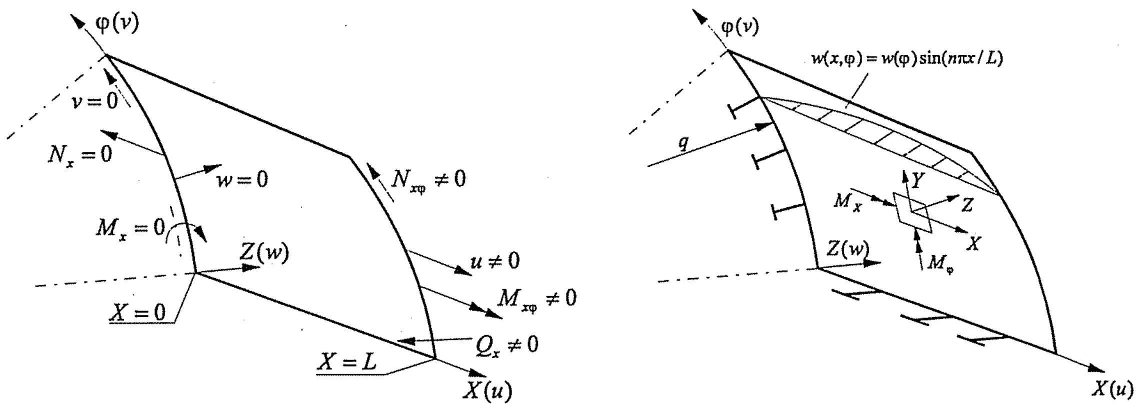

Fig.9: Fourier series expansion and boundary conditions

4.3 STEP 3: Loads' Fourier series expansion

Having decided to expand the displacements using Fourier series (STEP 2) to solve the governing differential equations means that the $Z(x, \varphi)$ loads, Eqs.(15), have to also satisfy the Fourier series expansion's shapes:

$$Z(x, \varphi) = Z^*(\varphi) \cdot Q(x) = Z^*(\varphi) \cdot \sum a \sin(\lambda x)$$

The way to implement the actual loads in the analytical procedure is explained in STEP 5.

Consider a load function $Q(x)$, which consists of a uniform load Q_0 between x_1 and x_2, and zero elsewhere. The Fourier expansion with sine and cosine series of this load is:

sine expansion: $$Q(x) = \sum_{n=1}^{\infty} \left[\frac{4Q}{n\pi} \sin\left(\frac{n\pi}{2L}(x_1 + x_2)\right) \cdot \sin\left(\frac{n\pi}{2L}(x_2 - x_1)\right) \right] \cdot \sin\frac{n\pi x}{L} \quad (20)$$

cosine expansion: $$Q(x) = \sum_{n=1}^{\infty} \left[\frac{4Q}{n\pi} \cos\left(\frac{n\pi}{2L}(x_1 + x_2)\right) \cdot \sin\left(\frac{n\pi}{2L}(x_2 - x_1)\right) \right] \cdot \cos\frac{n\pi x}{L} \quad (21)$$

Hydrostatic pressure is usually uniformly distributed along ox and varies linearly along $o\varphi$. The variation along $o\varphi$ is considered in STEP 5. The expansion along ox for such a symmetric load uses only the odd terms in a sine series:

$$Q(x) = \sum_{n=1}^{\infty} \frac{4Q}{(2n-1)\pi} \sin\frac{(2n-1)\pi x}{2L} \quad (22)$$

In practice, the first three terms of the series are enough to model such loads with sufficient accuracy.

Cargo loads and weight distribution can be approximated by step functions. In such cases, 7 to 13 usually suffice to model the loads with sufficient accuracy.

To model the primary bending moment, it is necessary to apply axial longitudinal loads at the both ends of each panel. As concentrated loads cannot be expanded with the Fourier series, these end loads are applied on a small zone on each side. The width of these zones is taken in LBR-5 as 1/20 of the span L. This is a compromise between computational effort and accuracy. The applied forces can differ between right and left end of the panel, because the primary bending moment changes along the hull girder. For such expansions, cosine Fourier series are used truncated typically after 7 to 13 terms.

4.4 STEP 4: Homogeneous solution of differential equation

From the solution of the 8^{th} order polynomial differential equation with a single variable (φ), Eq.(17), and keeping in mind that $w(x,\varphi) = w(\varphi)\sin\lambda x$, Eq.(19), we obtain:

$$w(x,\varphi) = [e^{\alpha_i q\varphi}(A_i\cos\beta_i q\varphi + B_i\sin\beta_i q\varphi) + e^{\alpha_i q(\varphi_0-\varphi)}(C_i\cos\beta_i q(2\pi-\varphi) + D_i\sin\beta_i q(2\pi-\varphi)) + ...] \cdot \sin(\lambda x) \quad (23)$$

with $i = 1$ to 2, 3, or 4.

If	then	i.e.	solutions
β_1 and $\beta_2 \neq 0$	$i = 1$ to 2	$(\alpha_1, \pm\beta_1)$, $(\alpha_2, \pm\beta_2)$	2 complex
$\beta_1 \neq 0$ and $\beta_2 = 0$	$i = 1$ to 3	$(\alpha_1, \pm\beta_1)$, $(\alpha_2, 0)$, $(\alpha_3, 0)$	1 complex, 2 real
$\beta_1 = 0$ and $\beta_2 \neq 0$	$i = 1$ to 3	$(\alpha_1, \pm\beta_1)$, $(\alpha_2, 0)$, $(\alpha_3, 0)$	1 complex, 2 real
$\beta_1 = \beta_2 = 0$	$i = 1$ to 4	$(\alpha_1, 0)$, $(\alpha_2, 0)$, $(\alpha_3, 0)$, $(\alpha_4, 0)$	4 real

A_i, B_i, C_i, D_i are the eight integration constants included in Eq.(23). These constants are determined through the boundary conditions (STEP 6).

For $u(\varphi)$ and $v(\varphi)$ similar equations can be written. The $u(\varphi)$ and $v(\varphi)$ equations contain other integration constants that depend directly on the eight integration constants of w (A_i, B_i, C_i, D_i.). This means that once these eight constants are fixed for w, the equations for u and v are also completely defined. In addition, using the 'resultant-displacement' relationships (like Eqs.(13)) the resultant and displacement derivatives (e.g. w° = slope) are also known. These will be required later (STEP6) to find the boundary forces to apply along the panel boundary edges ($\varphi = 0$ and $\varphi = \varphi_0$).

Eq.(23) for $w(\varphi)$ and those for $u(\varphi)$ and $v(\varphi)$ are the 'homogeneous solution' of the differential equations. The 'homogeneous solution' is our basic solution to determine the solution of the actual panel.

4.5 STEP 5: Superposition principle

At this stage it is valuable to resume briefly the general philosophy to solve analytically the governing differential equations of structures composed of cylindrical stiffened shells (plates):

A. We decompose (mesh modeling) the global structure in a series of stiffened cylindrical shells and stiffened plates, Fig.10.

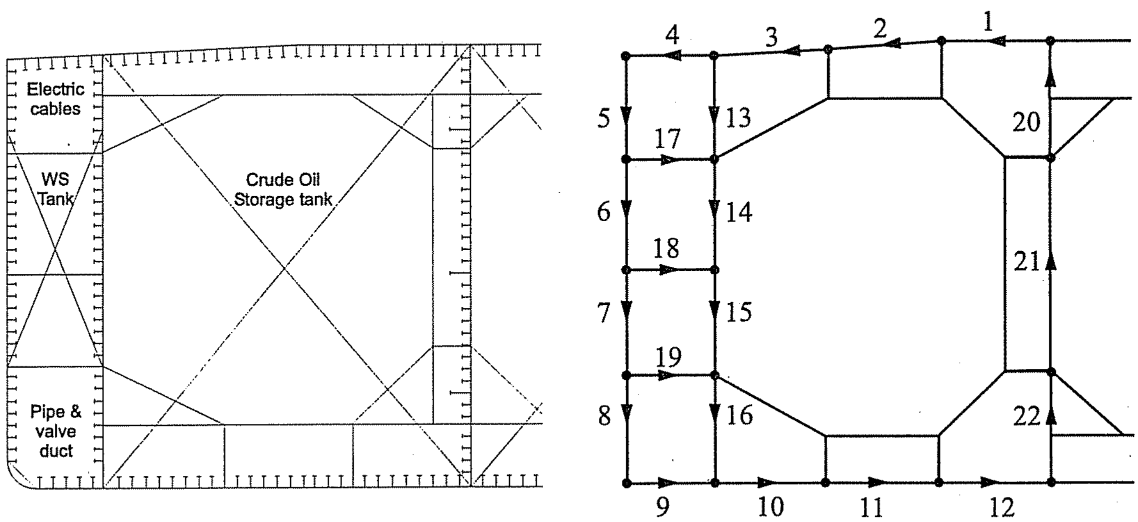

Fig.10: Modeling of the structure with stiffened panels

B. Using the displacement shape of the Fourier series expansion, we solve for each panel the governing differential equations without second member (homogeneous solution). At that stage, the $f(\varphi)$ concentrated terms of the differential equations, Eqs.(14) corresponding to the girders are discarded. For each panel, Eq.(23) gives the homogeneous solution, which includes the eight unknown integration constants.
This procedure (STEP 2 and STEP 8) is repeated for each term of the Fourier series expansion. At the end, the superposition principle is applied by summing all the solutions (one per term) to get the actual solution. The number of terms to use depends on the problem's complexity in terms of the load patterns. Usually 3 to 13 terms are required.
Each panel (cylindrical shell) is considered as a complete 360° cylinder (i.e. the shell opening angle is 360°). The actual opening angle φ_0 will be considered later.

C. Definition of the four 'basic unitary load lines': X_u, Y_u, Z_u, M_u. The principle is to find the eight integration constants for the four 'basic unitary load lines' applied on the complete cylinder. Four sets of integration constants are (one per unitary load line). The superposition principle allows then to find the solution (u,v,w) for the actual stiffened panels (actual opening angle φ_0 and loads) that compose the structure. The 4 'basic unitary load lines' applied on the complete cylinder are:

$$\begin{aligned} X_u &= 10000\cos(\lambda x) && [\text{N/m}] \\ Y_u &= 10000\sin(\lambda x) && [\text{N/m}] \\ Z_u &= 10000\sin(\lambda x) && [\text{N/m}] \\ M_u &= 10000\sin(\lambda x) && [\text{Nm/m}] \end{aligned} \tag{24}$$

Their forms are compatible with the Fourier series expansions of the actual loads. These unitary load lines are applied at $\varphi = 0$ (and $\varphi = 360°$).
For each of these 'unitary load lines', eight integration constants are obtained through the boundary conditions at $\varphi = 0$ and $\varphi = 360°$. To satisfy the boundary conditions, we can define four equations (equilibrium and/or compatibility). In addition, the symmetry or the anti-symmetry of the resultants and displacements induced by the load line provides four other equations. E.g. for the Z_u load case, the conditions are, Fig.11:

v	=	0.0	$\sin \lambda x$	in $\varphi = 0$ (per symmetry)
$N_{\varphi x}$	=	0.0	$\cos \lambda x$	in $\varphi = 0$ (per symmetry)
w°	=	0.0	$\sin \lambda x$	in $\varphi = 0$ (per symmetry)
R_φ	=	-5000	$\sin \lambda x$	in $\varphi = +\varepsilon$

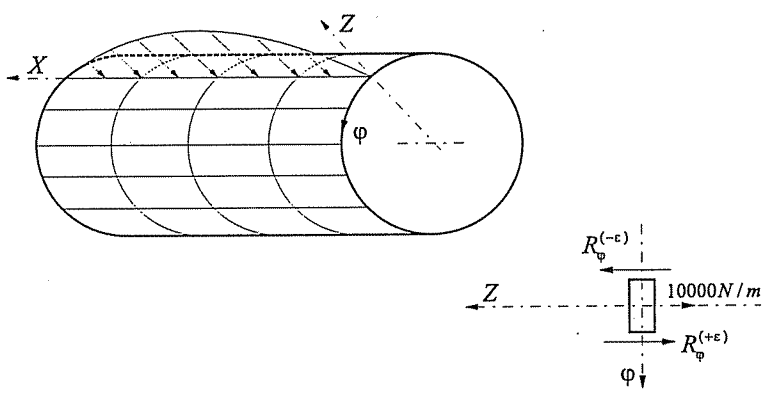

Fig.11: $Z_u = 9810 \sin \lambda x$ (N/m) basic unitary load line

At this stage, only the stiffeners and the frames are explicitly considered. The effects of the girders on the global solutions is added by introducing additional load lines (X_0, Y_0, Z_0, U_{0X}, and U_{0Z}), corresponding to the reactions between the panel and the girders. Considered as external loads, they are applied on the cylindrical shell (plate) as explained in STEP 7.
Lateral pressure (varying along $o\varphi$), the deadweight, and the longitudinal axial compression

(induced by the primary bending moment) can also be considered using the basic unitary load lines. The unitary load lines, Fig.12, are assumed to be applied on a small surface ($L \cdot dy$ or $L \cdot qd\varphi$) at $z = 0$ (L is the panel length along ox). Integrating the solutions obtained for the basic load lines according to the actual load distribution, we get the solutions (u, v and w) for a complete cylinder under the real load conditions. Only the girder contribution remains to be added (STEP 7).

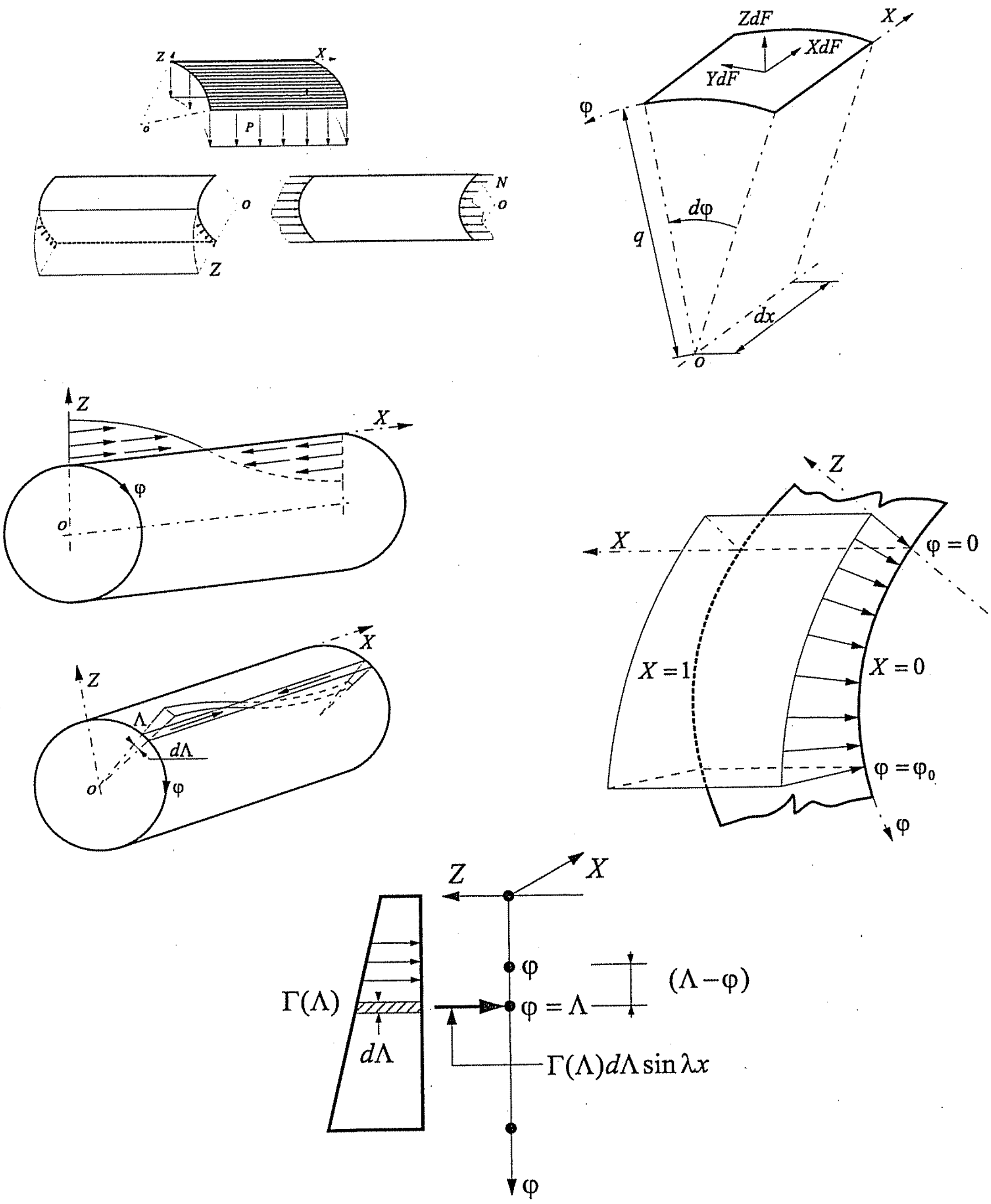

Fig.12: External loads (pressure, deadweight) and integration procedure

4.6 STEP 6: Actual panel

In order to get the solution of the real panel (for the actual shell opening angle φ_0) we have to consider the actual boundary conditions imposed along the two longitudinal edges ($\varphi = 0$ and $\varphi = \varphi_0$). To satisfy these boundary conditions, we apply along each edge a set of four basic load lines (X_u, Y_u, Z_u, and M_u). The problem is to find the amplitude of these load lines. For each panel, the unknowns are the 'edge amplification factors' of these load lines. Conditions to determine these 'edge amplification factors' are:

- For a free edge: $M_\varphi = N_\varphi = N_{x\varphi} = R_\varphi = 0$
- For a clamped edge: $w = v = u = dw/dy = 0$
- For a simply supported edge: $w = u = M_\varphi = N_\varphi = 0$
- For an edge (node) corresponding to the junction between two panels, we impose four compatibility conditions between the displacements of the two panels and four equilibrium equations.
- For an edge (node) corresponding to the junction between three panels, we impose eight compatibility conditions between the displacements of the three panels and four equilibrium equations.

The 'edge amplification factors' for all panels are determined in STEP 8. For a structure with N panels, there are $8 \cdot N$ unknowns corresponding to the eight 'edge multiplication factors' per panel. They are determined by solving a system of $8 \cdot N$ linear equations.

The equations (compatibility or equilibrium) at the panel edges require the displacements (u, v, w, w°) and the resultants (M_φ, N_φ, $N_{x\varphi}$, R_φ) acting along the edge ($\varphi = 0$) and the edge ($\varphi = \varphi_0$). These are determined for the nine 'standard loading cases', Fig.13:

- The actual external loads:
  - pressures (quasi-static): Z type
  - gravity loads (deadweight, cargo,...) having component along $o\varphi$ and along oz: Y and Z types
  - axial compression (induced by the primary bending moment) : X type
- the four basic unitary load lines (X_u, Y_u, Z_u, and M_u) acting at ($\varphi = 0$)
- the four basic unitary load lines (X_u, Y_u, Z_u, and M_u) acting at ($\varphi = \varphi_0$)

All these displacements and forces are calculated from the solutions of the homogeneous differential equations for the four basic load lines applied on the 360° cylinder (see STEPS 2 and 3).

4.7 STEP 7: Girders

In order to include the effects of the girders on the stiffened panels it is necessary to evaluate their individual contributions (each girder individually) on the complete cylinder submitted to the nine 'standard loading cases'. For each 'standard loading case' we establish the displacement compatibility and the force equilibrium between the girders and the cylindrical shell (which includes stiffeners and frames). So, a linear equations system has to be solved including $5 \cdot M_T$ equations, where M_T is the number of girders of the panel). There is one equation for each girder and for each girder load line (X_0dx, Y_0dx, Z_0dx, U_{0x}, U_{0z}). 'dx' is width of the strip where the X_0, Y_0, Z_0 girder forces act. It corresponds to the width where $f(\varphi) = 1$. Units of X_0, Y_0, Z_0 are [N/m^2] and those of X_0dx, Y_0dx, Z_0dx are [N/m]. We obtain from Eq.(14):

$$
\begin{array}{lcll}
 & & & \text{if flange shear neglected} \\
X_0\,dx & = & [\Omega_x^t u'' - H_x^t w + S_x^t(v^{\circ\prime} + u^{\circ\circ})]\,dx & = [\Omega_x^t u'' - H_x^t w]\,dx \\
Y_0\,dx & = & [S_x^t(v'' + u^{\circ\prime})]\,dx & = 0 \\
Z_0\,dx & = & [T_x^t w^{\circ\circ\prime\prime} - H_x^t u''' + R_x^t w'''' + L_x^t(v^{\circ\prime\prime} + u^{\circ\circ\prime})]\,dx & = [T_x^t w^{\circ\circ\prime\prime} - H_x^t u''' + R_x^t w'''']\,dx \\
U_{0x} & = & [T_x^t w^{\circ\prime\prime} + L_x^t(v'' + u^{\circ\prime})]\,dx & = [T_x^t w^{\circ\prime\prime}]\,dx \\
U_{0z} & = & [S_x^t(v' + u^\circ)]\,dx & = 0
\end{array}
\qquad (25)
$$

The X_0dx, Y_0dx, Z_0dx, U_{0x} load lines are proportional to, respectively, X_U, Y_U, Z_U, M_U. No load line is defined to represent U_{0z}, because U_{0z} is here discarded as negligible. However, since in principle U_{0z} can also be considered, it is retained in the following formulae.

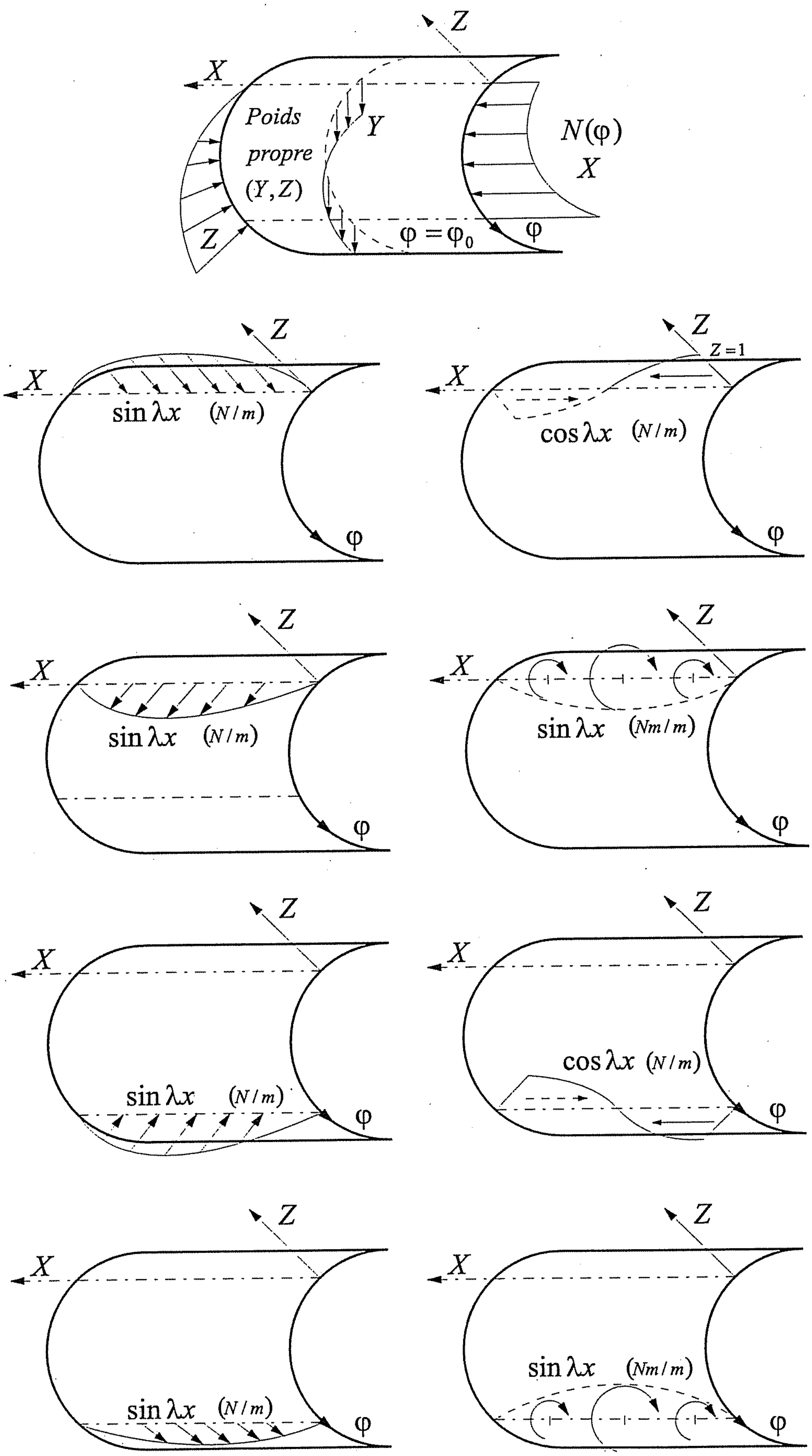

Fig.13: The nine 'standard loading cases' (applied on the 360° cylinder)
'poids propre' = own weight

Let us consider:

- $\varphi = Y$, the coordinate of the current line where displacements are calculated
- $\varphi = \Lambda$, the coordinate of the current line where a load line is applied
- M_T girders located at $\Lambda = K_j$, $j = 1...M_T$
- $\Psi_{Xo}(K_j)$, $\Psi_{Yo}(K_j)$, $\Psi_{Zo}(K_j)$, $\Psi_{Uox}(K_j)$, $\Psi_{Uoz}(K_j)$ the unknown amplitudes of the X_0dx, Y_0dx, Z_0dx, U_{0x}, U_{0z} girder load lines (located at $\Lambda = K_j$),
- $U_o(\varphi)$, $V_o(\varphi)$, $W_o(\varphi)$ the shell (plate) displacements due to one 'standard loading case'
- $U_{Xo}(\varphi)$, $V_{Xo}(\varphi)$, $W_{Xo}(\varphi)$ the shell (plate) displacements due to a unitary load ($X_u = 10000\cos\lambda x$) that is similar to X_0dx (at one constant)
- $U_{Yo}(\varphi)$, $V_{Yo}(\varphi)$, $W_{Yo}(\varphi)$ the shell (plate) displacements due to a unitary load ($Y_u = 10000\sin\lambda x$) that is similar to Y_0dx (at one constant)
- similarly for $[U_{Zo}(\varphi), V_{Zo}(\varphi), W_{Zo}(\varphi)]$, $[U_{Uox}(\varphi), V_{Uox}(\varphi), W_{Uox}(\varphi)]$ and $[U_{Uoz}(\varphi), V_{Uoz}(\varphi), W_{Uoz}(\varphi)]$
- $(X_0dx)_{Xo,Y=K_i,\Lambda=K_j}$, $(Y_0dx)_{Xo,Y=K_i,\Lambda=K_j}$, $(Z_0dx)_{Xo,Y=K_i,\Lambda=K_j}$, $(U_{0x})_{Xo,Y=K_i,\Lambda=K_j}$, and $(U_{0z})_{Xo,Y=K_i,\Lambda=K_j}$ are the values of the girder load lines at the location of the 'i' girder ($Y = K_i$) due to a unitary load $X_u = 10000\cos\lambda x$ applied at the location of the 'j' girder ($\Lambda = K_j$). They are calculated by replacing the displacements $[U_{Xo}(\varphi), V_{Xo}(\varphi), W_{Xo}(\varphi)]$ in Eq.(25) for $\varphi = K_i - K_j$
- $(X_0dx)_{Yo,Y=K_i,\Lambda=K_j}$, $(Y_0dx)_{Yo,Y=K_i,\Lambda=K_j}$, $(Z_0dx)_{Yo,Y=K_i,\Lambda=K_j}$, etc. are the girder load lines at the location of the 'i' girder ($Y = K_i$) due to a unitary load $Y_u = 10000\sin\lambda x$ applied at the location of the the 'j' girder ($\Lambda = K_j$). They are calculated by replacing the displacements $[U_{Yo}(\varphi), V_{Yo}(\varphi), W_{Yo}(\varphi)]$ in Eq.(25) for $\varphi = K_i - K_j$.
- similarly for $(X_0dx)_{Zo,Y=K_i,\Lambda=K_j}$, $(Y_0dx)_{Zo,Y=K_i,\Lambda=K_j}$, $(Z_0dx)_{Zo,Y=K_i,\Lambda=K_j}$, etc.
  $(X_0dx)_{Uox,Y=K_i,\Lambda=K_j}$, $(Y_0dx)_{Uox,Y=K_i,\Lambda=K_j}$, $(Z_0dx)_{Uox,Y=K_i,\Lambda=K_j}$, etc.
  $(X_0dx)_{Uoz,Y=K_i,\Lambda=K_j}$, $(Y_0dx)_{Uoz,Y=K_i,\Lambda=K_j}$, $(Z_0dx)_{Uoz,Y=K_i,\Lambda=K_j}$, etc.
- $(X_0dx)_{0,Y=K_i}$, $(Y_0dx)_{0,Y=K_i}$, $(Z_0dx)_{0,Y=K_i}$, $(U_{0x}dx)_{0,Y=K_i}$, and $(U_{0z}dx)_{0,Y=K_i}$ are the values of the girder load lines at the location of the 'i' girder ($Y = K_i$) due to the considered 'standard loading case'. They are calculated by replacing the displacements $[U_0(\varphi), V_0(\varphi), W_0(\varphi)]$ in Eq.(25) for $\varphi = K_i$.

Based on this a typical equation is:

$$\begin{aligned}
\Psi_{Xo}(K_i) = (X_0dx)_{0,Y=K_i} \quad & + \sum_{j=1}^{MT} \frac{\Psi_{Xo}(K_i)}{10000}(X_0dx)_{Xo,Y=K_i,\Lambda=K_j} \\
& + \sum_{j=1}^{MT} \frac{\Psi_{Yo}(K_i)}{10000}(X_0dx)_{Yo,Y=K_i,\Lambda=K_j} \\
& + \sum_{j=1}^{MT} \frac{\Psi_{Zo}(K_i)}{10000}(X_0dx)_{Zo,Y=K_i,\Lambda=K_j} + ... \\
\Psi_{Zo}(K_i) = (Z_0dx)_{0,Y=K_i} \quad & + \sum_{j=1}^{MT} \frac{\Psi_{Xo}(K_i)}{10000}(Z_0dx)_{Xo,Y=K_i,\Lambda=K_j} \\
& + \sum_{j=1}^{MT} \frac{\Psi_{Yo}(K_i)}{10000}(Z_0dx)_{Yo,Y=K_i,\Lambda=K_j} + ...
\end{aligned} \tag{26}$$

In principle the five components should be considered, but the LBR-5 program only considers the X_0 and Z_0 components, assuming the three others as negligible. The rigidity matrix of the system Eq.(26) does not change with the load case. So the system has to be solved only once but with none different sets of independent terms. For LBR-5, one set of (X_0, Z_0) is obtained for each 'standard loading case'.

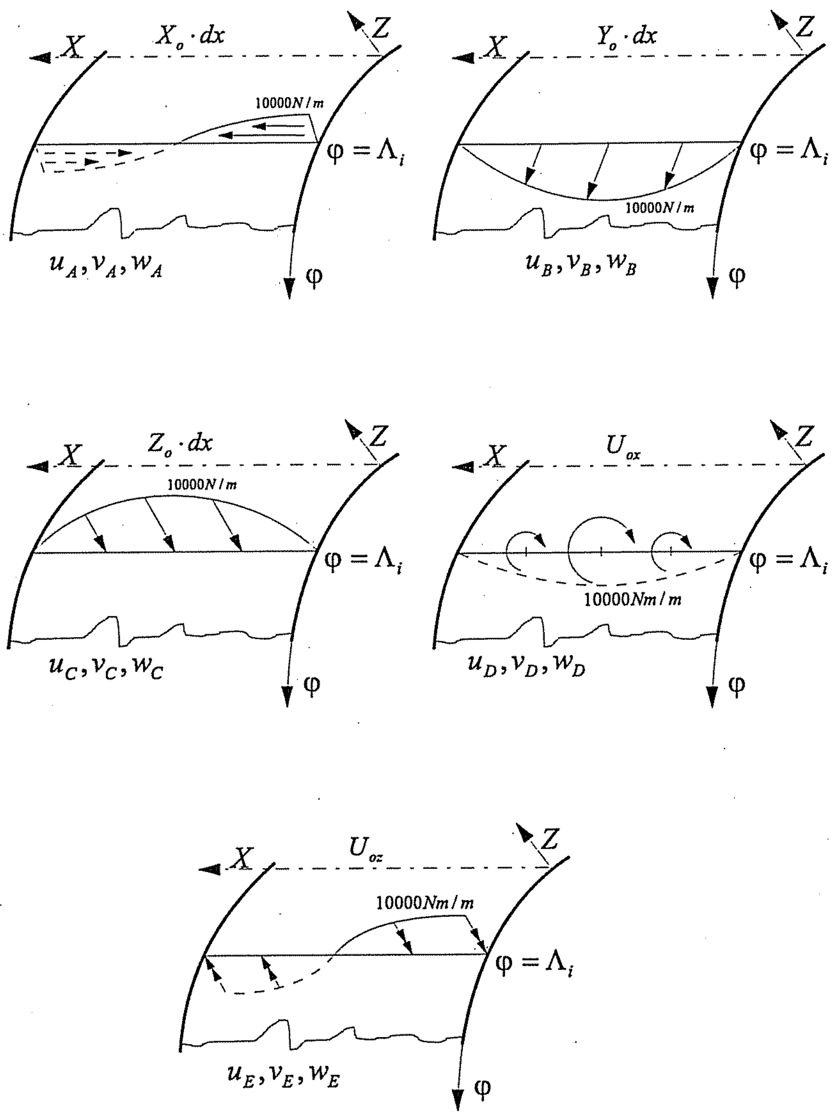

Fig.14: The five 'girder unitary load lines'

In conclusion, one needs for each girder nine sets of (X_0, Z_0) 'girder load lines'. Each set corresponds to one of the nine 'standard loading cases'. Then, these girder load lines are applied on the 360° cylindrical shell in addition to the standard loading cases in order to determine the displacements (u, v, w, w°) and the resultants (N_φ, M_φ, $N_{\varphi x}$, R_φ) along the two edges ($\varphi = 0$ and $\varphi = \varphi_0$).

Having now determined along the boundary edges of each panel all the displacements and resultants (including the girder contribution), we can solve the global system for the complete structure and get the amplification factors (STEP 6) to apply at the eight 'unitary load lines' on each panel (STEP 8).

4.8 STEP 8: Final solution

At the final stage, each panel is a 360° cylindrical shell including stiffener, frame, and girder contributions. For these panels we know the displacements (u, v, w, w°) and the resultants (N_φ, M_φ, $N_{\varphi x}$, R_φ) along their two boundary edges ($\varphi = 0$ and $\varphi = \varphi_0$) for the nine standard loading cases. To satisfy the actual boundary conditions of each panel, we determine the 'amplification factors' of the four 'unitary load lines' applied at $\varphi = 0$ and the four 'unitary load lines' applied at $\varphi = \varphi_0$. This is done through the compatibility and the equilibrium equations between panels, see STEP6. By solving the global system including all these equations (8 per panel) we get the amplification factors.

Then, the final solutions (u, v, w) of a panel of the structure is obtain by adding nine different solutions of the same 360° cylindrical panel (including stiffeners, frames and girders):

- the 360° cylindrical panel under actual external loads,
- at $\varphi = 0$, the 360° cylindrical panel under the X_u, Y_u, Z_u, and M_u 'unitary load lines' multiplied by their respective 'amplification factor',
- at $\varphi = \varphi_0$, the 360° cylindrical panel under the X_u, Y_u, Z_u, and M_u 'unitary load lines' multiplied by their respective 'amplification factor'.

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