A refinement of the S^ν-based multifractal formalism

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LIÈGE université

Road map

- The context
- The S^ν spaces and their generalization
- In practice
- More evolved examples
- A real life application

Information concerning the global smoothness of a signal can be grasped via its Hölder spectrum, which relies on the Hölder spaces.

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A locally bounded function f belongs to $\Lambda^\alpha(x_0)$ (with $\alpha \geq 0$ and $x_0 \in \mathbb{R}^n$) if there exist a constant C and a polynomial P_{x_0} of degree less than α such that

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in a neighborhood of x_0.

The Hölder exponent of f at x_0 is defined as

$$h_f(x_0) = \sup\{\alpha \geq 0 : f \in \Lambda^\alpha(x_0)\}.$$

The sample path $B = \{B_x\}_{x\in\mathbb{R}}$ of a Brownian motion belongs to the Hölder space $\Lambda^{1/2-\epsilon}(\mathbb{R})$ almost surely for any $\epsilon > 0$, but not to $\Lambda^{1/2}(\mathbb{R})$.

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The Khintchin law of the iterated logarithm implies that for almost every $x_0 \in \mathbb{R}$, there exists a constant $C > 0$ such that, for any x in a neighborhood of x_0, one has

$$|B_{x_0} - B_x| \leq C|x_0 - x|^{1/2} w(|x - x_0|),$$

with $w(h) = \sqrt{|\log|\log h^{-1}||}$.

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Is it possible to numerically detect this correction w?

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Under some general assumptions, there exist a function ϕ and $2^n - 1$ functions $(\psi^{(i)})_{1 \leq i < 2^n}$, called wavelets, such that

$$\{\phi(x-k) : k \in \mathbb{Z}^n\} \cup \{\psi^{(i)}(2^j x - k) : 1 \leq i < 2^n, k \in \mathbb{Z}^n, j \in \mathbb{N}\}$$

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Any function $f \in L^2(\mathbb{R}^n)$ can be decomposed as follows,

$$f(x) = \sum_{k \in \mathbb{Z}^n} C_k \phi(x-k) + \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{Z}^n} \sum_{1 \leq i < 2^n} c_{j,k}^{(i)} \psi^{(i)}(2^j x - k),$$

where

$$c_{j,k}^{(i)} = 2^{nj} \int_{\mathbb{R}^n} f(x) \psi^{(i)}(2^j x - k) \, dx$$

and

$$C_k = \int_{\mathbb{R}^n} f(x) \phi(x-k) \, dx.$$

On the torus $\mathbb{R}^n/\mathbb{Z}^n$, we will use the periodized wavelets

$$\psi_p^{(i)}(2^j x - k) = \sum_{l \in \mathbb{Z}^n} \psi^{(i)}(2^j(x-l)-k) \quad (j \in \mathbb{N},\ k \in \{0, \dots, 2^j-1\}^n)$$

to form a basis of the one-periodic functions on $\mathbb{R}^n$ which locally belong to $L^2(\mathbb{R}^n)$.

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We will write $c_{j,k}$ instead of $c_{j,k}^{(i)}$; the sequence $(c_{j,k})$ will be denoted by c.

In this talk, ν will refer to

- a right-continuous increasing function
- for which there exists $\alpha_{\min} \in \mathbb{R}$ such that
$$\nu(\alpha) \in \begin{cases} \{-\infty\} & \text{if } \alpha < \alpha_{\min} \\ [0, n] & \text{if } \alpha \geq \alpha_{\min}. \end{cases}$$

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With these notations being fixed, one define the S^ν space as follows:

$$S^\nu = \{c : \forall \alpha \in \mathbb{R}\ \forall \epsilon > 0\ \forall C > 0$$
$$\exists J > 0\ \forall j \geq J,\ \#E_j(C, \alpha)(c) \leq 2^{(\nu(\alpha)+\epsilon)j}\},$$

where

$$E_j(C, \alpha)(c) = \{k : |c_{j,k}| \geq C2^{-\alpha j}\}.$$

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If one considers the wavelet coefficients $c_{j,k}$ as a sequence, the space S^ν is a sequence space and one can study its topological properties.

Definition

For any $\alpha \in \mathbb{R}$, let $\sigma^{(\alpha)} = (\sigma_j^{(\alpha)})_{j\in\mathbb{N}}$ be a sequence of positive real numbers. We define

$$S^{\nu,\sigma^{(\cdot)}} = \{c : \forall \alpha \in \mathbb{R}\ \forall \epsilon > 0\ \forall C > 0 \\ \exists J > 0\ \forall j \geq J,\ \# E_j(C, \sigma^{(\alpha)})(c) \leq 2^{(\nu(\alpha)+\epsilon)j}\},$$

where

$$E_j(C, \sigma^{(\alpha)})(c) = \{k : |c_{j,k}| \geq C\sigma_j^{(\alpha)}\}.$$

For $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R} \cup \{-\infty\}$, we first define the metric spaces $(E(\sigma^{(\alpha)}, \beta), d_{\sigma^{(\alpha)},\beta})$ by

$$E(\sigma^{(\alpha)}, \beta) = \{c : \exists C, C' > 0 \ \# E_j(C, \sigma^{(\alpha)}) \leq C' 2^{\beta j} \text{ for any } j \in \mathbb{N}\}$$

and set

$$d_{\sigma^{(\alpha)},\beta}(c,d) = \inf\{C + C' : C, C' \geq 0$$
$$\# E_j(C, \sigma^{(\alpha)})(c-d) \leq C' 2^{\beta j} \text{ for any } j \in \mathbb{N}\}.$$

Proposition

We have the following properties:

1. the space $E(\sigma^{(\alpha)}, \beta)$ is a vector space,
2. the sum is a continuous operation in $(E(\sigma^{(\alpha)}, \beta), d_{\sigma^{(\alpha)},\beta})$, while the product is not necessarily continuous,
3. the metric $d_{\sigma^{(\alpha)},\beta}$ is invariant by translation and satisfies the inequality for any $\lambda \in \mathbb{C}$,
   $d_{\sigma^{(\alpha)},\beta}(\lambda c, 0) \leq \sup\{1, |\lambda|\} d_{\sigma^{(\alpha)},\beta}(c, 0)$,
4. if $\beta' \leq \beta$ and if there exists $J \in \mathbb{N}$ such that $\sigma_j^{(\alpha')} \leq \sigma_j^{(\alpha)}$ for any $j \geq J$, then $E(\sigma^{(\alpha')}, \beta')$ is included in $E(\sigma^{(\alpha)}, \beta)$,
5. suppose that $\sigma_j^{(\alpha')}/\sigma_j^{(\alpha)} \to 0$ as $j \to +\infty$ and $\beta' < \beta$. If the sequence $(\lambda_m)_{m\in\mathbb{N}}$ converges to λ in $\mathbb{C}$ and if $(c^{(m)})_{m\in\mathbb{N}}$ is a sequence of $E(\sigma^{(\alpha)}, \beta)$ which converges to $c \in E(\sigma^{(\alpha')}, \beta')$ for $d_{\sigma^{(\alpha)},\beta}$, then the sequences $(\lambda_m c^{(m)})_{m\in\mathbb{N}}$ converges to λc for $d_{\sigma^{(\alpha)},\beta}$.

Proposition

The space $E(\sigma^{(\alpha)}, \beta)$ is complete.

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Theorem

Suppose that $\alpha < \alpha'$ implies $\sigma_j^{(\alpha')}/\sigma_j^{(\alpha)} \to 0$ as $j \to +\infty$. For any sequence $(\alpha_n)_{n\in\mathbb{N}}$ dense in $\mathbb{R}$ and any sequence $(\epsilon_m)_{m\in\mathbb{N}}$ of strictly positive real numbers which converges to 0, we have

$$S^{\nu,\sigma^{(\cdot)}} = \bigcap_{m\in\mathbb{N}} \bigcap_{n\in\mathbb{N}} E(\sigma^{(\alpha_n)}, \nu(\alpha_n) + \epsilon_m).$$

Theorem

Under the hypothesis of the previous Theorem, if we set

$$d_{m,n} = d_{\sigma(\alpha_n),\nu(\alpha_n)+\epsilon_m},$$

then the application

$$d : (c,d) \in S^{\nu,\sigma^{(\cdot)}} \times S^{\nu,\sigma^{(\cdot)}} \mapsto \sum_{m=1}^{+\infty} \sum_{n=1}^{+\infty} \frac{1}{2^{m+n}} \frac{d_{m,n}(c,d)}{1 + d_{m,n}(c,d)}$$

is a metric on $S^{\nu,\sigma^{(\cdot)}}$.

This application is invariant by translation and the space $(S^{\nu,\sigma^{(\cdot)}}, d)$ is a complete topological vector space. The induced topology is independent of the sequences $(\alpha_n)_{n\in\mathbb{N}}$ and $(\epsilon_m)_{m\in\mathbb{N}}$.

Definition

The generalized profile of a sequence c is defined by

$$\nu_{c,\sigma^{(\cdot)}} : \alpha \in \mathbb{R} \mapsto \lim_{\epsilon \to 0^+} \limsup_{j \to +\infty} \frac{\log \# E_j(1, \sigma^{(\alpha+\epsilon)})(c)}{\log 2^j}.$$

This definition is well-founded if we suppose that for any $\alpha < \alpha'$ there exists $J \in \mathbb{N}$ such that $\sigma_j^{(\alpha')} \leq \sigma_j^{(\alpha)}$ for any $j \geq J$.

Theorem

Suppose that $\alpha < \alpha'$ implies $\sigma_j^{(\alpha')}/\sigma_j^{(\alpha)} \to 0$ as $j \to +\infty$. We have the following properties:

1. the function $\nu_{c,\sigma^{(\cdot)}}$ is right-continuous and increasing; moreover, we have $\nu_{c,\sigma^{(\cdot)}}(\alpha) \in [0, n] \cup \{-\infty\}$,
2. the constant 1 appearing in the definition of $\nu_{c,\sigma^{(\cdot)}}$ is arbitrary,
3. a sequence c belongs to $S^{\nu,\sigma^{(\cdot)}}$ if and only if $\nu_{c,\sigma^{(\cdot)}}(\alpha) \leq \nu(\alpha)$ for any $\alpha \in \mathbb{R}$,
4. if for any $\alpha < \beta$, we have $\sigma_j^{(\beta)} < \sigma_j^{(\alpha)}$ for any $j \in \mathbb{N}$, then there exists $c \in S^{\nu,\sigma^{(\cdot)}}$ such that $\nu_{c,\sigma^{(\cdot)}} = \nu$.

Theorem

Suppose that $\alpha < \alpha'$ implies $\sigma_j^{(\alpha')}/\sigma_j^{(\alpha)} \to 0$ as $j \to +\infty$. If for any $\alpha \in \mathbb{R}$, the sequence $\sigma^{(\alpha)}$ is admissible, then $S^{\nu,\sigma^{(\cdot)}}$ is a linear robust space.

Besides, for any $c \in S^{\nu,\sigma^{(\cdot)}}$, the function $\nu_{c,\sigma^{(\cdot)}}$ is robust, i.e. $\nu_{c,\sigma^{(\cdot)}} = \nu_{Ac,\sigma^{(\cdot)}}$ for any quasidiagonal matrix A.

Definition

Let σ be an admissible sequence and $0 < p, q \leq \infty$. The discrete counterpart of the generalized Besov space $B^{\sigma}_{p,q}([0,1]^n)$ is defined by

$$b^{\sigma}_{p,q} = \{c : \left(\sum_{i \in \{0,\ldots,2^n-1\},\ j \in \mathbb{N}} (\sigma_j 2^{-jn/p})^q \left(\sum_{k \in \{0,\ldots,2^j-1\}^d} |c_{j,k}|^p \right)^{q/p} \right)^{1/q} < \infty\}$$

with the usual modification if $p = \infty$ and/or $q = \infty$.

Theorem

For any $\alpha \in \mathbb{R}$, let $\sigma^{(\alpha)}$ be an admissible sequence and let us suppose that

- $\alpha < \alpha'$ implies $\sigma_j^{(\alpha')}/\sigma_j^{(\alpha)} \to 0$ as $j \to +\infty$,
- $\overline{s}(\sigma^{(\alpha)}) \to -\infty$ as $\alpha \to +\infty$.

For any $p > 0$, let $\theta^{(p)}$ be an admissible sequence. We have

$$S^{\nu,\sigma^{(\cdot)}} \subseteq \bigcap_{p>0} \bigcap_{\epsilon>0} b_{p,\infty}^{(\theta_j^{(p)} 2^{-j\epsilon/p})_j}$$

if and only if for any $p, \epsilon > 0$ and for any $\alpha \geq \alpha_{\min}$, there exists $C > 0$ such that

$$\theta_j^{(p)} 2^{-j\epsilon/p} \leq C 2^{jn/p} 2^{-j\nu(\alpha)/p} (\sigma_j^{(\alpha)})^{-1},$$

for any j.

Definition

The function $\tilde{\nu}$ is defined by

$$\tilde{\nu}(\alpha) = \begin{cases} \lim\limits_{\eta \to 0^+} \inf\limits_{p>0} \limsup\limits_{j \to +\infty} n - p \dfrac{\log(\theta_j^{(p)} \sigma_j^{(\alpha+\eta)})}{\log 2^j} & \text{if } \alpha \geq \alpha_{\min} \\ -\infty & \text{else} \end{cases} .$$

Theorem

Under the hypothesis of the previous Theorem, if $\tilde{\nu} \leq n$ and if for any $\alpha < \alpha_{\min}$, there exist $p, \epsilon > 0$ such that $2^{-jn/p}\sigma_j^{(\alpha)}\theta_j^{(p)}2^{-j\epsilon/p} \to +\infty$ as $j \to +\infty$, then we have

$$\bigcap_{p>0}\bigcap_{\epsilon>0} b_{p,\infty}^{(\theta_j^{(p)}2^{-j\epsilon/p})_j} \subset S^{\tilde{\nu},\sigma^{(\cdot)}}.$$

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Corollary

Under the hypothesis of the previous theorem, if for any $p, \epsilon > 0$ and for any $\alpha \geq \alpha_{\min}$, there exists $C > 0$ such that

$$\theta_j^{(p)}2^{-j\epsilon/p} \leq C2^{jn/p}2^{-j\tilde{\nu}(\alpha)/p}(\sigma_j^{(\alpha)})^{-1},$$

for any j and if for any $\alpha < \beta$, we have $\sigma_j^{(\beta)} \leq \sigma_j^{(\alpha)}$ for any j, then we have

$$S^{\nu,\sigma^{(\cdot)}} = \bigcap_{p>0}\bigcap_{\epsilon>0} b_{p,\infty}^{(\theta_j^{(p)}2^{-j\epsilon/p})_j}$$

if and only if $\nu = \tilde{\nu}$.

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We approximate $\nu_{c,\sigma^{(\cdot)}}(\alpha)$ with the slope of

$$j \mapsto \frac{\log \# E_j(C, \sigma^{(\alpha+\epsilon)})(c)}{\log 2},$$

for large values of j as soon as $\alpha \geq \alpha_{\mathrm{min}}$.

This slope will be denoted $\nu^{C}_{c,\sigma^{(\cdot)}}(\alpha)$.

In practice, the constant C is not arbitrary because we only have access to a finite number of wavelet coefficients.

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If the typical value of these coefficients is too large (resp. too small) with respect to C, not enough (resp. too many) of them will be taken into account; the detected value of $\nu^{C}_{c,\sigma(\cdot)}(\alpha)$ will thus be very different from the theoretical value $\nu_{c,\sigma(\cdot)}(\alpha)$.

Consequently, for a fixed α, we construct the function

$$C > 0 \mapsto \nu^{C}_{c,\sigma(\cdot)}(\alpha)$$

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If $\alpha < \alpha_{\min}$, this function should be decreasing. If $\alpha \geq \alpha_{\min}$, there should exist an interval I for which the values $\nu^{C}_{c,\sigma(\cdot)}(\alpha)$ with $C \in I$ are close to each other.

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We use a gradient descent to detect this interval.

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We use a gradient descent to detect this interval.

We chose the length of the interval I to be at least the median of the values $|c_{j,k}|/\sigma_j^{(\alpha)}$ (the worthwhile wavelet coefficients $c_{j,k}$ satisfy $|c_{j,k}|/\sigma_j^{(\alpha)} \geq C$).

We intend to build a function f with a prescribed Hölder exponent $h_f(x_0)$ at every point x_0 for which there exists a function w such that

$$|f(x_0) - f(x)| \leq C|x - x_0|^{h_f(x_0)} w(|x - x_0|),$$

for any x in a neighborhood of x_0.

Such a function f does not belong to $\Lambda^{h_f(x_0)}(x_0)$.

Let us denote by $\underline{\mathcal{H}}_K$ the set of the functions from $[0,1]$ to the compact K which are the lower limit of a sequence of continuous functions. For any $H \in \underline{\mathcal{H}}_K$, there exists a sequence $(Q_j)_{j\in\mathbb{N}}$ of polynomials such that

$$\left\{ \begin{array}{ll} H(t) = \liminf_{j\to+\infty} Q_j(t) & \forall t \in [0,1] \\ ||Q_j'||_\infty \leq j & \forall j \in \mathbb{N} \end{array} \right. , \qquad (1)$$

We have a similar result if one replaces the lower limit by a limit in the definition of $\underline{\mathcal{H}}_K$. In this case, the set is denoted by $\mathcal{H}_K$ and the lower limit in relation (1) becomes a limit.

Proposition

Let $K \subset (0,1)$ be a compact set, $H \in \mathcal{H}_K$ and $(Q_j)_{j\in\mathbb{N}}$ be a sequence of polynomials satisfying Relations (1), where the lower limit is replaced by a limit. For any $(j,k) \in \mathbb{N} \times \{0,\dots,2^j-1\}$, set

$$H_{j,k} = \max\left(\frac{1}{\log j}, Q_j\left(\frac{k}{2^j}\right)\right).$$

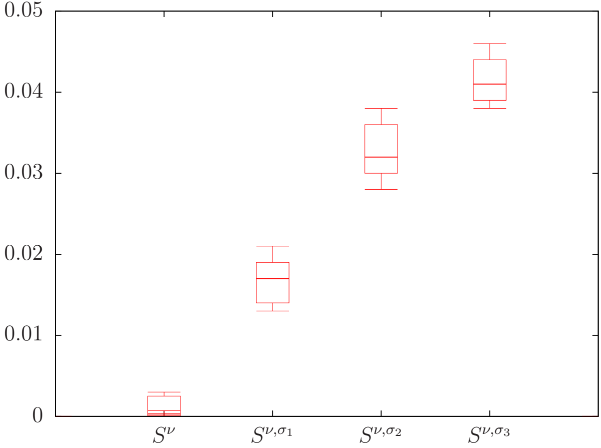
If $(a_j)_{j\in\mathbb{N}}$ is a real sequence such that $\lim_{j\to+\infty} \frac{\log a_j}{\log 2^{-j}} = 0$, then the function f defined as

$$f(x) = \sum_{j=0}^{+\infty} \sum_{k=0}^{2^j-1} 2^{-H_{j,k}j} a_j \psi_{j,k}(x)$$

satisfies $h_f(x) = H(x)$ for any $x \in [0,1]$.

First test: a monofractal function
with 20 simulations and Hölder exponent $H \in \{0.3, 0.35, \ldots, 0.7\}$.

First case: no correction ($w = 1$)



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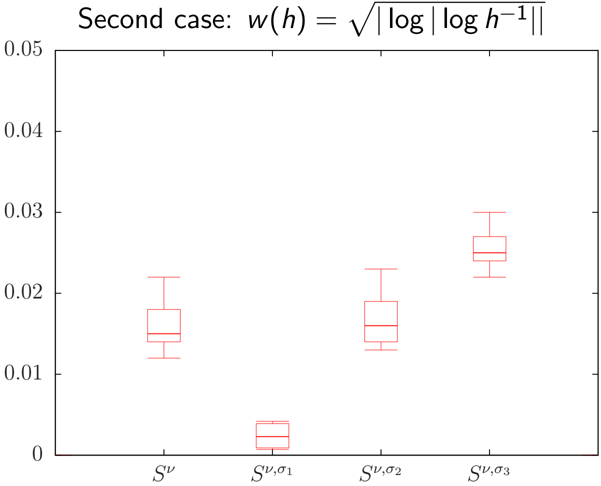

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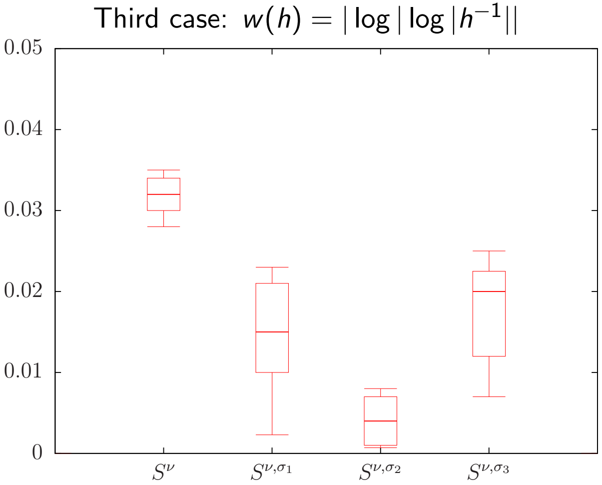

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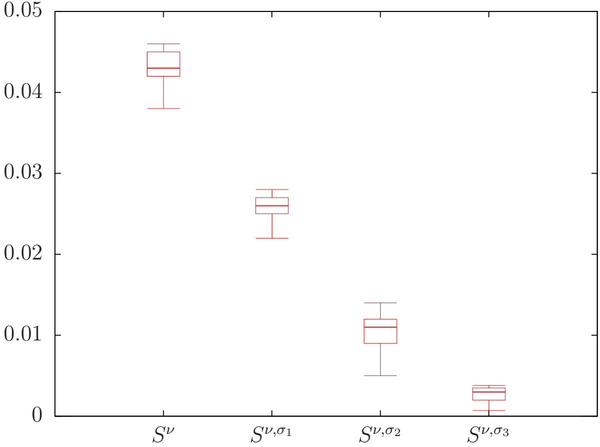
Fourth case: $w(h) = \sqrt{|\log h^{-1}|}$



Second test: a bifractal function
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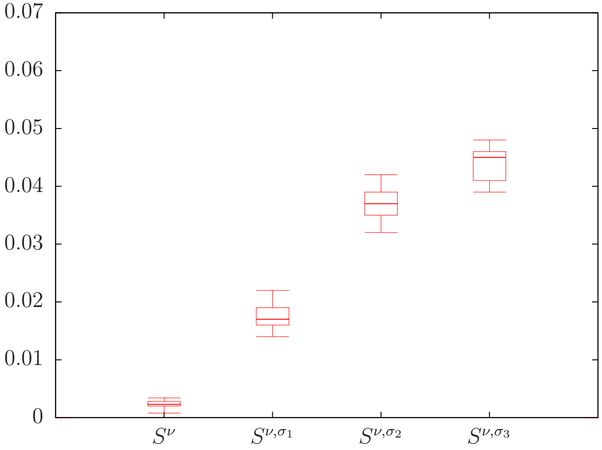

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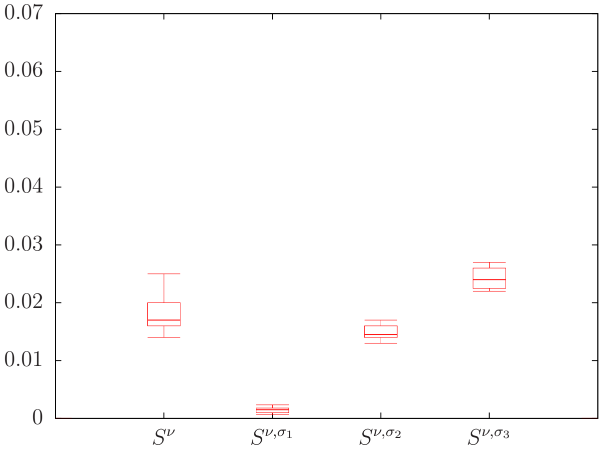

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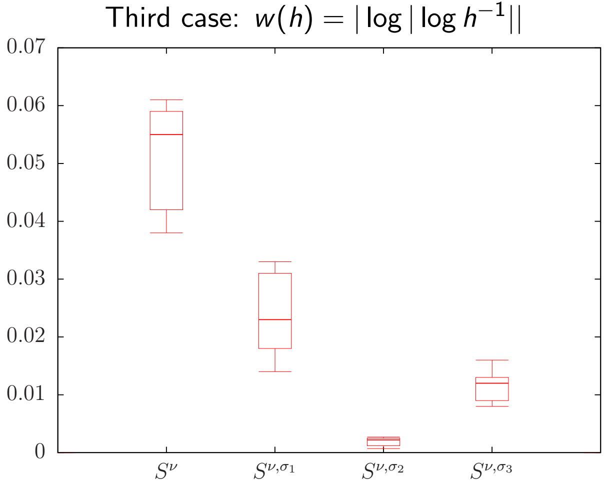

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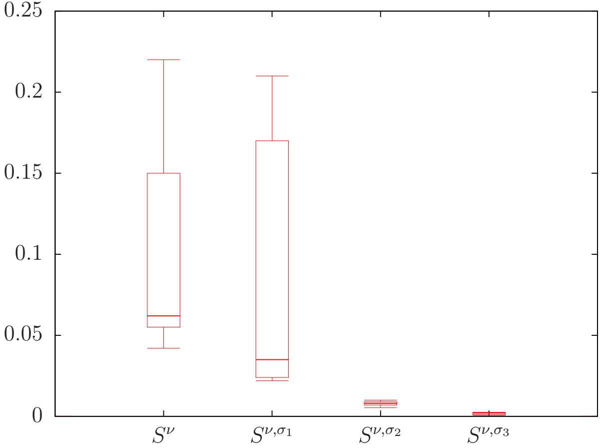

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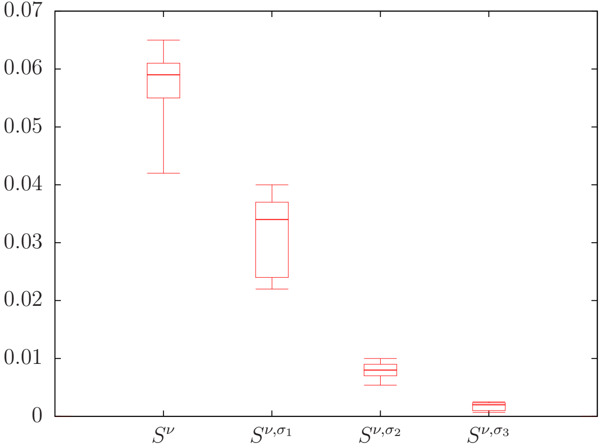
Fourth case: $w(h) = \sqrt{|\log h^{-1}|}$ with $|H_1 - H_2| > 0.3$



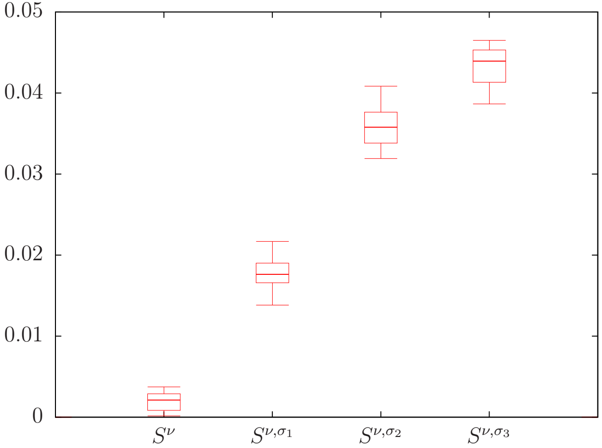
Third test: a multifractal function such that

$$H(x) = \begin{cases} \frac{c-a}{b}x + a & \text{if } x < b \\ c & \text{if } x \geq b \end{cases},$$

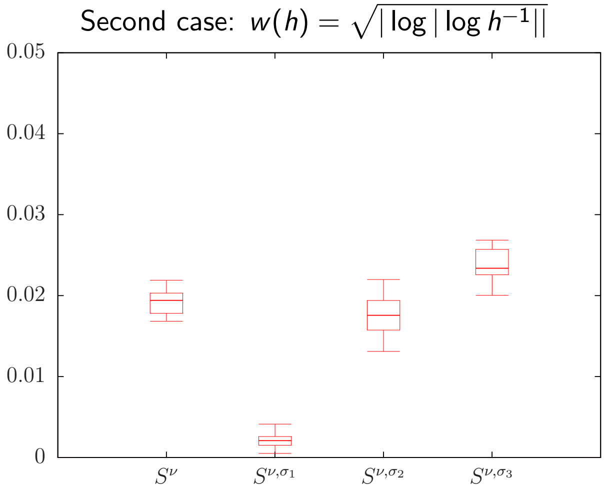
with $a \in \{0, 0.1, \ldots 0.5\}$, $b \in \{0.1, 0.2, \ldots, 0.5\}$ and $c \in \{0.2, 0.3, \ldots, 0.8\}$ $(a < c)$.

Third test: a multifractal function with 20 simulations.

First case: no correction ($w = 1$)



Third test: a multifractal function with 20 simulations.



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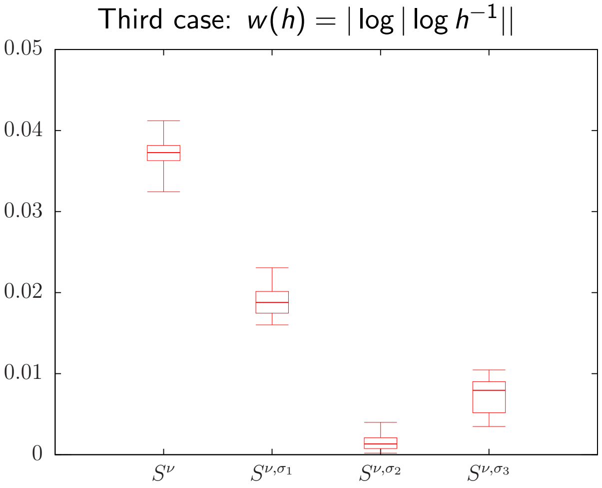

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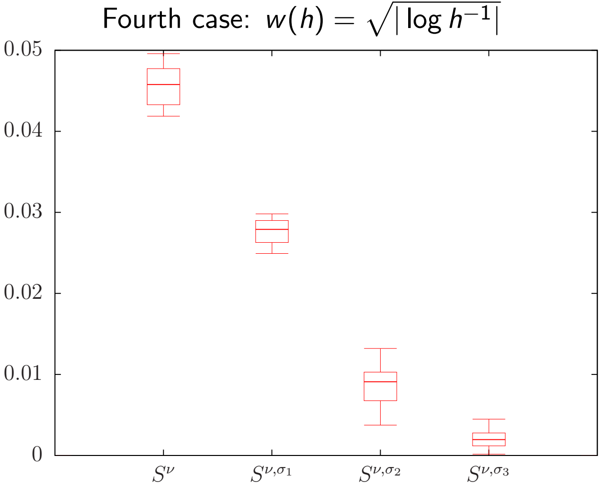

Road map

The Weierstraß function

$$W(x) = \sum_{j=0}^{\infty} \frac{1}{2^j} \cos(2^{2j} x\pi)$$

belongs to $\Lambda^{1/2}(\mathbb{R})$.

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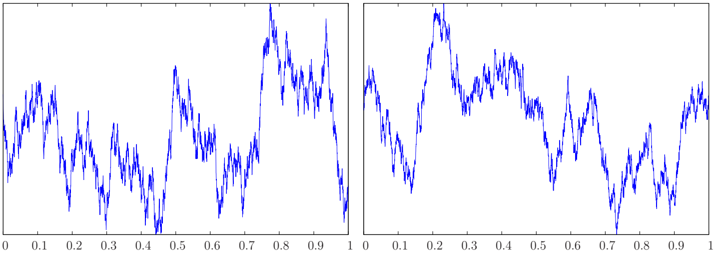
The uniform Weierstraß function of parameters (a, b) is the classical Weierstraß function coupled with a random phase. More precisely, this process is defined by

$$W(x) = \sum_{n=0}^{+\infty} a^n \cos((b^n x + U_n)\pi),$$

where $0 < a < 1 < b$ with $ab \geq 1$ and where each U_n is chosen independently with respect to the uniform probability measure on $[0, 1]$.

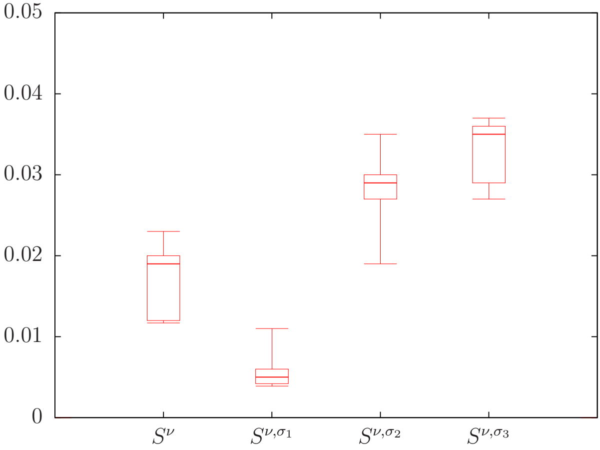
The Brownian motion vs the uniform Weierstraß function.



for W (right), we set $a = 0.8$ and $b = 1.6$.

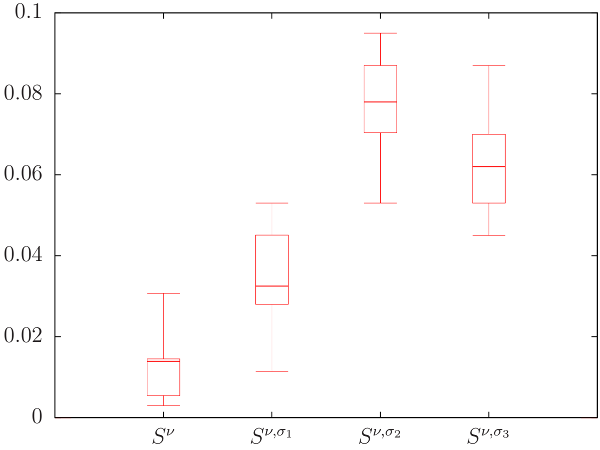
The Brownian motion vs the uniform Weierstraß function.

For 20 simulations of a BM, we get



The Brownian motion vs the uniform Weierstraß function.

For 20 simulations of W, we get



Let us define a process based on the Lévy-Ciesielski construction (that allows to decompose the Brownian motion in the Schauder basis) to obtain a multifractal process which share the same local regularity as the Brownian motion.

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The Schauder functions evaluated at t are the integrates of the Haar wavelets on $[0, t]$. More precisely, let us set

$$F_0(t) = \begin{cases} 0 & \text{if } t < 0 \\ t & \text{if } t \in [0,1] \\ 1 & \text{else} \end{cases},$$

and for any $(j, k) \in \mathbb{N} \times \{0, \dots, 2^j - 1\}$,

$$F_{j,k}(t) = \begin{cases} t - k2^{-j} & \text{if } t \in [k2^{-j}, k2^{-j} + 2^{-(j+1)}] \\ -t + (k+1)2^{-j} & \text{if } t \in [k2^{-j} + 2^{-(j+1)}, (k+1)2^{-j}] \\ 0 & \text{else} \end{cases}.$$

Let us recall that we have the following properties:

1. let $(a_{j,k})_{(j,k)\in\mathbb{N}\times\{0,\dots,2^j-1\}}$ be a real sequence, $a_0 \in \mathbb{R}$ and $\epsilon \in (0,1/2)$. If $\max_{k\in\{0,\dots,2^j-1\}} |a_{j,k}| = O(2^{j\epsilon})$ as $j \to +\infty$ then the function f defined by

$$t \mapsto a_0 F_0(t) + \sum_{j=0}^{+\infty} \sum_{k=0}^{2^j-1} a_{j,k} 2^{j/2} F_{j,k}(t) \qquad (2)$$

   is uniformly absolutely-convergent on $[0,1]$. Besides, f is a real continuous function such that $f(0)=0$,

2. any continuous function f from $[0,1]$ to $\mathbb{R}$ such that $f(0)=0$ can be written in the form (2). Besides, if $f \in \Lambda^\alpha(x_0)$ then there exists a constant $C>0$ such that

$$|a_{j,k} 2^{-j/2}| \leq C(2^{-j} + |k2^{-j} - x_0|)^\alpha$$

   for any $(j,k) \in \mathbb{N} \times \{0,\dots,2^j-1\}$.

Let $(Z_{j,k})_{(j,k)\in\mathbb{N}\times\{0,\dots,2^j-1\}}$ be a sequence of independents real-valued $\mathcal{N}(0,1)$ Gaussian random variables defined on the probability space Ω. Then, there exists an event $\Omega^* \subset \Omega$ of probability 1 such that, for any $\omega \in \Omega^*$, the function $B_.(\omega)$ defined by

$$B_.(\omega) : t \mapsto Z_0(\omega)F_0(t) + \sum_{j=0}^{+\infty} \sum_{k=0}^{2^j-1} Z_{j,k}(\omega) 2^{j/2} F_{j,k}(t)$$

is uniformly absolutely-convergent on $[0,1]$. Besides, the process $B = \{B_t\}_t$ is a Brownian motion.

Let K be a compact of $(-1/2, 1/2)$, $H \in \underline{\mathcal{H}}_K$ and $(Q_j)_{j\in\mathbb{N}}$ be a sequence of polynomials satisfying Relation (1). For any $(j,k) \in \mathbb{N} \times \{0, \ldots, 2^j - 1\}$, set

$$H_{j,k} = Q_j\left(\frac{k}{2^j}\right).$$

Let $(Z_{j,k})_{(j,k)\in\mathbb{N}\times\{0,\ldots,2^j-1\}}$ be a sequence of independents real-valued $\mathcal{N}(0,1)$ Gaussian random variables defined on the probability space Ω and let us define

$$B_t^H(\omega) = Z_0(\omega)F_0(t) + \sum_{j=0}^{+\infty} \sum_{k=0}^{2^j-1} 2^{-jH_{j,k}} Z_{j,k}(\omega) 2^{j/2} F_{j,k}(t).$$

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Theorem

There exists an event $\Omega^* \subset \Omega$ of probability 1 such that, for any $\omega \in \Omega^*$, we have the following properties:

1. the function $t \mapsto B_t^H(\omega)$ is a continuous function defined on $[0, 1]$,
2. we have the following relation: $h_{B_\cdot^H(\omega)}(t) = 1/2 + H(t)$, for any $t \in [0, 1]$,
3. let $t \in [0, 1]$; if there exists $C > 0$ such that $H(t) - Q_j(t) \leq Cj^{-1}$, for any $j \in \mathbb{N}$ then there exist a constant $C' > 0$ independent of t such that

$$|B_{t+h}^H(\omega) - B_t^H(\omega)| \leq C' 2^C |h|^{1/2+H(t)} \sqrt{\log h^{-1}},$$

for any h in a neighborhood of 0.

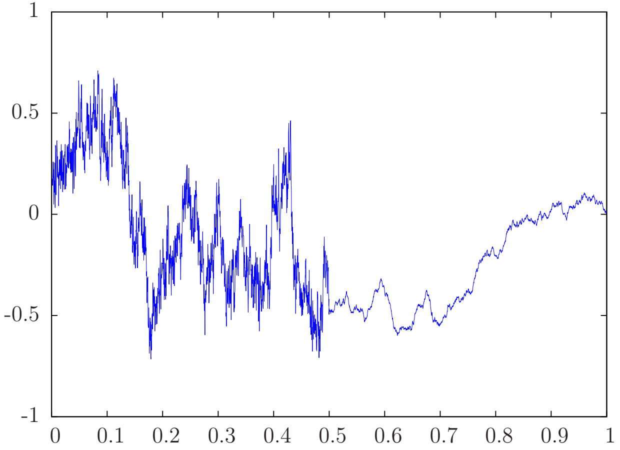
Proposition

Under hypothesis of the previous theorem, there exists an event $\Omega^* \subset \Omega$ of probability 1 such that, for any $\omega \in \Omega^*$ and for almost every $t \in [0,1]$, there exists a constant $C > 0$ such that

$$|B^H_{t+h}(\omega) - B^H_t(\omega)| \leq C|h|^{1/2+H(t)}\sqrt{|\log|\log h^{-1}||},$$

for any h small enough.

Let us define a bifractal process, with $H_1 \in \{0.2, 0.25, \dots, 0.4\}$ and $H_2 \in \{0.6, 0.65, \dots, 0.8\}$.



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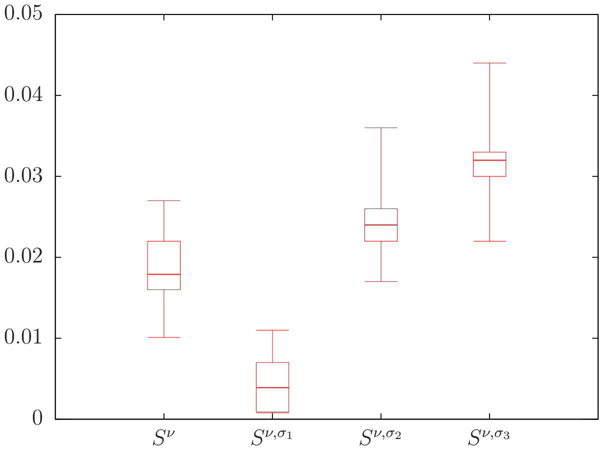

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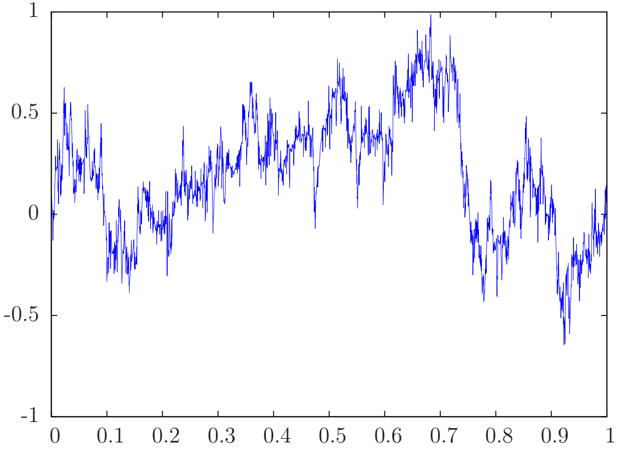

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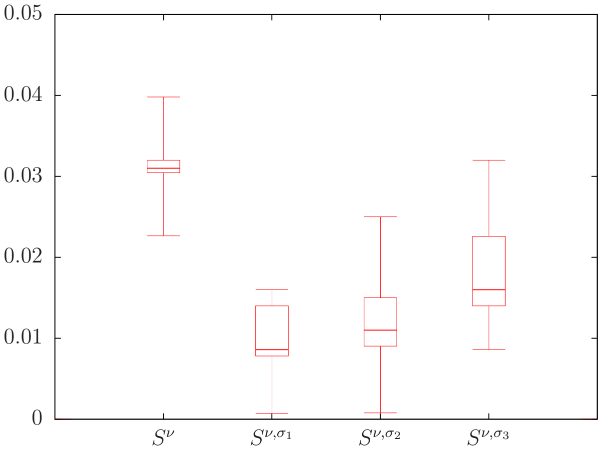
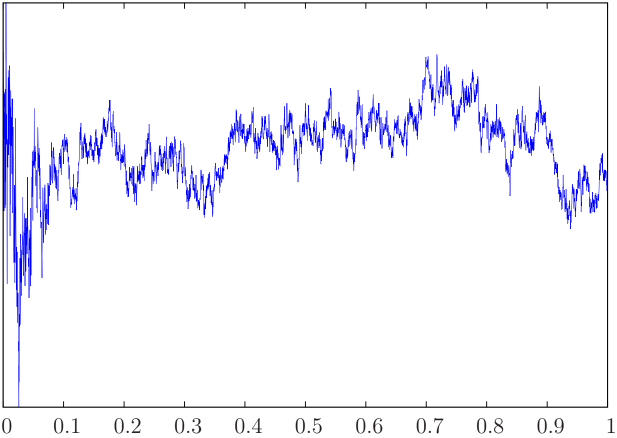


Let us set again

$$H(x) = \begin{cases} \frac{c-a}{b}x + a & \text{if } x < b \\ c & \text{if } x \geq b \end{cases},$$



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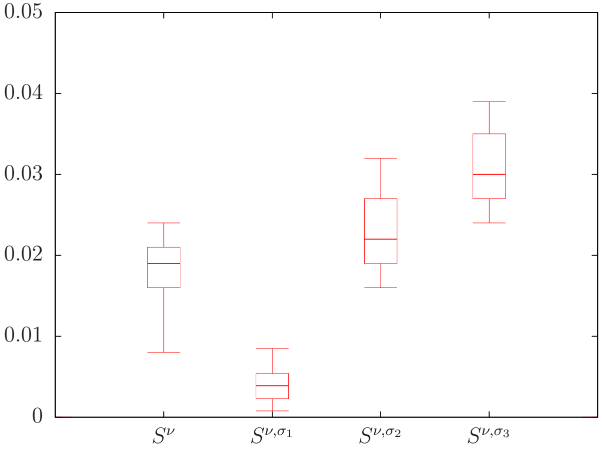

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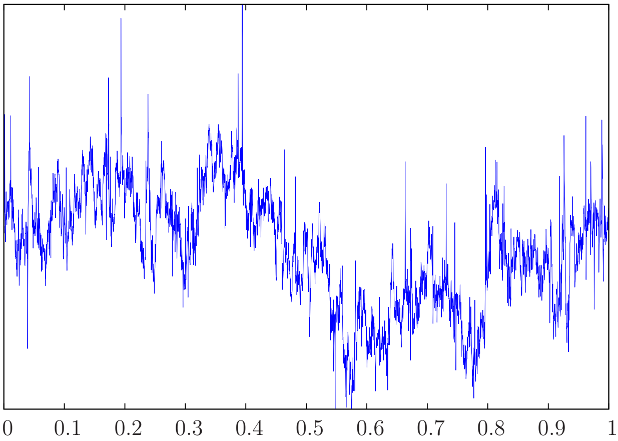

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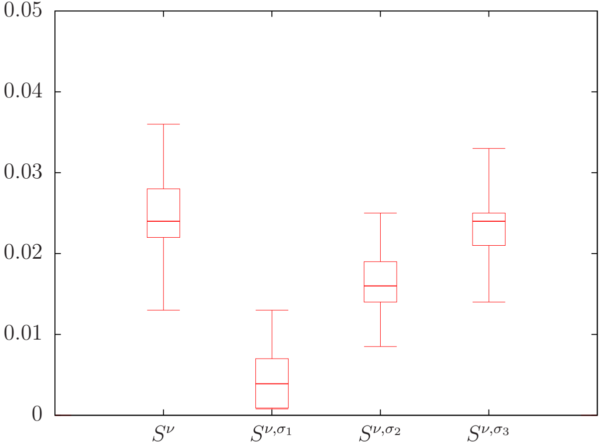


Road map

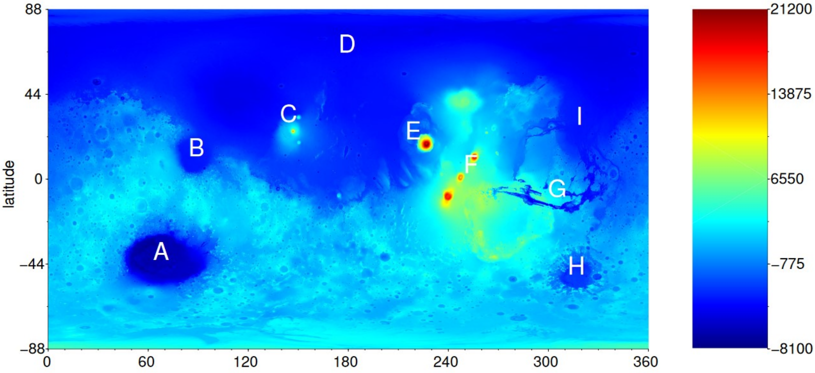
- The context
- The S^ν spaces and their generalization
- In practice
- More evolved examples
- A real life application

Application: a 2D study of Mars' topography.

We used the 128-pixel-per-degree map from the MOLA experiment.

This map almost represents the whole planet; the latitude ranges from 88°S to 88°N.

The main Hölder exponent



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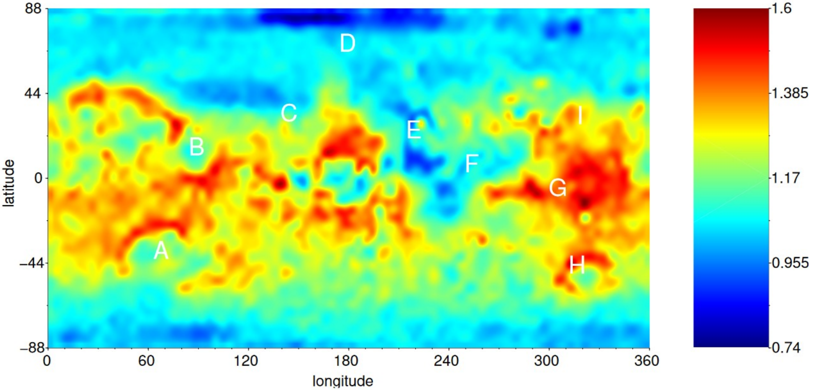
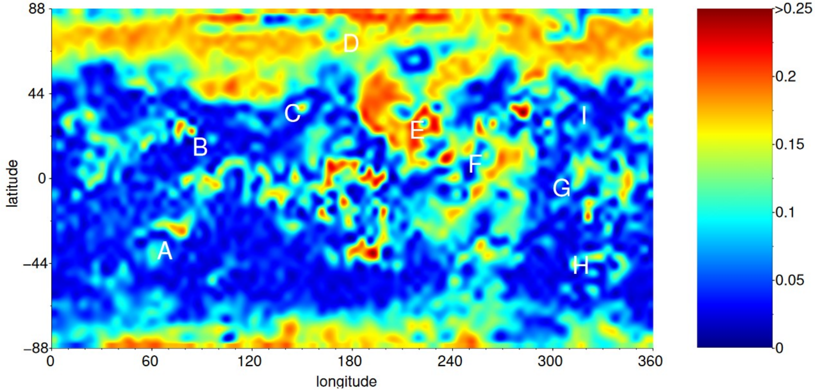


A 2D investigation of the multifractility



Let us give a method for detecting the existence of a Hölder exponent h such that $d_f(h) < n$ (where d_f denotes the multifractal spectrum).

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It suffices to find a h such that

$$C > 0 \mapsto \nu^{C}_{C,\sigma(\cdot)}(h)$$

has a stabilisation associated to a value strictly smaller than n.

When looking at longitudinal and latitudinal bands, most of these signals seem to be multifractal.