

PROPER ORTHOGONAL DECOMPOSITION AND MODEL REDUCTION OF NONLINEAR SYSTEMS

G. Kerschen, V. Lenaerts and J.C. Golinval

*Université de Liège, LTAS – Vibrations et identification des structures,
Chemin des chevreuils, 1 (Bât. B52), 4000 Liège
Email : g.kerschen@ulg.ac.be*

SUMMARY : This work deals with the application of the proper orthogonal decomposition (POD) to structural dynamics. On one hand, a physical interpretation of the proper orthogonal modes (POM) is given using the singular value decomposition (SVD). It is shown that the POM converge to the normal modes under some circumstances. On the other hand, the POM are exploited for model reduction purposes. The efficiency of the reduced model for the prediction of the response is tested on a nonlinear beam. The comparison with another reduced model obtained via the normal modes of the linearised system points out that the POM are optimal for the reconstruction of the dynamics of a system.

KEYWORDS : proper orthogonal decomposition, nonlinear dynamics, singular value decomposition, proper orthogonal modes, normal modes, model reduction.

INTRODUCTION

In many domains of applied sciences and in structural dynamics particularly, we have to deal with large-scale structural models. When nonlinearities occur, seeking for the solution by use of the finite element method may become computationally expensive. Thus, due to the high-dimensionality of such systems, it is necessary to reduce the complexity of the system while retaining its intrinsic properties. The motivation for reducing the dimensionality of dynamical system models is obvious : models of lower dimension are easier to work with for various purposes such as simulation, optimisation and control.

A general approach to model reduction is to find a coordinate transformation in the state space in order to sort out the state components in terms of their influence on the system behaviour. Then, the components of the transformed system with relatively small influence may be truncated without substantially degrading the predictive capability of the model.

The Proper Orthogonal Decomposition (POD), also known as Karhunen-Loève Transform (K-L), allows to obtain such a coordinate transformation. It is a statistical pattern analysis technique for finding the dominant structures in an ensemble of spatially distributed data. These structures called the proper orthogonal modes (POM) may be exploited as an

orthogonal basis for efficient representation of the ensemble. A key advantage of the decomposition is that each POM is associated with a proper orthogonal value (POV) which immediately provides the relative energy, i.e. mean square fluctuation, captured by the mode. Thus, it serves as a well-defined measure of a mode influence on the system behaviour.

The POD technique, described in the next section, was first applied to turbulence problems by Lumley [1]. Since then the method has also been applied successfully in the fields of thermics [2] and signal processing [3]. It is now emerging as a useful tool in structural dynamics for dynamical characterisation of nonlinear systems [4], for model reduction [5,6] and for updating of nonlinear structures [7].

PROPER ORTHOGONAL DECOMPOSITION (POD)

The definitions and formulation presented here closely follow the ones used in [5]. Let us define a random field $y(x,t)$ on a domain Ω . This field is decomposed into mean $Y(x) = \langle y \rangle$ and time varying $v(x,t)$ parts :

$$y(x,t) = Y(x) + v(x,t) \quad (1)$$

Suppose that the different fields are sampled at finite number of instants in time. At a fixed time t_n , the system displays a snapshot v_n which is a continuous function of x with $x \in \Omega$. The aim is to determine the coherent structures in the data. By coherent structures, we mean functions of the spatial variables that have maximum energy content. It can be argued [5] that these coherent structures $\sigma(x)$ maximise the following expression :

$$\frac{\langle (\sigma, v_n)^2 \rangle}{(\sigma, \sigma)} \quad (2)$$

where $(f, g) \equiv \int_{\Omega} f(x)g(x) d\Omega$ and $\langle v_n \rangle \equiv \frac{1}{N} \sum_{n=1}^N v_n(x)$

A necessary condition for $\sigma(x)$ to maximise expression (2) is that it is a solution of the following integral eigenvalue problem :

$$\int_{\Omega} K(x, x') \sigma(x') dx' = \lambda \sigma(x) \quad (3)$$

where the two point spatial correlation function K is defined as

$$K(x, x') = \frac{1}{N} \sum_{n=1}^N v_n(x) v_n(x') \quad (4)$$

In practice, the snapshots are available at discrete measurement points x_i where $i = 1, \dots, M$ and the integral eigenvalue problem (3) reduces to find the eigenfunctions of a $M \times M$ space correlation tensor :

$$\mathbf{R} \boldsymbol{\sigma} = \lambda \boldsymbol{\sigma} \text{ where } \mathbf{R} = \begin{bmatrix} K(x_1, x_1) & \dots & K(x_1, x_M) \\ \dots & & \dots \\ K(x_M, x_1) & \dots & K(x_M, x_M) \end{bmatrix} \quad (5)$$

This is referred as the *direct problem*. When the number of measurement points M is much higher than the number of snapshots N , solving directly the latter eigenvalue problem may become computationally expensive. The *method of snapshots* suggested by Sirovich [8] offers an efficient means of reducing the initial $M \times M$ problem to an $N \times N$ problem. However, in structural dynamics, the number of degrees of freedom is often much lower than the number of samples. This is the reason why the *direct method* is preferred in this paper.

To summarise, if the displacements of a discrete dynamical system with M degrees of freedom are sampled N times and if the $M \times N$ matrix

$$\mathbf{Y} = \begin{bmatrix} y_1(t_1) & \dots & y_1(t_N) \\ \dots & & \dots \\ y_M(t_1) & \dots & y_M(t_N) \end{bmatrix} = [\mathbf{y}(t_1) \quad \dots \quad \mathbf{y}(t_N)] \quad (6)$$

is formed, then the coherent structures associated with the data are merely the eigenvectors of $\mathbf{R} = \frac{1}{N} \mathbf{Y} \mathbf{Y}^T$. These coherent structures are called the proper orthogonal modes (POM) and the corresponding eigenvalues are the proper orthogonal values (POV). A POV measures the relative energy of the system dynamics contained in the associated POM. It is worth noticing that they have to be normalised to correspond to a percentage.

If the system response is expanded in terms of an orthonormal set of basis functions as

$$\mathbf{y}(t) = \sum_{j=1}^M a_j(t) \boldsymbol{\phi}_j \quad (7)$$

the purpose of the reconstruction procedure is to predict the response using a minimum number of basis functions :

$$\mathbf{y}(t) \approx \tilde{\mathbf{y}}(t) = \sum_{j=1}^n a_j(t) \boldsymbol{\phi}_j \text{ where } n < M \quad (8)$$

The most important feature of the POD is that the first n POM capture more energy on average than the first n functions of any other basis [9]. This property is the basis for the claim that the POD is optimal for reconstructing a signal. Thus, the best orthonormal set of basis functions to be used in a reconstruction procedure is obviously the subspace spanned by the first n POM which capture 99.99% or more of the signal energy.

Finally, it is important to establish the closed relationship between the POD and the singular value decomposition (SVD). The matrices considered here are assumed to be real. Any $M \times N$ matrix \mathbf{Y} can be decomposed as the product of an $M \times M$ orthogonal matrix \mathbf{U} , an $M \times N$ pseudo-diagonal matrix $\boldsymbol{\Sigma}$ and the transpose of an $N \times N$ orthogonal matrix \mathbf{V} :

$$\mathbf{Y} = \mathbf{U} \mathbf{\Sigma} \mathbf{V}^T \quad (9)$$

The columns of \mathbf{U} and \mathbf{V} forms respectively the left and right singular vectors of \mathbf{Y} and the diagonal entries of $\mathbf{\Sigma}$ are the singular values. It is easy to verify [10] that the singular values of \mathbf{Y} are found as the square roots of the eigenvalues of $\mathbf{Y}\mathbf{Y}^T$ or $\mathbf{Y}^T\mathbf{Y}$ while the left and right singular vectors are respectively the eigenfunctions of $\mathbf{Y}\mathbf{Y}^T$ and $\mathbf{Y}^T\mathbf{Y}$. Hence, the POD can be carried out directly by means of a SVD of the displacement matrix \mathbf{Y} . The left singular vectors are identical to the POM while the singular values are the square roots of the POV multiplied by the number of samples.

POM AND NORMAL MODES

1. Undamped and unforced systems

The aim of this section is to find the existing relationships between the POM and normal modes. Let first consider an undamped and unforced linear system with M degrees-of-freedom for which equation of motion is given as follows

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{K}\mathbf{y} = 0 \quad (10)$$

The system response due to initial conditions may be expressed as

$$\mathbf{y}(t) = \sum_{i=1}^M (A_i \cos \omega_i t + B_i \sin \omega_i t) \mathbf{x}_{(i)} = \sum_{i=1}^M e_i(t) \mathbf{x}_{(i)} \quad (11)$$

where $(\omega_i, \mathbf{x}_{(i)})$ are respectively the eigenvalues and eigenvectors of the system ;

A_i and B_i are constants depending on the initial conditions ;

$e_i(t) = A_i \cos \omega_i t + B_i \sin \omega_i t$ represents the time modulation of the mode $\mathbf{x}_{(i)}$.

In discrete time domain, we take N sampled values of the time functions and form a M by N matrix whose columns are the members of the data ensemble

$$\mathbf{Y} = [\mathbf{y}(t_1) \quad \dots \quad \mathbf{y}(t_N)] \quad (12)$$

$$\mathbf{Y} = \left[\sum_{i=1}^M e_i(t_1) \mathbf{x}_{(i)} \quad \dots \quad \sum_{i=1}^M e_i(t_N) \mathbf{x}_{(i)} \right] \quad (13)$$

which can also be written as

$$\begin{aligned} \mathbf{Y} &= [\mathbf{x}_{(1)} \quad \dots \quad \mathbf{x}_{(M)}] \begin{bmatrix} e_1(t_1) & \dots & e_1(t_N) \\ \dots & & \dots \\ e_M(t_1) & \dots & e_M(t_N) \end{bmatrix} = [\mathbf{x}_{(1)} \quad \dots \quad \mathbf{x}_{(M)}] \begin{bmatrix} \mathbf{e}_1^T \\ \dots \\ \mathbf{e}_M^T \end{bmatrix} \\ &= [\mathbf{x}_{(1)} \quad \dots \quad \mathbf{x}_{(M)}] [\mathbf{e}_1 \quad \dots \quad \mathbf{e}_M]^T \\ &= \mathbf{A} \mathbf{B}^T \\ &= \mathbf{A} [\mathbf{I} \quad \mathbf{Z}] [\mathbf{B} \quad \mathbf{R}]^T \end{aligned} \quad (14)$$

where \mathbf{A} is a $M \times M$ matrix whose columns are the normal modes of the system ;
 \mathbf{B} is a $N \times M$ matrix whose columns are the functions $e_i(t)$ at times t_1, \dots, t_N ;
 \mathbf{I} is a $M \times M$ identity matrix ;
 \mathbf{Z} is a $M \times (N - M)$ matrix of zeros ;
 \mathbf{R} is a $N \times (N - M)$ matrix.

Attention is directed toward the fact that \mathbf{R} does not influence \mathbf{Y} since it is multiplied by a matrix full of zeros.

Equation (14) can be expressed in a more familiar form, i.e. :

$$\mathbf{Y} = \mathbf{A} [\mathbf{I} \quad \mathbf{Z}] [\mathbf{B} \quad \mathbf{R}]^T = \mathbf{U} \Sigma \mathbf{V}^T \quad (15)$$

Thus, the above decomposition of \mathbf{Y} may be thought as the SVD of this matrix. However, this decomposition requires matrices \mathbf{U} and \mathbf{V} to be orthogonal (see previous section). The aim now is to find when the columns of \mathbf{U} and \mathbf{V} are orthogonal.

1. The columns of $\mathbf{U} (\equiv \mathbf{A})$ are formed by the normal modes. Since the normal modes are orthogonal to each other with respect to the mass and stiffness matrices, if the mass matrix is proportional to the identity matrix, it turns out that $\mathbf{x}_{(i)}^T \mathbf{x}_{(j)} = \delta_{ij}$. Consequently, \mathbf{A} is orthogonal if the mass matrix is proportional to the identity matrix.
2. It remains to find when the columns of $\mathbf{V} (\equiv [\mathbf{B} \quad \mathbf{R}])$ are orthogonal. That is why equation (6) is rewritten as follows

$$\begin{aligned} \mathbf{Y} &= \mathbf{A} [\mathbf{I} \quad \mathbf{Z}] [\mathbf{B} \quad \mathbf{R}]^T \\ &= \begin{bmatrix} \mathbf{x}_{(1)} & \dots & \mathbf{x}_{(M)} \end{bmatrix} \begin{bmatrix} \|\mathbf{e}_1\| & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \|\mathbf{e}_2\| & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \|\mathbf{e}_M\| & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \dots & \mathbf{e}_N \\ \|\mathbf{e}_1\| & \|\mathbf{e}_2\| & \dots & \|\mathbf{e}_M\| \mathbf{R} \end{bmatrix}^T \quad (16) \end{aligned}$$

If the frequencies are distinct, it can be easily argued that the columns of \mathbf{B} are orthogonal if we consider an infinite set of sampled values, i.e. $\frac{\mathbf{e}_i}{\|\mathbf{e}_i\|} \cdot \frac{\mathbf{e}_j}{\|\mathbf{e}_j\|} \rightarrow 0$ if $N \rightarrow \infty, i \neq j$. Since

\mathbf{R} does not have an influence on \mathbf{Y} , its columns can be computed in order that they are orthogonal to those of \mathbf{B} .

To summarise, if the mass matrix is proportional to the identity matrix and if the number of samples is infinite, the singular value decomposition of \mathbf{Y} is such that :

- the columns of \mathbf{U} are the normal modes ;
- the N first columns of \mathbf{V} are the normalised time modulations of the modes.

As stated in the previous section, the POD basis vectors are just the columns of the matrix \mathbf{U} in the singular value decomposition of the displacement matrix. Therefore, it can be

concluded that **the POM converge to the normal modes of an undamped and unforced linear system whose mass matrix is proportional to identity if a “sufficient” number of samples is considered.**

Feeny and Kappagantu [11] obtained previously the same conclusions by a different way. They based their demonstration on the fact that the POM are the eigenfunctions of the covariance matrix.

In the case of a mass matrix not proportional to identity, the proper orthogonal modes do not converge to the normal modes any more since the former are orthogonal to each other while the latter are orthogonal with respect to the mass matrix. However, knowing the mass matrix, it is still possible to retrieve the normal modes. The system has to be rewritten through the coordinate transformation $\mathbf{y} = \mathbf{M}^{-1/2}\mathbf{q}$ as

$$\ddot{\mathbf{q}} + \mathbf{K}^* \mathbf{q} = 0 \quad (17)$$

Since the mass matrix of system (17) is equal to identity, the left singular vectors of $\mathbf{Q} = [\mathbf{q}(t_1) \ \dots \ \mathbf{q}(t_N)]$ and consequently the POM converge to the normal modes $\mathbf{p}_{(i)}$ of (17). Then it is a simple matter to demonstrate that the normal modes $\mathbf{x}_{(i)}$ of system (10) are related to those of system (17) by the following equation :

$$\mathbf{x}_{(i)} = \mathbf{M}^{-1/2} \mathbf{p}_{(i)} \quad (18)$$

2. Damped and unforced systems

Consider now a damped but still unforced system with M degrees-of-freedom for which equation of motion is given as follows

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{C}\dot{\mathbf{y}} + \mathbf{K}\mathbf{y} = 0 \quad (19)$$

If the structure is lightly damped or with the assumption of modal damping, the system response can be readily written :

$$\mathbf{y}(t) = \sum_{i=1}^M C_i e^{-\varepsilon_i \omega_i t} \cos(\sqrt{1 - \varepsilon_i^2} \omega_i t + \Phi_i) \mathbf{x}_{(i)} = \sum_{i=1}^M e_i(t) \mathbf{x}_{(i)} \quad (20)$$

Using the same procedure as the one described in the previous section, one shows that

$$\mathbf{Y} = \begin{bmatrix} \mathbf{x}_{(1)} & \dots & \mathbf{x}_{(M)} \end{bmatrix} \begin{bmatrix} \|\mathbf{e}_1\| & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \|\mathbf{e}_2\| & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \|\mathbf{e}_M\| & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \dots & \mathbf{e}_M \\ \|\mathbf{e}_1\| & \|\mathbf{e}_2\| & \dots & \|\mathbf{e}_M\| \mathbf{R} \end{bmatrix}^T \quad (21)$$

where $\mathbf{e}_i = \left[C_i e^{-\varepsilon_i \omega_i t_1} \cos(\sqrt{1-\varepsilon_i^2} \omega_i t_1 + \Phi_i) \quad \dots \quad C_i e^{-\varepsilon_i \omega_i t_M} \cos(\sqrt{1-\varepsilon_i^2} \omega_i t_M + \Phi_i) \right]^T$. It is now imperative to find when the time modulations \mathbf{e}_i are orthogonal. The difference with an undamped system is that $\|\mathbf{e}_i\| \neq \infty$ as $N \rightarrow \infty$ since the system returns to the equilibrium position in a finite time. Thus, the time modulations will never be perfectly orthogonal to each other what leads to a set of POM different from the normal modes. However, it can be concluded that if the damping is low and if a sufficient number of points are retained, the POM are a very good approximation of the normal modes.

3. More general cases

The relationship between the POM and the normal modes in the case of forced vibrations of linear systems using the SVD was not studied yet. However, Feeny and Kappagantu [11] showed that if one mode is in resonance then the POM with the largest POV is an excellent approximation of the resonating mode shape. The nonresonating mode shapes are not detectable by the POD. They also showed that “*if the motion is a single synchronous nonlinear mode, the POM could be viewed in the least-squares sense as an optimisation of the error in the choice of a linear representation of the nonlinear normal mode under the constraint that the linear representation goes through the origin of the coordinate system*”.

MODEL REDUCTION

The aim of this section is to present one possible application of POD to structural dynamics. It consists in building a reduced model for a system and in reconstructing the system response through this reduced model. Model reduction is not the only application of POD to structural dynamics. For instance, identification and updating of nonlinear systems can be achieved [7].

The system considered here is a clamped aluminium beam with a local nonlinearity at the free end (see Figure 1). The steel beam is one meter length and has a section of $5.10^{-4} m^2$. It is modelled with six finite elements and the whole structure is characterised by 12 degrees-of-freedom. The nonlinearity is a spring that exhibits a cubic stiffness equal to $10^{12} N/m^3$. It is worth pointing out that proportional damping is added in the model.

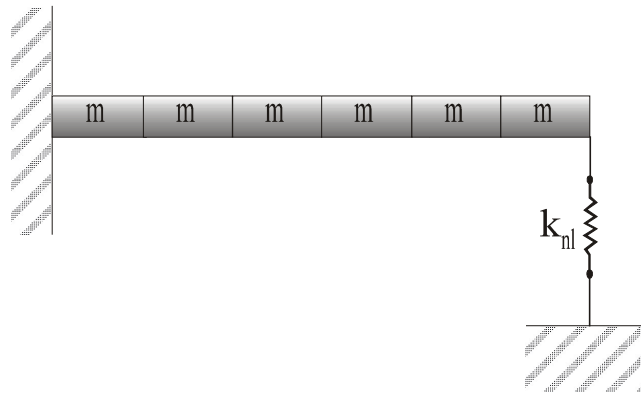


Figure 1 : Clamped beam with a local nonlinearity

The beam response is first simulated using a modified Newmark method [7]. The POM are then computed through the SVD of the displacement matrix and are exploited as an orthonormal basis to reconstruct the system response. The error between the exact signal x_i and the reconstructed signal \hat{x}_i is evaluated by the mean square error factor (MSE) defined as

$$MSE(x) = \frac{100}{N \sigma_x^2} \sum_i (x_i - \hat{x}_i)^2 \quad (22)$$

where N is the total number of samples and σ_x^2 is the variance of the measured input. Roughly speaking, a MSE of less than 5 % indicates good results while a value below 1 % is the indication of an excellent correlation.

Two types of excitation are considered : an initial load of 5 N and a sinusoidal force. Both excitations are applied on the degree of freedom associated with the vertical deflection at the end of the beam. The time step is equal to 0.0001 s and 2000 points are considered for the reconstruction. In the case of the sinusoidal force, the reconstruction is realised on the stationary part of the signal. The error obtained when the POM are the basis functions is compared with the error when normal modes of the system are used. It is worth pointing out that we choose the normal modes which participate the most and not the n first modes.

Table 1 presents the error when the POM are used for the reconstruction. The number of POM is chosen such that they capture 99.99 % or more of the total energy. Note that the error listed in this table is the sum of the errors associated with each degree-of-freedom.

Table 1 : Error for the reconstruction using the POM

	Number of POM	Captured energy (%)	Error (%)
Initial load 5 N	3	99.997	8.31
Sine force ampl.=5 N freq.=30 Hz	3	99.990	12.82
Sine force ampl.=5 N freq.=70 Hz	4	99.999	2.10⁻³

Table 2 illustrates the evolution of the error as a function of the number of normal modes included in the reconstruction procedure. The comparison between both tables illustrates that the error is always smaller with the POM if the same number of modes is considered. For the third excitation, the error associated with 4 POM is even better than that for 5 normal modes. This is confirmed by Figure 2 which represents the evolution of the total error as a function of the number of POM/normal modes retained for the reconstruction.

Figure 3 compares the displacements (at the first degree-of-freedom of translation from the clamped end) predicted by the reduced models. If the displacement computed with the POM merges perfectly the exact signal, the displacement obtained via the normal modes still presents slight distortions.

Table 2 : Error for the reconstruction using the normal modes

	Error (%) 2 modes	Error (%) 3 modes	Error (%) 4 modes	Error (%) 5 modes
Initial load 5 N	94.48	11.06	2.20	0.69
Sine force ampl.=5 N freq.=30 Hz	192.37	12.98	1.93	0.59
Sine force ampl.=5 N freq.=70 Hz	100.39	24.71	3.86	1.00

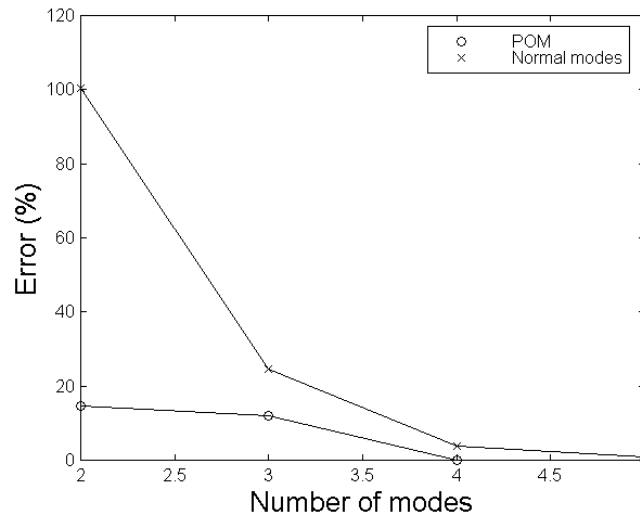


Figure 2 : Evolution of the total error vs. number of POM/normal modes (sine force 70 Hz)

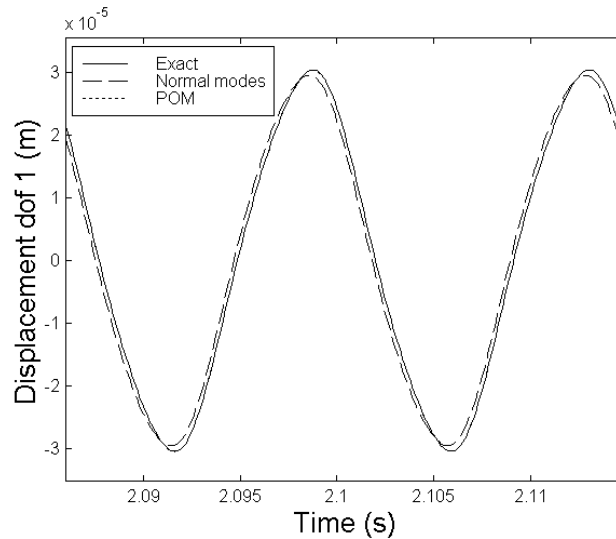


Figure 3 : Comparison between reconstructed displacements (sine force 70 Hz)

CONCLUSIONS

This paper has presented a new way of demonstrating that the POM converge to the normal modes in the case of undamped, unforced linear systems with a mass matrix proportional to identity. The proof is based on the fact that POM are the left singular vectors of the displacement matrix. Model reduction of nonlinear systems using the POD has also been introduced. A simple example has illustrated that the POM are optimal for the reconstruction of the dynamics of a system.

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