Advanced Macroeconomics
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An Overlapping Generations Model with Two Periods and Perfect Foresight

1 Introduction

The overlapping generations (OLG) model together with the Ramsey model (neoclassical growth model) are the main models used in macroeconomics. The Ramsey model is said to be an optimal macroeconomic model because the general equilibrium of the decentralized economy verifies the First Welfare Theorem (FWT) (= satisfies the Pareto optimal criterion). The specificity of the OLG model is its demographic structure: in each period, one generation dies and another is born and the young generation trades with the older on the markets. Apart from their age difference, these two groups are distinguished by their resources. The young live on labour income while the old live on capital income (= their saving + the interest on their saving). In an OLG model, the general equilibrium of the decentralized economy may not satisfy the FWT. When the economy is dynamically inefficient (overaccumulation of capital), there is room for public intervention to improve the welfare of economic agents (= Second Welfare Theorem (SWT)).

The OLG model is due to Allais (1947), Samuelson (1958) and Diamond (1965).

2 Setup

Let us consider a closed economy with perfectly competitive markets. Time is discrete. The economy starts at time $t = 0$ and its horizon is infinite. We will assume that at $t = 0$, the economy starts with positive endowments in physical capital and labor (people born old) reflecting its history. We will also assume that there is no money, no government and no bequest between generations. Finally, we consider that the future is certain. Therefore, the agents can perfectly forecast the future values of all variables. We say that the agents have perfect foresight.

At each period t, there exist three goods: physical capital, labor and an aggregate output produced from physical capital and labor. Each of these goods is traded and priced on a market: the capital market for the physical capital, the labor market for labor and the goods market for the output.

The demographic structure
At each period t, a generation indexed by t of N_t individuals is born and lives for two periods. At time t, the individuals of generation t are young, they work, consume and save for their old days. At time $t+1$, the individuals of generation t are old, they give birth to the generation $t+1$, they consume their saving and the accrued interests, and at the end of the period, they die without leaving any bequest to their offspring. We assume that the size of each generation is larger by a constant rate n than the previous generation:

$$N_{t+1} = (1+n)N_t \tag{1}$$

where N_t is the number of individuals of generation t. We assume that $n \in]-1, +\infty[$ so the population can possibly shrink at a constant rate. At each period t, the total population is composed of two generations: a generation t of young individuals and a generation of $t-1$ of old individuals. The growth rate of the total population is n:

$$N_{t+1} + N_t = (1+n)\left(N_t + N_{t-1}\right) \tag{2}$$

Proof: $N_{t+1} + N_t = (1+n)N_t + (1+n)N_{t-1} = (1+n)\left(N_t + N_{t-1}\right)$ ■

3 Consumption and Saving

The consumers' preferences
The preferences of the consumers[1] over their consumption bundle in the two periods of their life can be represented by the same additively separable life-cycle utility function:

$$U(c_t, d_{t+1}) = u(c_t) + \beta u(d_{t+1}) \tag{3}$$

where $U(.)$ is the life-cycle utility, $u(.)$ is the instantaneous utility, c_t is consumption when the representative consumer is young, d_{t+1} is consumption when she is old and β is a parameter, called a pyschological discount factor, indicating the degree of her impatience. Impatience implies that $\beta < 1$, i.e. the representative consumer values the present consumption more than the future consumption.

Assumptions on the instantaneous utility function:
For all $c, d > 0$, $u'(.) > 0$, $u''(.) < 0$ and $\lim_{c,d \longrightarrow 0} u'(.) = +\infty$.

These assumptions mean that the instantaneous utility function is monotonically increasing and concave.

[1] In the neoclassical growth model, it is assumed that there are a representative consumer and a representative firm. In the OLG model, there is no representative consumer but there is a representative firm.

The representative consumer lives on income throughout her life: labor income when young and capital income when old. At each period, she therefore has a budget constraint:

$$c_t + s_t = w_t \quad (4)$$
$$d_{t+1} = R_{t+1} s_t \quad (5)$$

On the right-hand side of the two equations, there are the incomes: w_t (labor income) and $R_{t+1}s_t$ (capital income) where s_t is the individual saving and $R_{t+1} = 1 + r_{t+1}$ is the interest factor (r_{t+1} is the interest rate). On the left-hand side, there are the spendings: consumption c_t and saving s_t during the first period of life and consumption d_{t+1} in the second period. By using the expression of s_t of the second-period budget constraint and plugging it in the first-period budget constraint, we obtain the intertemporal budget constraint:

$$c_t + \frac{d_{t+1}}{R_{t+1}} = w_t \quad (6)$$

Equation (6) is the present value at time t of the life-cycle consumption, which is equal to the labor income earned in the first period. In other words, the representative consumer has to allocate her labor income between consumption in her young age and consumption in her old days. How is the intertemporal allocation of labor income determined? By the utility function (3) representing her preferences between the present and the future.

The consumers' optimization problem
The optimization problem of the representative consumer consists in allocating her labor income over her life cyle to reach an optimal level of utility. Mathematically, this problem is the maximization of the utility function (3) with respect to the budget constraints, or equivalently, to the intertemporal budget constraint. The optimal solution to this problem is to find the optimal level of saving s_t, i.e. the part of the wage that will be transfered to the second period. There are two ways to calculate the optimal saving: by substitution or by using the Lagrangian.

Solving by substitution
We substitute c_t and d_{t+1} in the utility function (3):

$$u(w_t - s_t) + \beta u(R_{t+1}s_t) \quad (7)$$

which now is a function of the unknown s_t, which we want to calculate at the optimum. The solution to this maximization problem is the optimal saving. For a given instantaneous utility function, the optimal saving

$$s_t = \phi(w_t, R_{t+1}) \quad (8)$$

is a function of two prices: the real wage (price of labor) and the real interest factor (the price of physical capital). Maximizing (7) with respect to s_t gives the first-order condition:

$$-u'(w_t - s_t) + \beta R_{t+1} u'(R_{t+1}s_t) = 0 \quad (9)$$

which leads to

$$u'(w_t - s_t) = \beta R_{t+1} u'(R_{t+1} s_t) \tag{10}$$

which means that the marginal utility of first-period consumption is equal to the marginal utility of second-period consumption multiplied by a factor equal to βR_{t+1}.

Example: the log-utility function

$$U(c_t, d_{t+1}) = \ln(c_t) + \beta \ln(d_{t+1}) \tag{11}$$

By substituting c_t and d_{t+1} in the utility function, we obtain

$$U(c_t, d_{t+1}) = \ln(w_t - s_t) + \beta \ln(R_{t+1} s_t) \tag{12}$$

Maximizing utility with respect to s_t yields

$$\frac{-1}{w_t - s_t} + \frac{\beta R_{t+1}}{R_{t+1} s_t} = 0 \tag{13}$$

which leads to

$$s_t = \frac{\beta}{1+\beta} w_t \tag{14}$$

When utility is logarithmic, the optimal saving depends on the wage but not on the interest factor. The reason is that the income effect and the substitution effect of an increase in the interest factor on saving offset each other exactly.

Solving by Lagrange's method
We use the intertemporal budget constraint (6) and build the Lagrangian:

$$\mathcal{L}(c_t, d_{t+1}, \lambda_t) = u(c_t) + \beta u(d_{t+1}) + \lambda_t \left(w_t - c_t - \frac{d_{t+1}}{R_{t+1}} \right) \tag{15}$$

The first-order conditions are

$$\frac{\partial \mathcal{L}}{\partial c_t} = u'(c_t) - \lambda_t = 0 \tag{16}$$

$$\frac{\partial \mathcal{L}}{\partial d_{t+1}} = \beta u'(d_{t+1}) - \frac{\lambda_t}{R_{t+1}} = 0 \tag{17}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t} = w_t - c_t - \frac{d_{t+1}}{R_{t+1}} = 0 \tag{18}$$

By eliminating the Lagrange multiplier λ_t, we obtain[2]

$$u'(c_t) = \beta R_{t+1} u'(d_{t+1}) \tag{19}$$

which is the same as Equation (10).

[2]This first-order condition is often referred to as the Euler equation.

Example: the log-utility function

$$\mathcal{L}(c_t, d_{t+1}, \lambda_t) = \ln(c_t) + \beta \ln(d_{t+1}) + \lambda_t \left(w_t - c_t - \frac{d_{t+1}}{R_{t+1}} \right) \tag{20}$$

The first-order conditions are

$$\frac{\partial \mathcal{L}}{\partial c_t} = \frac{1}{c_t} - \lambda_t = 0 \tag{21}$$

$$\frac{\partial \mathcal{L}}{\partial d_{t+1}} = \frac{\beta}{d_{t+1}} - \frac{\lambda_t}{R_{t+1}} = 0 \tag{22}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t} = w_t - c_t - \frac{d_{t+1}}{R_{t+1}} = 0 \tag{23}$$

By eliminating the Lagrange multiplier λ_t, we obtain

$$d_{t+1} = \beta R_{t+1} c_t \tag{24}$$

By using the two budget constraints to substitute c_t and d_{t+1}, we obtain

$$s_t = \frac{\beta}{1+\beta} w_t \tag{25}$$

which is the same as Equation (14).

Ambiguity of the saving function's response to factor price movements
Equation (10) or, equivalently, Equation (19) give the condition for the consumer to reach the optimal level of utility during her life cycle. The solution satisfying this condition is the optimal saving. Once the instantanous utility is specified, the optimal saving is a function of two prices (see Equation (8)): the level of the wage and the level of the interest factor. The effects of the wage and the interest factor on optimal saving thus depends on the form of the saving function, which is itself a function of the instantanous utility.

Since saving is the fundamental element of our dynamic model, it is crucial to know the properties of the saving function. In particular, we would like to know how optimal saving responds to an increase in the wage or in the interest factor. Unfortunately, we do not have, for a general utility function such as (3), an explicit function between s_t and its arguments, w_t and R_{t+1}, to answer our question. However, we can use the implicit function theorem to do so. This powerful theorem allows to compute the partial derivatives we want without knowing the explicit formula between s_t and w_t and R_{t+1}. Two conditions are necessary for this theorem to apply. The first condition is the existence of a continuous relation $\Omega(w_t, R_{t+1}, s_t)$ such that, in the neighborhood of a particular optimal solution $(w_t^o, R_{t+1}^o, s_t^o)$, we have $\Omega(w_t^o, R_{t+1}^o, s_t^o) = 0$.[3] This first condition is met thanks to Equation

[3]where s_t^o is the value of the optimal saving when the wage and the interest factor have the particular values w_t^o and R_{t+1}^o respectively.

(10). Let us define this relation as

$$\Omega(w_t, R_{t+1}, s_t) \equiv -u'(w_t - s_t) + \beta R_{t+1} u'(R_{t+1} s_t) \tag{26}$$

which is equal to zero whenever s_t is an optimum. For instance, assume that s_t^o is an optimum when the wage and the interest factor are w_t^o and R_{t+1}^o respectively. Then, $\Omega(w_t^o, R_{t+1}^o, s_t^o) = 0$. The second condition is that the partial derivative of $\Omega(w_t, R_{t+1}, s_t)$ with respect to the variable of interest, s_t, evaluated at the optimal solution $(w_t^o, R_{t+1}^o, s_t^o)$ is nonzero. We can check that

$$\left.\frac{\partial \Omega(w_t, R_{t+1}, s_t)}{\partial s_t}\right|_{w_t^o, R_{t+1}^o, s_t^o} = u''(w_t^o - s_t^o) + \beta (R_{t+1}^o)^2 u''(R_{t+1}^o s_t^o) < 0 \tag{27}$$

The second condition is met since the required partial derivative is nonzero. As a result, we can apply the implicit function theorem, which guarantees the existence of a unique continuous function $s_t = \phi(w_t, R_{t+1})$ in the neighborhood of (w_t^o, R_{t+1}^o) such that, for all (w_t, R_{t+1}) near (w_t^o, R_{t+1}^o),

$$\Omega\left[w_t, R_{t+1}, \phi(w_t, R_{t+1})\right] = 0 \tag{28}$$

By using the chaine rule, we can now differentiate Equation (28) with respect to w_t and R_{t+1}:

$$\Omega'_{w_t} + \Omega'_{s_t} \frac{\partial s_t}{\partial w_t} = 0 \tag{29}$$

$$\Omega'_{R_{t+1}} + \Omega'_{s_t} \frac{\partial s_t}{\partial R_{t+1}} = 0 \tag{30}$$

where $\Omega'_{s_t} = \Omega'_{\phi(w_t, R_{t+1})}$ and

$$\Omega'_{s_t} = \frac{\partial\, \Omega\left[w_t, R_{t+1}, \phi(w_t, R_{t+1})\right]}{\partial \phi(w_t, R_{t+1})} = u''(w_t - s_t) + \beta R_{t+1}^2 u''(R_{t+1} s_t) \tag{31}$$

$$\Omega'_{w_t} = \frac{\partial\, \Omega\left[w_t, R_{t+1}, \phi(w_t, R_{t+1})\right]}{\partial w_t} = -u''(w_t - s_t) \tag{32}$$

$$\Omega'_{R_{t+1}} = \frac{\partial\, \Omega\left[w_t, R_{t+1}, \phi(w_t, R_{t+1})\right]}{\partial R_{t+1}} = \beta u'(R_{t+1} s_t) + \beta R_{t+1} s_t u''(R_{t+1} s_t) \tag{33}$$

Recall that $u'(.) > 0$ and $u''(.) < 0$ by assumption. It is then straightforward to obtain the signs of the partial derivatives of the relation $\Omega\left[w_t, R_{t+1}, \phi(w_t, R_{t+1})\right]$:

$$\Omega'_{s_t} < 0 \tag{34}$$

$$\Omega'_{w_t} > 0 \tag{35}$$

$$\Omega'_{R_{t+1}} \lesseqgtr 0 \tag{36}$$

From Equations (29) and (30), we can find the partial derivatives of the saving function we have been looking for[4]:

$$\frac{\partial s_t}{\partial w_t} = \phi'_{w_t} = -\frac{\Omega'_{w_t}}{\Omega'_{s_t}} = \frac{1}{1+\frac{\beta R_{t+1}^2 u''(R_{t+1}s_t)}{u''(w_t-s_t)}} \tag{37}$$

$$\frac{\partial s_t}{\partial R_{t+1}} = \phi'_{R_{t+1}} = -\frac{\Omega'_{R_{t+1}}}{\Omega'_{s_t}} = -\frac{\beta u'(R_{t+1}s_t)+\beta R_{t+1}s_t u''(R_{t+1}s_t)}{u''(w_t-s_t)+\beta R_{t+1}^2 u''(R_{t+1}s_t)} \tag{38}$$

The partial derivatives of the saving function $s_t = \phi(w_t, R_{t+1})$, defined implicitly, can now be characterized:

$$0 < \phi'_{w_t} < 1 \tag{39}$$

$$\phi'_{R_{t+1}} \lesseqgtr 0 \tag{40}$$

The inequality (39) says that the effect of the wage on optimal saving is strictly positive and bounded above by the value 1. The function ϕ'_{w_t} is called the marginal propensity to save and the inequality (39) says that an optimizing consumer must save a positive amount but less than her labor income.

The inequality (40) says that the effect of the interest factor on optimal saving is ambiguous. Observe that the denominator of (38) is always negative. Therefore, the ambiguity comes from the numerator. This numerator can be rewritten as

$$\beta u'(R_{t+1}s_t)+\beta R_{t+1}s_t u''(R_{t+1}s_t) = \beta u'(R_{t+1}s_t)\left(1-\frac{1}{\sigma(R_{t+1}s_t)}\right) \tag{41}$$

where $\sigma(R_{t+1}s_t) \equiv -\frac{u'(R_{t+1}s_t)}{u''(R_{t+1}s_t)R_{t+1}s_t}$ is the intertemporal elasticity of substitution evaluated at $d_{t+1} = R_{t+1}s_t$. Therefore, there are three possible cases for the inequality (40):

$$\text{If } \sigma(R_{t+1}s_t) < 1 \implies \phi'_{R_{t+1}} < 0 \tag{42}$$
$$\text{If } \sigma(R_{t+1}s_t) = 1 \implies \phi'_{R_{t+1}} = 0 \tag{43}$$
$$\text{If } \sigma(R_{t+1}s_t) > 1 \implies \phi'_{R_{t+1}} > 0 \tag{44}$$

Case (43) deserves a comment. This case corresponds to the situation where a change in the interest factor R_{t+1} has no effect on optimal saving. It may seem strange that the interest factor has no effect on saving. In fact, the interest factor has two opposite effects. The first is an incentive effect: the higher the interest factor, the more profitable it is to save more, since the return on saving is higher. This effect is called the income effect of the interest factor and it is positive on saving. The second effect is a windfall effect:

[4]Observe that $\Omega'_{s_t} \neq 0$ is required for the solutions to exist. This is precisely the second condition of the implicit function theorem that was mentioned earlier.

the higher the interest factor, the higher the future income and therefore the greater the opportunity to save less and consume more today. This effect is called the substitution effect of the interest factor and it is negative on saving. In Case (43), these two effects cancel each other out, which explains why the interest factor has no effect on saving.

Because of the ambiguity of the inequality (40), the total effect of the wage and the interest factor on optimal saving is ambiguous and depends on the specification of the utility function. We can single out two cases where the saving function's response is not ambiguous:

$$\textbf{Case 1:} \text{ If } \sigma(R_{t+1}s_t) > 1 \implies \phi'_{w_t}dw_t + \phi'_{R_{t+1}}dR_{t+1} > 0 \quad (45)$$
$$\textbf{Case 2:} \text{ If } \sigma(R_{t+1}s_t) = 1 \implies \phi'_{w_t}dw_t + \phi'_{R_{t+1}}dR_{t+1} = \phi'_{w_t}dw_t > 0 \quad (46)$$

The last possible case is the one for which the saving function's response is ambiguous:

$$\textbf{Case 3:} \text{ If } \sigma(R_{t+1}s_t) < 1 \Longrightarrow \phi'_{w_t}dw_t + \phi'_{R_{t+1}}dR_{t+1} \lesseqqgtr 0 \quad (47)$$

Depending on the specification of the instantaneous utility function, the substitution effect of the interest factor on saving, which is negative, will be more or less strong. The stronger it is, the more negative $\phi'_{R_{t+1}}$ will be, and the more likely the total effect on saving, $\phi'_{w_t}dw_t + \phi'_{R_{t+1}}dR_{t+1}$, will be negative. For some utility functions, satisfying the assumptions we have made, an increase in the interest factor reduces saving.

Example: the log-utility function
Equation (14) yields the explicit saving function when utility is logarithmic:

$$s_t = \frac{\beta}{1+\beta}w_t$$

This function shows that optimal saving depends positively on the wage but does not depend on the interest factor. In fact, the log-utility function corresponds to Case 2 due to the specific condition described by expression (43).

4 Production

We assume that the aggregate ouptut is produced by all the firms of the domestic economy. These firms can be represented by a representative firm. The technological frontier of the production possibility set can be represented by a production function. This production function is called the technological constraint of the firms. The production function is the following

$$Y_t = AF(K_t, L_t) \quad (48)$$

where Y_t is the output at time t, K_t is the physical capital stock, L_t is the labor force, A is the total factor productivity (constant), and $F(.)$ is the transformation function of inputs (production factors) into output Y_t.

Assumptions on the production function:
We assume that $F(.)$ is neoclassical, .i.e, it is twice differentiable, positively valued (its domain is $\mathbb{R}_+$), increasing in its arguments (first derivatives are positive: $\frac{\partial F(.)}{\partial K} > 0$, $\frac{\partial F(.)}{\partial L} > 0$) and concave (second derivatives are negative: $\frac{\partial^2 F(.)}{\partial K^2} < 0$, $\frac{\partial^2 F(.)}{\partial L^2} < 0$). We also assume that $F(.)$ is homogeneous of degree one, i.e. it has constant returns to scale: for any $\lambda > 0$, $F(\lambda K, \lambda L) = \lambda F(K, L)$.

Example: the Cobb-Douglas production function: $AK_t^\alpha L_t^{1-\alpha}$ *where* $0 < \alpha < 1$.

$$\frac{\partial AK_t^\alpha L_t^{1-\alpha}}{\partial K_t} = \alpha AK_t^{\alpha-1}L_t^{1-\alpha} > 0; \qquad \frac{\partial^2 AK_t^\alpha L_t^{1-\alpha}}{\partial K_t^2} = \alpha(\alpha-1)AK_t^{\alpha-2}L_t^{1-\alpha} < 0$$

$$\frac{\partial AK_t^\alpha L_t^{1-\alpha}}{\partial L_t} = (1-\alpha)AK_t^\alpha L_t^{-\alpha} > 0; \qquad \frac{\partial^2 AK_t^\alpha L_t^{1-\alpha}}{\partial L_t^2} = -\alpha(1-\alpha)AK_t^\alpha L_t^{\alpha-2} < 0$$

Therefore, the Cobb-Douglas production function is a concave function in its arguments K_t and L_t. As for the constant returns to scale assumption:

$$A(\lambda K_t)^\alpha (\lambda L_t)^{1-\alpha} = \lambda^{\alpha+(1-\alpha)}AK_t^\alpha L_t^{1-\alpha} = \lambda AK_t^\alpha L_t^{1-\alpha}$$

Therefore, the Cobb-Douglas production satisfies the constant returns to scale assumption.

The firm's optimization problem
The objective of the representative firm is to maximize profit. The profit function of the firm is

$$\pi_t = P_t Y_t - \tilde{R}_t K_t - \tilde{w}_t L_t \tag{49}$$

where π_t is profit at time t, P_t is the price of the output, $\tilde{R}_t$ is the nominal interest factor and $\tilde{w}_t$ is the nominal wage and $Y_t = AF(K_t, L_t)$. The condition for a maximum of profit is

$$\frac{\partial \pi_t}{\partial L_t} = 0 \tag{50}$$

$$\frac{\partial \pi_t}{\partial K_t} = 0 \tag{51}$$

These are the first-order conditions of profit maximization. They lead to

$$\frac{\tilde{w}_t}{P_t} = \frac{\partial AF(K_t, L_t)}{\partial L_t} \tag{52}$$

$$\frac{\tilde{R}_t}{P_t} = \frac{\partial AF(K_t, L_t)}{\partial K_t} \tag{53}$$

where $\frac{\tilde{w}_t}{P_t} \equiv w_t$ is the real wage and $\frac{\tilde{R}_t}{P_t} \equiv R_t$ is the real interest factor at time t. We generally consider the price of the ouptut P as the numéraire (we can set $P_t \equiv 1$ without loss of generality but it is not necessary). Therefore, we can rewrite the real wage and the real interest factor as

$$w_t = \frac{\partial AF(K_t, L_t)}{\partial L_t} \tag{54}$$

$$R_t = \frac{\partial AF(K_t, L_t)}{\partial K_t} \tag{55}$$

Since markets are perfectly competitive, the factor prices (54) and (55) cannot be influenced by any of the economic agents who are price-takers. Therefore, at given market factor prices, Equation (54) determines the labor demand and Equation (55) determines the demand for physical capital.

Example: the Cobb-Douglas production function: $AK_t^{\alpha}L_t^{1-\alpha}$ *where* $0 < \alpha < 1$.
When the production function is Cobb-Douglas, the factor prices are

$$w_t = (1-\alpha)A\left(\frac{K_t}{L_t}\right)^{\alpha} \tag{56}$$

$$R_t = \alpha A\left(\frac{K_t}{L_t}\right)^{\alpha-1} \tag{57}$$

Since markets are perfectly competitive, $\pi_t = 0$ for all t and then the whole output has to be redistributed to production factors K_t and L_t such as

$$Y_t = R_t K_t + w_t L_t \tag{58}$$

We can observe that the allocation of incomes is generational: labor income is allocated to the young generation who work and capital income is allocated to the old generation, retired, who own the capital, i.e. shares of the firms. This capital is purchased when young and is consumed when old.

Example: the Cobb-Douglas production function: $AK_t^{\alpha}L_t^{1-\alpha}$ *where* $0 < \alpha < 1$.
When the production function is Cobb-Douglas, it is possible to check that the remuneration of production factors equals the aggregate output:

$$\alpha A\left(\frac{K_t}{L_t}\right)^{\alpha-1}(K_t) + (1-\alpha)A\left(\frac{K_t}{L_t}\right)^{\alpha}(L_t) = AK_t^{\alpha}L_t^{1-\alpha} \tag{59}$$

which is equal to the aggregate production Y_t.

It is useful to express the production function and the marginal returns to production factors in intensive form, i.e. per worker.[5] There are two reasons. The first reason is

[5] In an OLG model, per-worker variables and per-capita variables are not equivalent since the old generation does not work. Variables per capita imply the division of variables by the total population composed of two generations.

mathematical. In aggregate form, production is a function of two variables: physical capital and labor. If we divide the output and capital by labor, the problem reduces to a two-dimensional problem, which is easier to study. The second reason is economic. The study of economic development and wealth makes sense if we can measure it at the level of a person or a worker. In macroeconomics, we are interested in the evolution of wealth of people over time. Therefore, by dividing the output and capital by labor, we study the income per worker over time.

The transformation of the aggregate production function (48) into the production function per worker is not straightforward. It is necessary that the aggregate production is homogeneous of degree 1. Recall that we assumed that the production function had constant returns to scale. Therefore, it is possible to divide the production factors K_t and L_t by L_t and obtain

$$y_t = AF(k_t, 1) \tag{60}$$

where $y_t \equiv \frac{Y_t}{L_t}$ is the income per worker and $k_t \equiv \frac{K_t}{L_t}$ is the so-called capital-labor ratio. We can rewrite the production function (60) as

$$y_t = Af(k_t) \tag{61}$$

where $f(k_t) \equiv F(k_t, 1)$. As for the marginal returns to production factors and by using the chain rule:

$$w_t = \frac{\partial Y_t}{\partial L_t} = \frac{\partial L_t A f(k_t)}{\partial L_t} = Af(k_t) + L_t\left(-\frac{K_t}{L_t^2}\right)Af'(k_t) = Af(k_t) - k_t A f'(k_t) \tag{62}$$

$$R_t = \frac{\partial Y_t}{\partial K_t} = \frac{\partial L_t A f(k_t)}{\partial K_t} = L_t\left(\frac{1}{L_t}\right)Af'(k_t) = Af'(k_t) \tag{63}$$

By assumption on the first derivatives of the production function (48), the marginal returns to production factors are positive. This means that we assume that the factor prices are positive, which is fairly realistic!!

Example: the Cobb-Douglas production function: $AK_t^{\alpha}L_t^{1-\alpha}$ where $0 < \alpha < 1$.
When the production function is Cobb-Douglas, the production function and its marginal returns to production factors in inetnsive form are:

$$y_t = Ak_t^{\alpha} \tag{64}$$

$$w_t = (1-\alpha)Ak_t^{\alpha} \tag{65}$$

$$R_t = \alpha A k_t^{\alpha-1} \tag{66}$$

5 Markets

The goods market The goods market at any time t must satisfy the national income accounts identity where resources are equal to spending:

$$Y_t = C_t + I_t \tag{67}$$

where C_t is aggregate consumption and I_t is aggregate investment (= flow in capital goods), which is the consumption of firms. The sum $C_t + I_t$ is total spending in the economy.

The capital market The equilibrium of the capital market (in a closed economy) is simply a rewriting of the national income accounts identity:

$$Y_t - C_t = I_t \tag{68}$$

where $Y_t - C_t = S_t$ is aggregate saving. In a closed economy, aggregate saving is always equal to aggregate investment. This is an accounting construction:

$$S_t = I_t \tag{69}$$

Aggregate saving is the sum of individuals' saving, i.e. the part of labor income that is saved by the young generation:[6]

$$S_t = N_t s_t \tag{70}$$

Aggregate investment is the difference between the stocks of capital at two different time periods:

$$I_t = K_{t+1} - (1 - \delta)K_t \tag{71}$$

where δ is the rate of physical depreciation of the capital stock. In the OLG model in discrete time, a period is assumed to be equal to a working life, i.e. approximately 30 years. Therefore, it is reasonable to assume that the capital stock fully depreciates at the end of a period. Therefore, we assume that $\delta = 1$, which means

$$I_t = K_{t+1} \tag{72}$$

where K_{t+1} is the physical capital stock at time $t + 1$. The capital market equilibrium is therefore given by the following condition:

$$K_{t+1} = N_t s_t \tag{73}$$

where s_t is the individual saving function (8). The equilibrium is obtained when the factor prices are such that the optimal saving of the representative consumer (determined by the consumer's optimization) corresponds to the optimal amount of capital used by the representative firm (determined by the firm's optimization).

The labor market We assume that the supply of labor is inelastic, which means that the supply of labor does not depend on the market wage. The labor market is in equilibrium if its supply is equal to the demand. When the supply of labor is inelastic, the wage depends only on the demand for labor, which is determined by the firms' optimization program, i.e., the maximization of profit.

[6] In the Solow model, aggregate saving is a constant fraction of total income. This simplyfing assumption is relaxed in the OLG model where agents adjust their saving depending on their wage and the interest rate.

6 The macroeconomic equilibrium

The macroeconomic equilibrium is the general equilibrium in the three interdependent markets: the goods market, the capital market and the labor market. The equilibrium condition in all three markets is given by the equality between supply and demand. Since the markets are perfectly competitive, the equilibrium prices of the three markets correspond to the marginal cost of production of the aggregate output good.

The macroeconomic equilibrium is a general equilibrium at one point in time (temporary equilibrium) and over time (intertemporal equilibrium). The temporary equilibrium is obtained when the three markets clear at one point in time. On the labor market, the workforce is employed and remunerated at the equilibrium real wage. On the goods market, the output produced is sold at the equilibrium price to buyers who consume part of it and save the remaining part. Saving is traded on the capital market at the expected return to investors who will use it to build a physical asset that will carry over time, though imperfectly due to physical depreciation. The capital market thus makes the link between the present and the future. This link is represented by the unique dynamic equation of the model, which is also the temporary equilibrium of the capital market (73). Given the individual optimal saving function (8), the dynamic equation of the OLG model with two periods and perfect foresight can be rewritten as

$$K_{t+1} = N_t\phi(w_t, R_{t+1}) \tag{74}$$

and, in terms of per worker,

$$(1+n)k_{t+1} = \phi(w_t, R_{t+1}) \tag{75}$$

where $k \equiv \frac{K}{L}$ is the physical capital stock per worker or, for short, the capital-labor ratio. Equation (75) describes the physical capital accumulation per worker over time. The solution to the dynamic equation (75) establishes the intertemporal equilibrium, i.e. the macroeconomic equilibrium between the present and the future. However, the existence and uniqueness of the solution to the dynamic equation (75) is not straightforward. Recall that R_{t+1} is a function of k_{t+1} (see Equation (55)) while w_t is a function of k_t (see Equation (54)). Then, observe that k_{t+1} is a function of the optimal saving function $\phi(.)$ and $\phi(.)$ is a function of k_{t+1}. Therefore, the solution consists in finding the value for k_{t+1} that guarantees the general equilibrium at period t and the general equilibrium at period $t+1$. In other words, for a given k_t, does k_{t+1} exist and is it unique? Since there is no explicit function between k_{t+1} and its argument k_t, we can define the implicit relation

$$\Delta(w_t, k_{t+1}) \equiv (1+n)k_{t+1} - \phi(w_t, R_{t+1}) \tag{76}$$

which is equal to zero whenever k_{t+1} is a solution (= intertemporal equilibrium) to Equation (75).

Existence of an intertemporal equilibrium
In order to conclude that there exists at least one solution to Equation (75), it is necessary to show that

$$\Delta(w_t, k_{t+1}) \equiv (1+n)k_{t+1} - \phi(w_t, R_{t+1}) = 0 \tag{77}$$

occurs at least once. For a given $w_t > 0$, it is possible to show that $\Delta(w_t, k_{t+1})$ takes negative values when $k_{t+1} \longrightarrow 0$ and takes positive values when $k_{t+1} \longrightarrow +\infty$. Since it is continuous with respect k_{t+1}, $\Delta(w_t, k_{t+1})$ necessarily equals 0 for a positive k_{t+1} (see the existence proof in de la Croix & Michel (2002), pp. 20-21).
This proof establishes the existence of the solution to the dynamic equation (75) without additional assumptions.

Uniqueness of the intertemporal equilibrium
A necessary condition for the uniqueness of the solution is that $\Delta(w_t, k_{t+1}) = 0$ is strictly increasing in k_{t+1} (see de la Croix & Michel (2002), pp. 22-23). However, the assumptions we have made so far do not ensure that it is always the case. Therefore, we have to make an additional assumption:

$$\Delta'_{k_{t+1}}(w_t, k_{t+1}) = 1 + n - \phi'_{k_{t+1}}(w_t, R_{t+1})Af''(k_{t+1}) > 0 \tag{78}$$

where, I recall, $R_{t+1} = Af'(k_{t+1})$. Observe that this assumption is verified if

$$\phi'_{k_{t+1}}(w_t, R_{t+1}) > \frac{1+n}{Af''(k_{t+1})} \tag{79}$$

where the right-hand side of the inequality is negative. This implies that no all concave utility functions qualify. For instance, a utility function leading to a reduction in saving higher than $\frac{1+n}{Af''(k_{t+1})}$ when the interest factor increases at the margin would not qualify.
Thanks to assumption (78), the intertemporal equilibrium is unique at any t, which implies that there is a unique trajectory of the macroeconomic equilibrium over time.

Unfortunately, the uniqueness condition (78) is not easily translated into assumptions on preferences and production technology. Therefore, a stronger but sufficient condition is to assume that the utility function must have an intertemporal elasticity of substitution greater or equal to one:

$$\sigma \equiv -\frac{u'(c)}{u''(c)c} \geqslant 1 \tag{80}$$

Example: OLG model with logarithmic utility function and Cobb-Douglas production function
By using Equation (14), we can write the dynamic equation of an OLG model with logarithmic utility function and Cobb-Douglas production function as

$$(1+n)k_{t+1} = \frac{\beta}{1+\beta}w_t \tag{81}$$

Since $w_t = (1-\alpha)Ak_t^\alpha$ when the production technology is Cobb-Douglas, the capital accumulation equation per worker can be rewritten as

$$(1+n)k_{t+1} = \frac{\beta}{1+\beta}(1-\alpha)Ak_t^\alpha \tag{82}$$

Equation (82) shows that k_{t+1} exists and is uniquely defined by the past value k_t and a bunch of parameters. This equation can be applied to all t and thus defines the dynamic trajectory of this economy.

The steady state

Is our dynamic model converging to a stationary (or steady) state? This is the fundamental question that must be answered when studying a dynamic model. In this section, we want to know whether a decentralized (=market) economy can converge to a steady state. In other words, does a decentralized economy follow a convergent trajectory or not? If yes, we can characterize this steady state and draw conclusions about the conditions and the characteristics of a stylized market economy.

First, the definition of a steady state. A steady state is characterized by the sequence of $k_{t+1} = k_t \equiv \bar{k}$. When $\bar{k} = 0$, we say that the steady state is a corner steady state. When $\bar{k} = +\infty$ the economy grows without limit. When $\bar{k} > 0$, we say that the steady state is an interior steady state.

Example: OLG model with logarithmic utility function and Cobb-Douglas production function

The interior steady state of an OLG model with logarithmic preferences and Cobb-Douglas technology is

$$\bar{k} = \left(\frac{\beta A(1-\alpha)}{(1+n)(1+\beta)}\right)^{\frac{1}{1-\alpha}} \tag{83}$$

Observe that the steady state only depends on parameters. Once we know the steady-state value $\bar{k}$, we can calculate the steady-state values of the wage, the interest rate and the consumption levels of the young and the old generations.

Note that such a dynamic model does not display long-term growth in income per capita because $k_{t+1} - k_t = 0$ at the steady state. The reason is the same as in the Solow model: as long as there are diminsihing returns to physical capital K_t, no long-term growth can emerge.

Dynamics

Once we have found the steady state of our economy, the next question is whether the economy converges to it. In other words, we have to show that the sequence of $(1+n)k_{t+1} = \phi(w_t, R_{t+1})$ is a monotonic sequence. The condition for monotonic dynamics is that the function $\phi(w_t, R_{t+1})$ is continuous and non-decreasing on the domain $k \in]0, +\infty[$.

If assumption (80) is verified, then the dynamics is monotonic and the economy converges to the steady state. The question is: which steady state does the economy converge to? Three possible cases: two cases at the boundaries of the domain ($\bar{k} = 0$ and $\bar{k} = +\infty$) and one case at the interior of the domain ($\bar{k} > 0$). The infinite case $\bar{k} = +\infty$ can be excluded because unlimited growth is impossible in an OLG model due to the fact that saving is bounded by the wage. The other boundary case, $\bar{k} = 0$, is possible if the saving function is convex for low values of k. Such a steady state is called a poverty trap and the OLG model is a relevant framework to study it. If saving is a strictly concave function of k, the steady state is interior ($\bar{k} > 0$), unique and globally stable.

Example: OLG model with logarithmic utility function and Cobb-Douglas production function
The logaritmic utility function satisfies assumption (80). In fact, the intertemporal elasticity of subsitution of a logarithmic utility function is one. Therefore, the dynamics of an OLG model with logarithmic preferences and Cobb-Douglas technology converges to its unique interior steady state.

7 Optimality

In this section, our aim is to study the optimality of a stationary equilibrium of an OLG model. Two questions are related to optimality:

1) Is the stationary equilibrium of the decentralized economy a social optimum?

2) Is the stationary equilibrium of the decentralized economy Pareto optimal?[7] In other words, does this stationary equilibrium verify the First Welfare Theorem (FWT)?

Until now, the optimizing behaviors have been considered from the point of view of a single generation who cared to maximize her own lifecycle welfare regardless of the welfare impact of her decisions to the other generations. We now turn to the welfare of the young and old generations living at the same time and investigate which saving decision would maximize the total welfare at every time period. If the saving decision made selfishly by a single generation (decentralized economy) turns out to be identical to the saving decision necessary to maximize the welfare of two generations living at the same time at any time period, then the steady state obtained in the decentralized economy is the Golden Rule (=social optimum). Put differently, is the private optimum (individual welfare maximization) the same as the social optimum (social welfare maximization)? If

[7]Pareto optimality is obtained when it is no longer possible to increase the welfare of one individual without worsening the welfare of another individual. Be aware that our optimality investigation compares welfare results across generations and *not* across individuals within a generation. In the present OLG model, the representative agent's assumption prevents us from studying distributional issues within a generation. The study of income inequalities requires a more complex OLG model, in which agents are heterogeneous within a generation.

not, is the private optimum Pareto optimal? If it is Pareto optimal, the private optimum satisfies the FWT and moving from the private optimum to the social optimum cannot be Pareto improving.

The individual welfare maximization is constrained by the individual resources. The social welfare maximization is constrained by the resources of the aggregate economy. At time t, the resource constraint of the aggregate economy is

$$Y_t = I_t + N_t c_t + N_{t-1} d_t \tag{84}$$

where the aggregate ouptut is on the left-hand side (resources of the aggregate economy) and the aggregate spending is on the right-hand side. The aggregate spending includes the aggregate investment I_t, the consumption by the young generation $N_t c_t$ and the consumption by the old generation $N_{t-1} d_t$ at time t. Using Equation (72) and taking the fact that $L_t = N_t$, we can rewrite (84) in per-worker terms:

$$y_t = (1+n)k_{t+1} + c_t + \frac{d_t}{1+n} \tag{85}$$

where $y_t \equiv \frac{Y_t}{N_t}$ is the income per worker.[8]

The condition for the steady state of the decentralized economy to be a social optimum is that a value of $\bar{k}$ solves the following system of equations:

$$(1+n)\bar{k}_{t+1} = \phi(\bar{w}_t, \bar{R}_{t+1}) \tag{86}$$

$$\bar{y}_t - (1+n)\bar{k}_{t+1} = \bar{c}_t + \frac{\bar{d}_t}{1+n} \tag{87}$$

$$\bar{y}_{t+1} - (1+n)\bar{k}_{t+2} = \bar{c}_{t+1} + \frac{\bar{d}_{t+1}}{1+n} \tag{88}$$

$$\bar{c}_t, \bar{d}_t, \bar{c}_{t+1}, \bar{d}_{t+1} > 0 \tag{89}$$

where Equation (86) is the dynamic equation of the decentralized OLG economy, i.e. the dynamic equation for the optimizing generation t, while Equations (87) and (88) are the resource constraints of the aggregate OLG economy during the lifecycle of generation t. Since the study of this economy from two points of view (one from the point of a single generation and one point of view of the total population living at a time) is carried out at the steady state, you can observe that Equations (87) and (88) are identical. Therefore, we can rewrite the condition for a value of $\bar{k}$ to result in a social optimum as

$$(1+n)\bar{k}_{t+1} = \phi(\bar{w}_t, \bar{R}_{t+1}) \tag{90}$$

$$\bar{y}_t - (1+n)\bar{k}_{t+1} = \bar{c}_t + \frac{\bar{d}_t}{1+n} \tag{91}$$

$$\bar{c}_t, \bar{d}_t, > 0 \tag{92}$$

[8] Observe that the resource constraint is not divided by the total population. In fact, we are interested in comparing the welfare of a single welfare-maximizing generation and the social welfare for two generations living at the same time. This is the reason why we expressed the aggregate resource per worker.

Equation (90) is the dynamic equation for the generation t whose optimal saving function is $\phi(\bar{w}_t, \bar{R}_{t+1})$. Once preferences and technology are given, the private optimum is known. But, what is the welfare optimum of the aggregate economy? The social optimum condition is the maximization of the welfare of two generations living at the same time, i.e. the maximization of (91) with respect to $\bar{c}_t$ and $\bar{d}_t$:

$$\max_{\bar{c}_t, \bar{d}_t} \ \bar{c}_t + \frac{\bar{d}_t}{1+n} \tag{93}$$

where $\bar{c}_t + \frac{\bar{d}_t}{1+n} = \bar{y}_t - (1+n)\bar{k}_{t+1}$. Therefore, we are looking for a value of $\bar{k}$ generating a social optimum, which we call $\bar{k}_{GR}$: the Golden-rule steady-state capital stock per worker. This value of $\bar{k} = \bar{k}_{GR}$ is the solution to the following social maximization problem:

$$\max_{\bar{k}_{GR}} \ Af(\bar{k}_{GR}) - (1+n)\bar{k}_{GR} \tag{94}$$

where $\bar{y}_{GR} = Af(\bar{k}_{GR})$ and $Af(\bar{k}_{GR}) - (1+n)\bar{k}_{GR} = \bar{c}_t + \frac{\bar{d}_t}{1+n}$. The first-order condition of (94) is

$$Af'(\bar{k}_{GR}) = 1+n \tag{95}$$

Equation (95) gives the condition for a steady-state capital stock per worker to maximize the welfare of the young and old generations living at the same time. This condition is called the Golden Rule.

Is the steady state capital stock per worker, resulting from the decentralized economy, a social optimum? The condition for it is that $\bar{k}_{GR}$ also solves Equation (90), the private welfare maximization:

$$(1+n)\bar{k}_{GR} = \phi(\bar{w}_{GR}, \bar{R}_{GR}) \tag{96}$$

If condition (96) is satisfied, then the long-term equilibrium of a market economy (i.e. the long-term equilibrium of the decentralized economy) is a social optimum. Observe condition (96). It is very unlikely that condition (96) is verified. To be verified, it is necessary that the competitive factor prices are $\bar{w}_{GR}$ and $\bar{R}_{GR}$. Since markets are perfectly competitive, no agent can fix factor prices equal to $\bar{w}_{GR}$ and $\bar{R}_{GR}$. Therefore, the chance for condition (96) to hold is one among many and no market forces, alone, can single it out.

However, if a government finds a way to redistribute wealth across generations as to generate saving equal to $\phi(\bar{w}_{GR}, \bar{R}_{GR})$, then the government intervention can help market forces to single out the steady state corresponding to the Golden Rule. Then the question is whether it is Pareto improving to move from the private optimum to the social optimum.

If $\bar{k} < \bar{k}_{GR}$, then the steady state of the decentralized economy $\bar{k}$ is Pareto optimal because it is not possible to reach the Golden Rule without imposing a decrease in total consumption. In fact, increasing $\bar{k}$ to reach $\bar{k}_{GR}$ requires higher saving from early generations, i.e. a decrease in their first-period consumption and a decrease in the return to physical capital (and, hence, saving). Therefore, the private optimum is Pareto optimal. We say that it verifies the FWT.

If $\bar{k} > \bar{k}_{GR}$, this is the opposite situation : the economy overaccumulates capital and is said to be dynamically inefficient. In other words, the steady state of the decentralized economy is not Pareto optimal because it is possible to decrease the steady-state capital-stock per worker by imposing less saving and more total consumption. We say that the private optimum does not verify the FWT. But, we can decrease $\bar{k}$ to reach $\bar{k}_{GR}$ by lowering saving from early generations, i.e. increasing their first-period consumption (and their welfare), which results in a higher return to physical capital (and, hence, saving) and, therefore, an increase in consumption for all generations. The Pareto optimum reached by this reallocation of ressources over time is the social optimum (Golden Rule). When the Pareto optimum is obtained by reallocating resources over time and, thus, by reducing the individual optimal saving level to increase total consumption, we say that this Pareto optimum verifies the Second Welfare Theorem (SWT).[9]

To conclude, the steady state of a decentralized economy of an OLG model with two periods and perfect foresight may violate the FWT but can be modified by an appropriate reallocation of ressources across generations to reach a Pareto optimum verifying the SWT. If the steady state of this decentralized economy is Pareto optimal (verifying the FWT), the process to reach the Golden Rule, i.e. increasing saving from early generations, is obviously not Pareto-improving.

Example: OLG model with logarithmic utility function and Cobb-Douglas production function

The Golden-Rule steady-state capital stock per worker with Cobb-Douglas technology is

$$\bar{k}_{GR} = \left(\frac{\alpha A}{1+n}\right)^{\frac{1}{1-\alpha}} \tag{97}$$

The value of the capital stock per worker solving Equation (82) is Equation (83), i.e.,

$$\bar{k}_{DE} = \left(\frac{\beta A(1-\alpha)}{(1+n)(1+\beta)}\right)^{\frac{1}{1-\alpha}} \tag{98}$$

where $\bar{k}_{DE}$ is the steady-state capital stock per worker of the decentralized economy with log utility and Cobb-Douglas production technology. This decentralized economy is a social optimum if and only if

$$\bar{k}_{DE} = \bar{k}_{GR} \tag{99}$$

[9] The Golden Rule should not be confused with the Pareto optimum. The Golden Rule is the social optimum while the Pareto optimum is a situation where the welfare of some can no longer be improved without worsening the others' welfare. For instance, a private optimum can be Pareto optimal although it is not a social optimum. This is the case in an OLG model when $\bar{k} < \bar{k}_{GR}$. However, the Golden Rule is always Pareto optimal. When it is possible to reach the Golden Rule, the macroeconomic equilibrium is a social optimum and Pareto optimal.

i.e.,

$$\frac{\beta}{1+\beta} = \frac{\alpha}{1-\alpha} \tag{100}$$

This condition is restrictive and it is unlikely that a real market economy verifies it. Empirical studies generally estimate $\beta = 0.97$ and $\alpha = 0.3$ for developed economies. So,

$$\begin{aligned} \frac{\beta}{1+\beta} &= 0.4923 \\ \frac{\alpha}{1-\alpha} &= 0.4285 \end{aligned}$$

We can conclude that the decentralized economy with log utility and Cobb-Douglas production technology will unlikely reach the Golden Rule. In our present example, $\frac{\beta}{1+\beta} > \frac{\alpha}{1-\alpha}$, so $\bar{k}_{DE} > \bar{k}_{GR}$, which means that the FWT is violated. There is thus room to improve the welfare of all generations by reaching a lower steady-state capital stock per worker corresponding to the Golden Rule.

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