

Saturation effects in pp scattering in the impact-parameter representation*

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The impact of unitarity is considered in different approaches to saturation in impact-parameter space. The energy and momentum-transfer dependence of the total and differential cross sections and of the ratio of the real to imaginary parts of the scattering amplitude are obtained in a model that includes soft and hard pomeron contributions, coupled to hadrons via the electromagnetic form factor. It is shown that the hard pomeron may significantly contribute to soft physics at the LHC. A similar conclusion can also be reached in the framework of non-linear approaches to unitarisation of the BFKL pomeron.

1. Introduction

One expects that non-linear effects will enter the BFKL equation in the non-perturbative infrared region, *i.e.* at large impact parameters. This is a different regime from that connected with the Black Disk Limit (BDL), in which saturation occurs at small impact parameters first.

The common viewpoint is that saturation will lead to a decrease of the growth of σ_{tot} . But the estimates of the energy after which saturation will be important vary a lot between different models.

Unitarity of the scattering matrix is connected with the properties of the scattering amplitude in the impact parameter representation, which is equivalent at high energy to a decomposition in partial waves. The scattering amplitude can then saturate the unitarity bound for impact parameters $b < b_i$. To satisfy the unitarity condition, there are different models. Two of them are based on the solution of the unitarity equation $SS^\dagger = 1$. First of all, in the U -matrix approach [1], one obtains a ratio $\sigma_{el}/\sigma_{tot} \rightarrow 1$, as $s \rightarrow \infty$. The second possible solution of the unitarity condition, which is the usual one, corresponds to

the eikonal representation

$$T(s, t) = i \int_0^\infty b db J_0(b\Delta) [1 - \exp(-\chi(s, b))] \quad (1)$$

with $t = \Delta^2$. If one takes the eikonal phase in a factorised form $\chi(s, b) = h(s) f(b)$, one usually supposes that, despite the fact that the energy dependence of $h(s)$ can be a power $h(s) \sim s^\Delta$, the total cross section will satisfy the Froissart bound $\sigma_{tot} \leq C \log^2(s)$.

We find in fact that the energy dependence of the imaginary part of the amplitude and hence of the total cross section depends on the form of $f(b)$, *i.e.* on the s and t dependence of the slope of the elastic scattering amplitude.

1.1. Eikonal representation and σ_{tot}

Let us first take a Gaussian for the eikonal phase $f(b) \sim \exp(-b^2/R^2)$. As was made first by Landau, let us introduce the new variable $y = \exp(-b^2/R^2)$. We can then calculate the integral exactly and obtain that

$$T(s, t) \sim i R^2 [\Gamma(0, s^\Delta) + \Delta \log s], \quad (2)$$

where $\Gamma(a, z) = \int_z^\infty t^{a-1} e^{-t} dt$ and, in our case, $\Gamma(0, s^\Delta) \rightarrow 0$ as $s \rightarrow \infty$. If R^2 is independent from s , we have $\sigma_{tot} \sim \log(s)$, whereas if R^2 grows not faster than $\log(s)$, the total cross section becomes proportional to the Froissart bound $\sigma_{tot} \sim \log^2(s)$.

However, let us now take a polynomial form $f(b) \sim s^\Delta/b^4$. Such a form comes, for example, from the BFKL equation. In that case, we can introduce the new variable $y = 1/b^4$ and obtain

$$\begin{aligned} T(s, t) &\sim i \int_0^\infty \frac{1}{y\sqrt{y}} [1 - \exp(-s^\Delta y)] \\ &= 2\sqrt{\pi} s^{\Delta/2}. \end{aligned} \quad (3)$$

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So, in this case, the scattering amplitude eventually violates the Froissart bound!

If we introduce an additional small constant radius r which removes the singular point $b = 0$ in $f(b)$ and take $f(b) \sim s^\Delta/[b^4 + r^4]$, the answer, after some complicated algebra, is

$$T(s, t = 0) \sim \frac{1}{4r^2} \left\{ \pi s^\Delta \exp \left[-\frac{s^\Delta}{2r^4} \right] \right\} \\ \times \left[I_0 \left(\frac{s^\Delta}{2r^4} \right) + I_1 \left(\frac{s^\Delta}{2r^4} \right) \right]. \quad (4)$$

The asymptotic value of the Modified Bessel functions is

$$I_{0,1} \left(s^\Delta/(2r^4) \right) \sim r^2 / \left(\sqrt{\pi} s^{\Delta/2} \right). \quad (5)$$

We again obtain for asymptotically high energies

$$T(s, t) \sim i \sqrt{\pi} s^{\Delta/2} / \left(2\sqrt{2} \right). \quad (6)$$

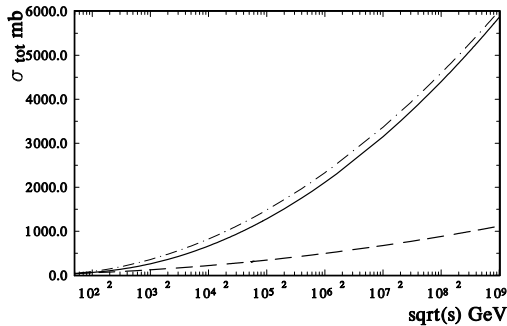


Figure 1. The total cross section of proton-proton scattering calculated in the eikonal representation (hard line: with an exponential form; dashed line: with a Gaussian form) compared with a $\log^2(s)$ dependence (dash-dotted line).

Finally, let us consider the more complicated (but most interesting) case, of an exponential form for $f(b) \sim \exp(-mb)$. The corresponding amplitude in the t representation is

$$T(s, t) \sim i s^\Delta {}_qF_p[(1, 1, 1), (2, 2, 2), -s^\Delta]. \quad (7)$$

with $p = q = 3$, and where the function ${}_qF_p$ is the hyper-geometric function.

A numerical estimate of this integral shows that we obtain for the exponential form of the

eikonal an energy dependence of the scattering amplitude in the t -representation which can be approximated as $T(s, t = 0) \sim i a \log^2(s/s_0)$, with large coefficients $a = 4.5$ and $s_0 = 135 \text{ GeV}^2$.

Hence, in this case, the scattering amplitude obeys the Froissart bound, but with a large scale s_0 and a large coefficient a . This leads to a weak energy dependence at moderate energies followed by a fast growth at super-high energies.

Of course the eikonal representation, which most of the time leads to a unitarity answer at every impact parameter, leads to amplitudes which do not saturate at finite energy: the eikonal representation for the scattering amplitude in b -space, in the form $1 - \exp(-\chi(s, b))$, reaches the BDL only asymptotically. However, this representation is not the only possibility, and it may be more useful to consider the effects of saturation by considering parametrisations in s and t , transforming them to impact parameter space, and imposing directly the BDL as an upper bound on the amplitude in s and b .

1.2. Non-linear effects and σ_{tot}

A different approach to saturation is found in the studies of the non-linear saturation processes, which have been considered in a perturbative QCD context [2,3]. Such processes lead to an infinite set of coupled evolution equations in energy for the correlation functions of multiple Wilson lines [4]. In the approximation where the correlation functions for more than two Wilson lines factorise, the problem reduces to the non-linear Balitsky-Kovchegov (BK) equation [4,5].

It is unclear how to extend these results to the non-perturbative region, but one will probably obtain a similar equation. In fact we found simple differential equations that reproduce either the U-matrix or the eikonal representation. We can first consider saturation equations of the general form

$$\partial N(\xi, b)/\partial \xi = \mathcal{S}(N) \quad (8)$$

with N the true (saturated) imaginary part of the amplitude. We shall impose that (a) $N \rightarrow 1$ as $s \rightarrow \infty$, (b) $\partial N/\partial \xi \rightarrow 0$ as $s \rightarrow \infty$, (c) $\mathcal{S}(N)$ has a Taylor expansion in N , with the hard pomeron $N_{bare} = f(b)s^\Delta$ as a first term. This enables us to fix the integration constant by demanding that the first term of

the expansion in s^Δ reduces to N_{bare} .

Inspired by the BK results, we shall use the evolution variable $\xi = \log s$. If we want to fulfil condition (c), then we need to take $\mathcal{S}(N) = \Delta N + O(N^2)$. Conditions (a) and (b) then give $\mathcal{S}(N) = \Delta(N - N^2)$ as the simplest saturating function. The resulting equation

$$\partial N / \partial \log s = \Delta(N - N^2) \quad (9)$$

has the solution

$$N = f(b)s^\Delta / (f(b)s^\Delta + 1) \quad (10)$$

One can in fact go further: eq. (9) has been written for the imaginary part of the amplitude. If we want to generalise it to a complex amplitude, so that it reduces to (9) when the real part vanishes, we must take:

$$\partial A / \partial \log s = \Delta(A + iA^2) \quad (11)$$

The solution of this is exactly the form obtained in the U-matrix formalism, for $\Im U(s, b) = s^\Delta f(b)$.

Many other unitarisation schemes are possible, depending on the function $\mathcal{S}(N)$. We shall simply indicate here that the eikonal scheme can be obtained as follows:

$$\partial N / \partial \log s = \Delta(1 - N)(-\log(1 - N)) \quad (12)$$

Other unitarisation equations can be easily obtained via another first-order equation. The idea here is that the saturation variable is the imaginary part of the bare amplitude. One can then write

$$\begin{aligned} \partial N / \partial N_{bare} &= \mathcal{S}'(N) \Rightarrow \\ \partial N / \partial \log s &= [\partial N_{bare} / \partial \log s] \mathcal{S}'(N) \end{aligned} \quad (13)$$

with N_{bare} the unsaturated amplitude.

This will trivially obey the conditions (a)-(c) above, and saturate at $N = 1$. Choosing $\mathcal{S}'(N) = 1 - N$ gives the eikonal solution whereas $\mathcal{S}'(N) = (1 - N)^2$ leads to the U-matrix representation (10). We have thus shown that the most usual unitarization schemes could be recast into differential equations which are reminiscent of saturation equations [4,5]. Such an approach can be used to build new unitarization schemes and may also shed some light on the physical processes underlying the saturation regime.

2. Conclusion

In the presence of the hard Pomeron [6], the saturation effects can change the behaviour

of some features of the cross sections already at LHC energies. Some forms of the eikonal phase in the factorising eikonal representation can lead to a violation of the Froissart bound. Non-linear effects which work in the whole energy region supply an acceptable growth of the total cross sections. Saturation leads to a relative growth of the contribution of peripheral interactions. The most usual unitarization schemes could be recast into differential equations which are reminiscent of saturation equations. Such an approach can be used to build new unitarization schemes and may also shed some light on the physical processes underlying the saturation regime.

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