Surface Impedance Boundary Conditions for the Modeling of Saturable Massive Conducting Volumes in Time-Domain Finite-Element Calculations

J. Gyselinck [1], P. Dular [2,3], C. Geuzaine [2], R. V. Sabariego [2]

[1] Dept. of Bio-, Electro- and Mechanical Systems (BEAMS), Université Libre de Bruxelles, Belgium, Johan.Gyselinck@ulb.ac.be
[2] Dept. of Electrical Engineering and Computer Science (ACE), University of Liège, [3]FNRS, Belgium

***Abstract* —The authors propose a novel nonlinear time-domain extension of the well-known frequency-domain surface-impedance method in computational magnetodynamics. Herein the 1-D eddy-current problem in a massive conducting region (semi-infinite slab) is considered via a number of exponentially decreasing trigonometric basis functions that cover the relevant frequency range of the application at hand. The resulting nonlinear equations are solved using the Newton-Raphson method. The method is validated by means of a simple 2-D test case.**

I. INTRODUCTION

Surface-impedance boundary conditions (SIBCs) are widely applied in frequency-domain eddy-current problems for considering massive conducting regions. The approach is based on the relation between the tangential components of the electric and the magnetic field at the surface of the region and allows to discretize only its surface. Several refinements concerning mostly the surface curvature [1] but also the material saturation [2] have been presented in literature.

The few time-domain extensions proposed to date are mostly based on the fast Fourier transform [1, 3], on the iterative coupling between the main 3-D finite element (FE) model and a large number of 1-D FE calculations (with classical nodal basis functions) [4].

The authors proposed recently a linear time-domain approach based on dedicated basis functions derived from the analytical frequency-domain solution [5]. This approach is hereafter extended to saturable regions.

II. 1-D NONLINEAR EDDY-CURRENT PROBLEM IN SEMI-INFINITE SLAB

The Maxwell equations and constitutive laws relevant in low-frequency eddy-current problems are:

$$\operatorname{div} \underline{b} = 0\,, \qquad \operatorname{curl} \underline{h} = \underline{j}\,, \qquad \operatorname{curl} \underline{e} = -\partial_t\, \underline{b}\,, \quad \text{(1 a-c)}$$

$$\underline{j} = \sigma\, \underline{e}\,, \qquad \underline{h} = \underline{h}(\underline{b})\,, \quad \text{(2 a b)}$$

where the vector fields $\underline{b}$, $\underline{h}$, $\underline{j}$ and $\underline{e}$ are the flux density (or induction), the magnetic field, the current density and the electric field, respectively; σ is the conductivity. For linear isotropic magnetic media, (2b) reduces to $\underline{h} = \nu\, \underline{b}$, the reluctivity ν and permeability $\mu = 1/\nu$ being constant scalars.

We introduce the magnetic vector potential $\underline{a}$ in order to strongly satisfy (1a,c), i.e. $\underline{b} = \operatorname{curl} \underline{a}$ and $\underline{e} = -\partial_t \underline{a}$. The remaining equations in (1-2) lead to the following nonlinear partial differential equation:

$$\operatorname{curl} \underline{h}(\operatorname{curl} \underline{a}) \;=\; \underline{j}_s - \sigma\, \partial_t\, \underline{a}\,, \tag{3}$$

where $\underline{j}_s$ is the prescribed source current density in a subdomain $\Omega_s \subset \Omega$; the current density $\underline{j} = -\sigma\, \partial_t\, \underline{a}$ is induced in a conducting domain Ω_c.

*This work was partly supported by the Belgian Science Policy (IAP P6/21).

A. 1-D eddy-current problem and FE model

We consider now the 1-D eddy-current problem in a semi-infinite slab ($0 \le x \le \infty$), with $\underline{b}(x,t)$ and $\underline{h}(x,t)$ parallel to the z-axis, and $\underline{j}(x,t)$ and $\underline{e}(x,t)$ parallel to the y-axis. The source current density $\underline{j}_s$ is zero. With the y-component of the vector potential $\underline{a}(x,t)$ denoted by $a(x,t)$, (3) leads to

$$\partial_x h(\partial_x a) \;=\; \sigma\, \partial_t a \quad \text{with} \quad a(x=\infty,t) = 0\,, \tag{4}$$

where the boundary condition at infinity ($x = \infty$) ensures the uniqueness of $a(x,t)$. The flux in the semi-infinite slab may be imposed via the boundary value $a(x=0,t)$. Alternatively the magnetomotive force may be imposed; which is easily done at the discrete level, as shown hereafter.

The FE discretisation of (4) by means of N basis functions $\alpha_i(x)$, $0 \le x < \infty$, $1 \le i \le N$, leads to a system of first-order differential equations in terms of the associated values of $a(x,t)$. In the linear case (constant ν) and prescribed magnetomotive force, the system of equations reads

$$[S][A(t)] + [M]\, \partial_t [A(t)] \;=\; [I(t)]\,, \tag{5}$$

where the column matrix $[A(t)]$ comprises the N degrees of freedom of $a(x,t)$; only the first element of the column matrix $[I(t)]$ is non-zero and equals the magnetomotive force; the elements of $[S]$ and of $[M]$ are given by

$$S_{ij} = \nu \int_0^\infty \partial_x \alpha_i(x)\, \partial_x \alpha_j(x)\, dx\,, \tag{6}$$

$$M_{ij} = \sigma \int_0^\infty \alpha_i(x)\, \alpha_j(x)\, dx\,. \tag{7}$$

In the nonlinear case, the nonlinear algebraic equations that result from the time discretisation of (5) can be solved by means of the Newton-Raphson scheme. The Jacobian matrix $[J]$ and the associated column matrix $[H]$, which is part of the residue, are given by:

$$J_{ij} = \int_0^\infty \frac{d\,h}{d\,b}\, \partial_x \alpha_i(x)\, \partial_x \alpha_j(x)\, dx\,, \tag{8}$$

$$H_i = \int_0^\infty h(b)\, \partial_x \alpha_i(x)\, dx\,. \tag{9}$$

The differential reluctivity $\frac{d\,h}{d\,b}$ in (8) can also be written in terms of $\nu(b^2)$ and its derivative with respect to b^2:

$$\frac{d\,h}{d\,b} = \nu + 2\, b^2 \frac{d\,\nu}{d\,b^2}\,. \tag{10}$$

The above FE scheme could be carried out with a classical first-order FE discretisation, i.e. with a truncated interval $0 \le x \le x_{max}$ split up (uniformly or not) into N line segments and with the basis functions $\alpha_i(x)$ associated to all nodes but the last (considering $a(x_{max},t) = 0$). However, it is far more efficient to use dedicated basis functions.

B. Dedicated basis functions

The sinusoidal steady-state analytical solution of (4) at frequency f (pulsation $\omega = 2\pi f$) with boundary condition

$a(x=0,t) = \hat{a}\,\cos(\omega t + \phi)$ is given by

$$a(x,t) = \hat{a}\,e^{-x/\delta}\,\cos(x/\delta - \omega t - \phi) \qquad (11)$$

$$= \hat{a}\,\cos(\omega t + \phi)\,e^{-x/\delta}\cos(x/\delta) + \hat{a}\,\sin(\omega t + \phi)\,e^{-x/\delta}\sin(x/\delta)\,, \qquad (12)$$

with δ the skin depth and ϕ an arbitrary phase angle.

This motivates the following choice of basis functions [5]. For a given nonlinear time-domain problem, a set of skin depths δ_k can be preset accounting for the frequency content of the magnetic fields, the level of saturation and the accuracy required. The corresponding n pairs of basis functions $\alpha_{ck}(x)$ and $\alpha_{sk}(x)$, $0 \leq k \leq n$, thus read:

$$\alpha_{c1}(x) = e^{-x/\delta_1}\cos(x/\delta_1)\,, \qquad (13)$$

$$\alpha_{ck}(x) = e^{-x/\delta_k}\cos(x/\delta_k) - \alpha_{c1}(x)\,,\ \ 2 \leq k \leq n\,, \qquad (14)$$

$$\alpha_{sk}(x) = e^{-x/\delta_k}\sin(x/\delta_k)\,,\ \ 1 \leq k \leq n\,. \qquad (15)$$

Note that all basis functions vanish at the boundary $x = 0$ except the first one, i.e. $\alpha_{c1}(x=0) = 1$.

The matrix $[M]$ can be evaluated analytically (assuming σ constant), and so can $[S]$ in the linear case [5]. In the general nonlinear case, one has to resort to numerical integration for evaluating $[J]$ and $[H]$ at each NR-iteration.

III. INTEGRATION IN FE MODEL

The application of the linear 1-D eddy current model to a massive conducting region Ω_m in a 2-D or 3-D FE model is discussed in [5]. The extension to a saturable region Ω_m will be elaborated in detail in the full paper.

For 2-D FE models, using the one-component magnetic vector potential and classical first-order triangular elements, $N = 2n$ degrees of freedom are associated to each node on the boundary of Ω_m. The first of these N degrees of freedom (with basis function $\alpha_{c1}(x)$, equal to 1 at the boundary $x = 0$) ensures the link between the vector potential outside Ω_m and inside the abstracted region Ω_m.

IV. APPLICATION EXAMPLE

We consider a steel cylinder (circular cross-section with radius R equal to 10 cm; $\sigma = 2\,10^6$ S/m, $\nu(b^2) = 100 + 10\exp(1.8\,b^2)$ with b in T and ν in m/H) placed inside an inductor (rectangular cross-section coil) with imposed sinusoidal current (50 Hz, amplitude 6000 At). Only one quarter of the geometry is modeled (see Fig. 1). A transformation method is used to account for the extension of space to infinity. A classical FE model with a very fine discretisation of the cylinder near its surface provides an accurate reference solution. Two typical flux patterns obtained are shown in Fig. 1. When using the SIBC, only the mesh outside the cylinder is effectively considered.

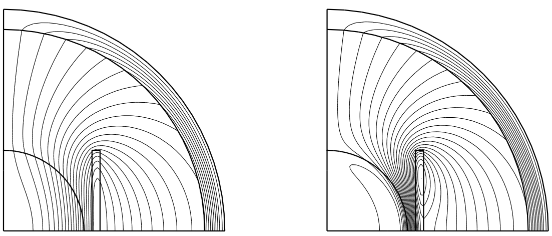

Figure 1. Flux pattern (in phase with imposed sinusoidal current) with skin depth equal to 0.5 (left) and 0.1 (right) times the radius of the cylinder

We adopt a low order approximation of the SIBC with $f_1 = 50$ kHz and further discrete frequencies being odd multiples of f_1, i.e. $f_k/f_1 = 2k-1$ with $1 \leq k \leq n$. The skin depths are thus related as $\delta_k/\delta_1 = 1/\sqrt{2k-1}$ with $\nu = 674$ m/H which corresponds to $b = 1.5$ T.

The induction at the point of the surface of the conducting cylinder closest to the inductor is depicted in Fig. 2. Fig. 3 shows the induction at a point halfway between the cylinder and the inductor. An excellent agreement is observed between the flux waveforms obtained with the reference FE model and the SIBC approach with $n = 3$. Even though in both cases, the approximation improves clearly with n, more terms are needed for increasing the precision at the surface of the cylinder.

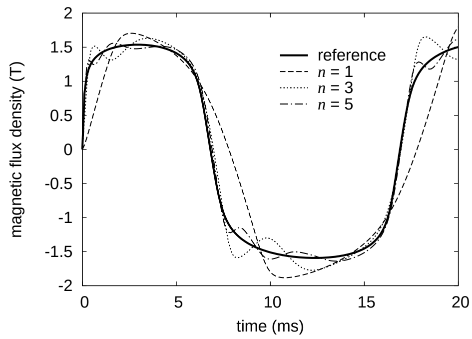

Figure 2. Induction waveform obtained with reference model and the SIBC approach ($n = 1, 3, 5$) at the surface of the cylinder

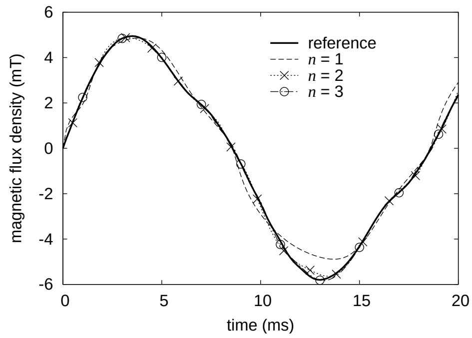

Figure 3. Induction waveform obtained with reference model and the SIBC approach ($n = 1, 2, 3$) at a point between the cylinder and the inductor

REFERENCES

[1] S. Barmada, L. Di Rienzo, N. Ida, S. Yuferev, "The use of surface impedance boundary conditions in time domain problems: numerical and experimental validation", *ACES Journal*, vol. 19, pp. 76–83, July 2004.

[2] L. Krähenbühl, O. Fabrègue, S. Wanser, M. De Sousa Dias, A. Nicolas, "Surface impedances, BIEM and FEM coupled with 1D non linear solutions to solve 3D high frequency eddy current problems", *IEEE Trans. Magn.*, vol. 33, pp. 1167–1172, March 1997.

[3] K.R. Davey, L. Turner, "Transient eddy current analysis for generalized structures using surface impedances and the fast Fourier transform", *IEEE Trans. Magn.*, vol. 26, pp. 1164–1170, May 1990.

[4] D. Rodger, H.C. Lai, "A surface impedance method for 3D time transient problems", *IEEE Trans. Magn.*, vol. 35, pp. 1369–1371, May 1999.

[5] J. Gyselinck, P. Dular, C. Geuzaine, R. V. Sabariego, "Surface-impedance boundary conditions in time-domain finite-element calculations using the magnetic-vector-potential formulation", *Proceedings of CEFC 2008*, p. 392.