# Multifractal analysis: on the trail of Cantor

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Journées internationales du GDR AFHP
Novembre 2020

# I. Davenport series

Let $\{x\} = x - [x] - 1/2$.

**Definition**

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We will suppose that $(a_n) \in l^1$.

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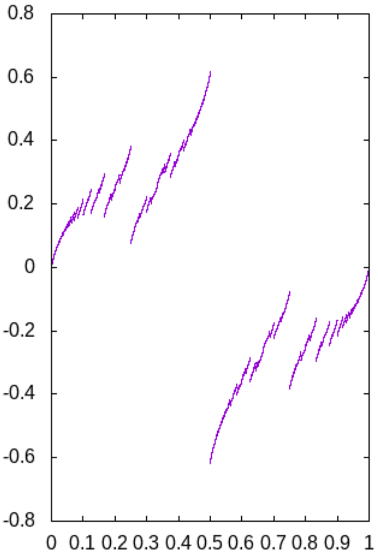
an example from Jordan:

$$\sum_{n=1}^{\infty} \frac{\{nx + 1/2\}}{n^3},$$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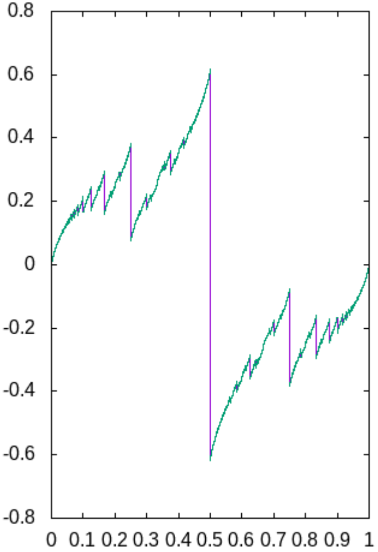
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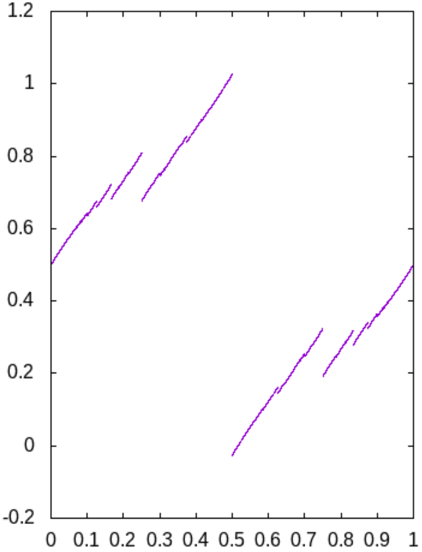
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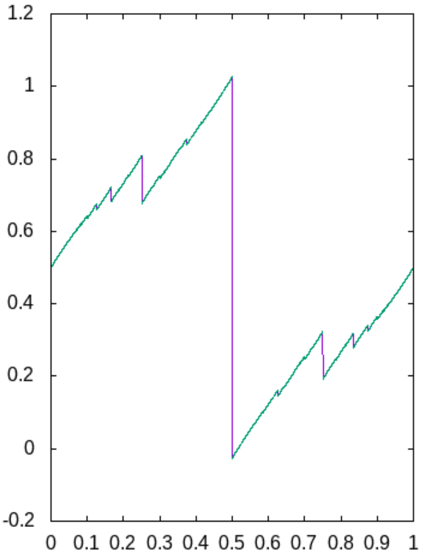
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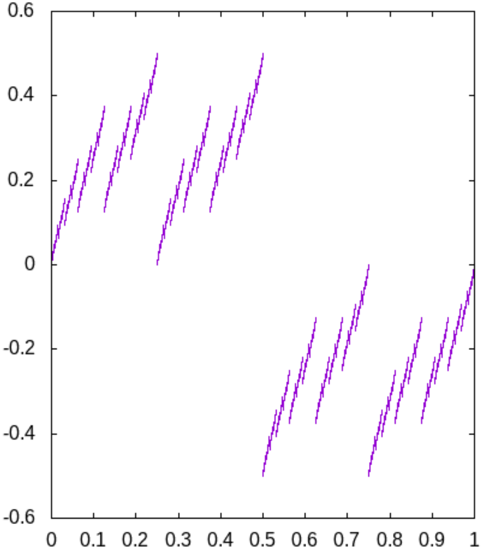
another example (Lebesgue-Davenport):

$$[0,1] \to [0,1] \quad x \mapsto \begin{cases} 1 & \text{if } x = 1 \\ x = \sum_{n=1}^{\infty} \frac{x_{2n}}{2^{2n}} & \text{if } x = \sum_{n=1}^{\infty} \frac{x_n}{2^n} \end{cases},$$

where $x = \sum_{n=1}^{\infty} x_n/2^n$ is the proper binary expansion of $x$.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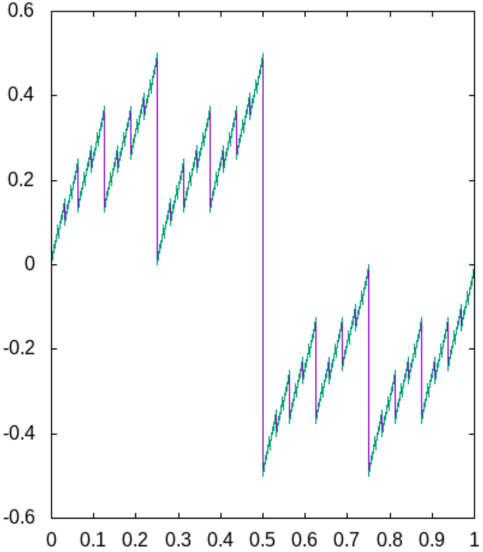
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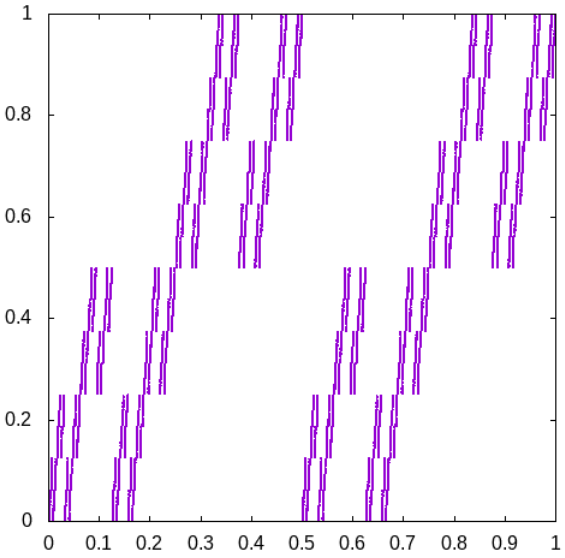

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## II. Multifractal analysis

### Definition

Given $\alpha \geq 0$ and $x_0 \in \mathbb{R}$, a locally bounded function $f : \mathbb{R} \to \mathbb{R}$ belongs to $\Lambda^\alpha(x_0)$ if there exist two constant $R, C > 0$ and a polynomial $P$ of degree less than $\alpha$ such that

$$\text{if } |x - x_0| < R \text{ then } |f(x) - P(x - x_0)| < C|x - x_0|^\alpha.$$

The Hölder exponent of $f$ at $x_0$ is

$$h_f(x_0) := \sup\{\alpha \geq 0 : f \in \Lambda^\alpha(x_0)\}.$$

The Hölder exponent of a function can vary from one point to another in a very erratic way. Since the Hölder function $x \mapsto h_f(x)$ is generally very irregular, one usually also tries to characterize the importance of a given Hölder value, i.e. to determine the size of the set of points sharing the same Hölder exponent.

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## Definition

The isoHölder sets of a locally bounded function $f$ are the sets

$$E_H = \{x : h_f(x) = H\}.$$

The spectrum of singularities of $f$ is then defined as

$$d_f : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{-\infty\} \quad h \mapsto \dim_{\mathcal{H}}(E_h),$$

where $\dim_{\mathcal{H}}$ stands for the Hausdorff dimension, with the standard convention $\dim_{\mathcal{H}}(\emptyset) = -\infty$.

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More precisely, it involves notions close to the irrationality exponent, which is the supremum of the real numbers $\mu > 0$ for which

$$0 < |x - \frac{k}{q}| < \frac{1}{q^\mu}$$

is satisfied for an infinite number of integer pairs $(k, q)$.

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However additional assumptions have to be made on the possible solutions $(k, q)$ in the previous inequality depending on the considered function or more precisely on the nature of the considered sequence $(a_n)$.

Given a number $x$, let

$$\mu_E(x) = \sup\{\mu : |x - \frac{k}{q}| < \frac{1}{q^\mu}$$

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**Theorem**

If $f$ is the Riemann function, $h_f(x) = 2/\mu_E(x)$, where $k/q \in E$ iff $q$ is even.
Moreover $d_f(h) = h$ if $h \in [0,1]$ and $d_f(h) = -\infty$ otherwise.

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In this case, the Hölder exponent at $x$ depends on how closely $x$ can be approximated by $p$-adic rationals.

Such a function is continuous at every non $p$-adic real number and has a right and left limit at every $p$-adic rational $k/p^n$ (where $k$ and $p$ are coprime) with a jump of amplitude $b_n := \sum_{l \geq n} a_l$.

The spectrum of singularity of such functions was first obtained as a consequence of general upper and lower bounds for the spectra.

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The determination of the Hölder function was later obtained as a generalisation of a result obtained for Lévy's function.

## Theorem

Let $f$ be a $p$-adic Davenport series with $(a_n) \in l^1$, and define $b_n := \sum_{l \geq n} a_l$.

If $x$ is not a $p$-adic rational,

$$h_f(x) = \liminf_{n \to \infty} \frac{\log |b_n|}{\log \operatorname{dist}(x, p^{-n}\mathbb{Z})}.$$

Otherwise, if $x = k/p^l$ with $k$ and $p$ coprime, $h_f(x) = 0$ if $b_l = 0$, else

$$h_f(x) = \liminf_{n \to \infty} \frac{\log |b_n|}{\log p^{-n}}.$$

# III. Approximation by $p$-adic rationals

If $x - [x] = (0; x_1, \ldots)_p$ is not a $p$-adic rational, let

$$\delta(n) = \sup\{l : \forall l' \leq l, x_{n+l'} = x_n\}$$

## III. Approximation by $p$-adic rationals

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$$\delta(n) = \sup\{l : \forall l' \leq l, x_{n+l'} = x_n\}$$

and $(m_n)_{n\in\mathbb{N}}$ be the sequence defined recursively as follows:

$$m_1 = \inf\{l : x_l = 0 \text{ or } x_l = p-1\}$$

and

$$m_n = \inf\{l > m_{n-1} + \delta(m_{n-1}) : x_l = 0 \text{ or } x_l = p-1\} \quad (n > 1)$$

if it makes sense, that is if $m_n$ is finite for every $n$.

One also defines the sequence $(\delta_k)_k$ by $\delta_n = \delta(m_n)$ if $m_n$ is finitie for every $n$. The number $\delta_n$ represents the size of the $n$-th "gap" made of numbers 0 or $p-1$, while $m_n$ points at the position of this gap.

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Finally, define

$$\rho_p(x) = \limsup_{n\to\infty} \frac{\delta_n}{m_n}$$

if $m_n$ is finitie for every $n$ and $\rho_p(x) = 0$ if there exists $k$ such that $m_k = \infty$.

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**Proposition**

If $x$ is not a $p$-adic rational, the supremum of the $\mu > 0$ such that the equation (depending on $k$ and $l$)

$$|x - \frac{k}{p^l}| < (\frac{1}{p^l})^\mu$$

has infinitely many solutions is $\rho_p(x) + 1$.

If $x$ is a $p$-adic rational, as $\delta_n = \infty$ for some $n$ if $x \in (0,1)$, one naturally sets $\rho_p(x) = \infty$.

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The previous proposition leads to the following definition.

**Definition**

The $p$-adic irrationality exponent $\mu^{(p)}(x)$ of a number $x$ in base $p$ is given by $\mu^{(p)}p(x) = \rho_p(x) + 1$.

One has $\mu^{(p)}(1/p) = \infty$ and $\mu^{(p)}(1/p+1) < \infty$.

If $x$ is the Liouville number

$$x = \sum_{k=1}^{\infty} \frac{1}{p^{k!}},$$

one easily checks that $\mu^{(p)}(x) = \infty$, although $x$ is transcendental.

Given $\alpha \in [1, \infty]$, let us define $\mathcal{M}_\alpha^p$ as the set of points whose critical exponent in base $p$ is $\alpha$:

$$\mathcal{M}_\alpha^p = \{x : \mu^{(p)}(x) = \alpha\}.$$

We will often omit the reference to the base $p$ and write $\mathcal{M}_\alpha$ instead of $\mathcal{M}_\alpha^p$.

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The Hausdorff dimension of $\mathcal{M}_\alpha$ is $1/\alpha$.

**Theorem**

We have

$$\dim_{\mathcal{H}}(\mathcal{M}_\alpha) = \frac{1}{\alpha},$$

for any $\alpha \in [1, \infty]$.

## IV. Regularity of $p$-adic Davenport series

Given $(a_n)$, let $b_n = \sum_{l \geq n} a_l$.

### Definition

Given $p \in \mathbb{N}$ and $l > 0$, a sequence $(a_n)$ is of order $l$ with respect to $p$ if $(p^{nl} a_n)$ is bounded. In this case, we will write $a_n \sim l$.

Given a $p$-adic Davenport series $\sum_{n=1}^{\infty} a_n \{p^n x\}$, lets us define

$$\mu_f^{(p)}(x) := 1 + \limsup_{\substack{n \to \infty \\ b_{m_n - 1} \neq 0}} \frac{\delta_n}{m_n}$$

Given a $p$-adic Davenport series $\sum_{n=1}^{\infty} a_n\{p^n x\}$, lets us define

$$\mu_f^{(p)}(x) := 1 + \limsup_{\substack{n \to \infty \\ b_{m_n - 1} \neq 0}} \frac{\delta_n}{m_n}$$

and

$$\mathcal{M}'_\alpha := \mathcal{M}'^{p}_{f,\alpha} := \{x : \mu_f^{(p)}(x) = \alpha\}.$$

## Theorem

Let $f$ be a $p$-adic Davenport series with $a_n \sim l$; if $x$ belongs to $\mathcal{M}'_\alpha$ with $\alpha \in [1, \infty]$ then $h_f(x) = l/\alpha$.

In particular, the isoHölder sets of $f$ are

$$E_H = \mathcal{M}'_{\frac{l}{H}}$$

## Corollary

If $f$ is a $p$-adic Davenport series with $a_n \sim 1/l$, the spectrum of singularities of $f$ is

$$d_f(h) = \begin{cases} lh & \text{if } h \in [0, 1/l] \\ -\infty & \text{otherwise} \end{cases} .$$

Since $\{x + 1/2\} = \{2x\} - \{x\}$, the spectrum of singularitues of

$$f = \sum_{n=1}^{\infty} \frac{\{2^n x + 1/2\}}{2^n}$$

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is $d_f(h) = h$ if $h \in [0,1]$ and $d_f(h) = -\infty$ otherwise.

We also have $\mathcal{M}'_\alpha = \mathcal{M}_\alpha$, so that $E_H = \mathcal{M}_{1/H}$ and $h_f(x) = 1/\mu^{(2)}(x)$.

The function

$$f : [0,1] \to [0,1] \quad x \mapsto \begin{cases} 1 & \text{if } x = 1 \\ x = \displaystyle\sum_{n=1}^{\infty} \frac{x_{2n}}{2^{2n}} & \text{if } x = \displaystyle\sum_{n=1}^{\infty} \frac{x_n}{2^n} \end{cases}$$

can be rewritten

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} a_n \{2^n x\},$$

with $a_{2n} = -2^{-n}$ and $a_{2n+1} = 2^{-n}$.

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Therefore $d_f(h) = 2h$ if $h \in [0, 1/2]$ and $d_f(h) = -\infty$ otherwise. However, $\mathcal{M}'_\alpha \neq \mathcal{M}_\alpha$ and

$$h_f(x) = 2/(1 + \limsup_{\substack{n \to \infty \\ m_n \text{ even}}} \frac{\delta_n}{m_n})$$

(one only takes dyadic rationals of the form $k/2^{2l-1}$).

# V. Cantor's bijection

The functions

$$f_1 : [0,1] \to [0,1] \quad x \mapsto \begin{cases} 1 & \text{if } x = 1 \\ x = \sum_{n=1}^{\infty} \frac{x_{2n-1}}{2^{2n}} & \text{if } x = \sum_{n=1}^{\infty} \frac{x_n}{2^n} \end{cases}$$

and

$$f_2 : [0,1] \to [0,1] \quad x \mapsto \begin{cases} 1 & \text{if } x = 1 \\ x = \sum_{n=1}^{\infty} \frac{x_{2n}}{2^{2n}} & \text{if } x = \sum_{n=1}^{\infty} \frac{x_n}{2^n} \end{cases}$$

was Cantor's first attempt to build a one-to-one mapping from $[0,1]$ to $[0,1]^2$.

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Let $I = [0,1] - \mathbb{Q}$; the function

$$I \to I^2 \quad [a_1, a_2, a_3 \ldots] \mapsto \begin{pmatrix} [a_1, a_3, \ldots] \\ [a_2, a_4, \ldots] \end{pmatrix}$$

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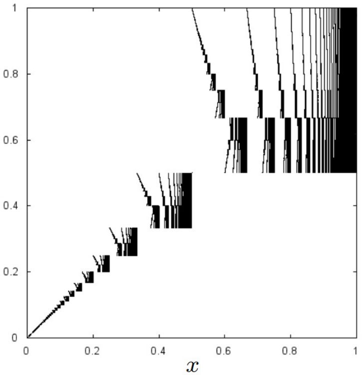
is one-to-one.

Interestingly, the first function gives rise to a one-to-one function thanks to the Schröder-Bernstein theorem, conjectured by Cantor a few years later.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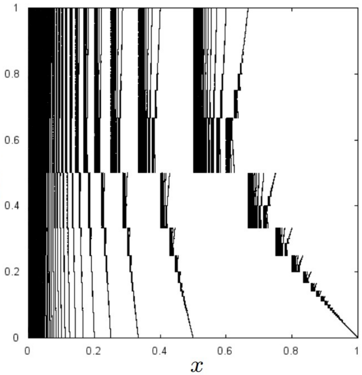
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Indeed it is discontinuous at any rational.

Let $f$ be the first component of Cantor's bijection (there is a similar result for the second component).

### Theorem

Let $n \in \mathbb{N}$; if $x = [a]$, for the "right" $y$, we have

$$\frac{\frac{1}{n}\sum_{k=1}^{\lceil n/2 \rceil} \log a_{2k-1}}{\frac{1}{n}\sum_{k=1}^{n+2} \log(a_k + 1) + C_1(n)/n} \leq \frac{\log|f(x) - f(y)|}{\log|x - y|} \leq \frac{\frac{1}{n}\sum_{k=1}^{\lceil n/2 \rceil + 3} \log(a_{2k-1} + 1) + C_2(n)/n}{\frac{1}{n}\sum_{k=1}^{n+2} \log a_k},$$

where $C_1(n) = \frac{\log 2}{2} + \log \max\{\frac{a_{n+2}+2}{a_{n+2}+1}, \frac{a_{n+3}+2}{a_{n+3}+1}\}$ and $C_2(n) = \frac{\log 2}{2} + \log \max\{\frac{a_{\lceil n/2 \rceil+3}+2}{a_{\lceil n/2 \rceil+3}+1}, \frac{a_{\lceil n/2 \rceil+5}+2}{a_{\lceil n/2 \rceil+5}+1}\}$.

- $h_f([1,2,1,4,1,8,1,16,\ldots]) = 1$,
- $h_f([2,1,4,1,8,1,16,1,\ldots]) = 0$,
- $h_f([2,4,8,16,\ldots]) = 1/2$.

Using the ergodic theorem on continued fractions, we get

## Theorem

We have $h_f([a]) \in [\frac{\log K_0}{2 \log K_1}, \frac{\log K_1}{2 \log K_0}]$ almost everywhere,

where $K_k = \prod_{l=1}^{\infty} (1 + \frac{1}{l(l+2)})^{\log(k+l)/\log 2}$.

## Proposition

Given $a \in \mathbb{N}^{\mathbb{N}}$, let $a' = (a_{2k-1})$; if

$$\lim_n \frac{1}{n} \log q_n(a) = \lim_n \frac{1}{n} \log q_n(a') \quad (\overset{\text{a.e.}}{=} \frac{\pi^2}{12 \log 2}),$$

then $h_f([a]) = 1/2$.

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Using Birkhoff's theoem?

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We have

$$\frac{1}{n} \log q_n(a) = -\frac{1}{n} \sum_{k=0}^{n-1} \log[\tau^k(a)] + R_n(a),$$

with $R_n(a) \to 0$ as $n \to \infty$.

From this, we get

$$\frac{1}{n}\log q_n(a') = -\frac{1}{n}\sum_{k=0}^{n-1}\log[\tau^{2k}(a)] + S_n(a) + R_n(a'),$$

with $S_n(a) = \dfrac{1}{n}\displaystyle\sum_{k=0}^{n-1}\log\frac{[\tau^{2k}(a)]}{f([\tau^{2k}(a)])}$.

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And thus

$$\lim_n \frac{1}{n}\log q_n(a') = \frac{\pi^2}{12\log 2} + \lim_n S_n(a)$$

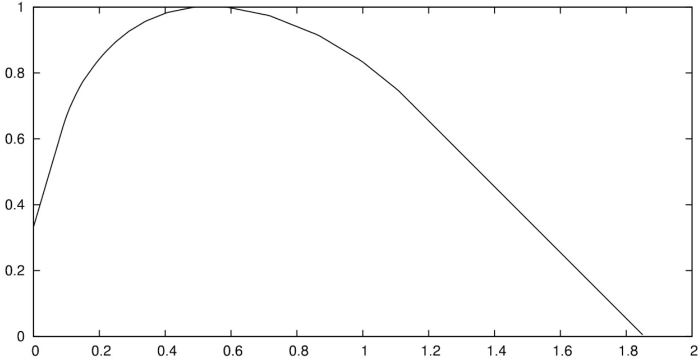
almost everywhere.

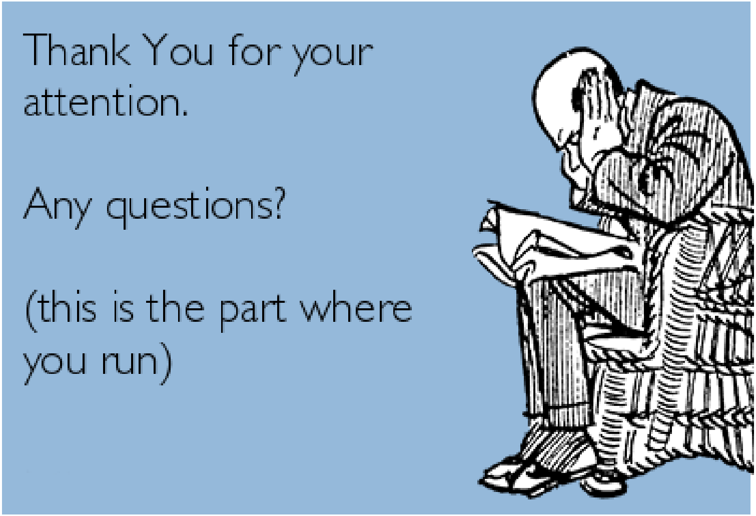
Therefore, if

$$\lim_n \frac{1}{n} \sum_{k=0}^{n-1} \log \frac{[\tau^{2k}(a)]}{f([\tau^{2k}(a)])} = 0$$

almost everywhere, then $h_f([a]) = 1/2$ almost everywhere.

The *numerical* spectrum of $f$:




Thank You for your attention.
Any questions?
(this is the part where you run)