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LAYOUT OPTIMIZATION : A MATHEMATICAL PROGRAMMING APPROACH

by

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ABSTRACT

This paper presents applications of specially tailored methods of the mathematical programming approach for solving topology design problems formulated as an optimal material distribution, providing a way to enlarge the scope of potential applications of topology optimization.
The first feature of the methodology is to resort to dual maximization to solve optimization problems with a huge number of design variables, using the concept of Sequential Convex Programming (SCP).

Here a central theme is the choice of convex approximations suited to the problems. First order approximations are firstly considered and compared. However, to increase the performances of the procedures (for example, to reduce the number of steps to arrive to a stationary design), a new approach to build un-expensive second approximation schemes, especially relevant for large scale problems, have been developed and validated.

To illustrate the efficiency of the mathematical programming approach, the final part of the paper deals with an efficient treatment of perimeter constraints. Such constraints are essential to regularize material distribution with non optimal microstructures, but perimeter constraints are very difficult to handle in a numerical procedure for practical designs. The paper presents a solution to this problem that has been implemented and validated on a wide range of applications.

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INTRODUCTION

It is today standard approach to formulate the layout and topology optimization of structural components as a material distribution problem, for example in the homogenization proposed by Bendsøe and Kikuchi (1988). The distribution of mechanical properties of a porous material is controlled through the knowledge of the local microstructural parameters and the angles of orthotropy. Up to now in topology optimization problems, the objective function and the constraints considered are usually either global structural response such as compliance or eigenfrequencies or geometrical characteristics such as volume or perimeter. Topology design problems present the usual difficulties of structural optimization. The structural responses are implicit and highly non-linear functions of the design variables. The numerical cost of their evaluation and, furthermore, of their derivatives is expensive since they require a finite element (F.E.) analysis. Nevertheless, the most stringent difficulty of formulation of the topology problems is the very large number of design variables. For common meshes, the discretization of the density introduces between 1.000 and 10.000 design variables.

In standard implementations of the homogenization method (e.g., Bendsøe (1989), Suzuki and Kikuchi (1991)), the solution procedure is split into two separate update problems. The first one improves the orientation angles. The new values are based on the knowledge of the principal strains using the analytic results of Pedersen (1990) or Díaz and Bendsøe (1992). The second problems is the update of the microstructural parameter field. To this end, one has to solve a constrained optimization problem with many design variables. In most of the previous research works, this optimization problem was solved with optimality criteria (Bendsøe and Kikuchi, 1988, Suzuki and Kikuchi, 1991, Zhou and Rozvany, 1992...). This strategy is quite efficient since the optimization problem statement is restricted to very simple formulations: minimizing an objective function such as the compliance with one constraint on the volume. But this strategy is limited by its lacks of generality. Extension to solution of problems with multiple constraints is difficult since, for example, an appropriate automatic selection of active constraints is not fully rigorous. This partly explains why treatment of multiple load cases or multiple eigenfrequencies design appears difficult and these problems must be reformulated with a unique objective function.

This paper shows that the mathematical programming approach can advantageously and efficiently replace these optimality criteria methods. The method is able to generally furnish a stationary solution in a restricted number of iterations. Furthermore, the scope of topology problems that can be treated is enlarged. Thus, the scope of the layout optimization is easily extended to formulations with several constraints. For example, it is possible to solve min-max formulations that could appear naturally in multiple static load cases or eigenfrequency crossing problems.

Although, presently, the mathematical programming method is not widely applied in topology design, applications of some of the concepts of sequential convex programming have been studied by different authors. Mlejnek *et al.* (1992) compared the efficiency of first and second order approximations in topology genesis. Olhoff *et al.* (1993) applied the Method of Moving Asymptotes (MMA) of Svanberg (1987) with its dual method to solve the topology design of bi-material structures. Krog and Olhoff (1996) and Duysinx (1996) treated max-min objectives

and Duysinx *et al.* (1995) showed that the solution of very large optimization problems can be undertaken with convex approximations of good quality.

This paper presents a synthesis of some application aspects of the sequential programming method to topology design. The attention will be focused on three aspects in applying the mathematical programming approach to solve efficiently the layout problems.

The first point will underline the use of dual solvers. The difficulty arising from the large number of design variables is reduced when working in the dual space. The dimension of the dual space is limited to the number of active constraints and thus is small in the applications considered here. As we will see, the advantage is effective if the primal-dual relations can be evaluated easily. This means that separable approximations have to be used.

The second point will concern the choice of structural approximations that leads to a reliable and quick convergence of partial solutions towards the optimal distribution. A careful selection of structural approximations is important to determine the optimal solution within a minimum number of iterations and computational time. In optimality criteria methods, one generally observes a quick reduction of the objective function during the first 10 iterations; they are then followed by a slow progression around the optimum so that more than 50 iterations are often required to satisfy precise termination criteria. To reduce this number of iterations, we tested first order approximations such as CONLIN (Fleury and Braibant, 1986) and MMA (Svanberg, 1987). But, for a further improvement of the performances, we have developed a high quality approximation procedure based on second order schemes and a Quasi-Newton technique preserving the diagonal structure of the curvature estimates. The BFGS update for diagonal matrices is derived from the general theory of Quasi-Newton updates preserving sparse structure (Thapa, 1981). It has then been adapted to structural problems to generate quickly convergent sequence of curvature estimates (Duysinx *et al.*, 1995, Duysinx and Nguyen, 1997). This procedure is inexpensive even for large problems. When the curvatures are combined with second order approximations such as second order MMA (Smaoui *et al.*, 1988) or quadratic separable linearizations (Fleury, 1989b), it provides high quality approximations that accelerate the convergence speed even for large optimization problems like those we have to treat in topology design.

The third point of interest is an efficient treatment of the perimeter constraint. This additional constraint is an alternative way to regularize the design problem if sub-optimal microstructures are used to improve the void-solid separation and the clearness of the material distribution (Ambrosio and Buttazzo, 1993, Haber, Jog and Bendsøe, 1996). This constraint is, however, not easy to take into account and to satisfy in a numerical optimization procedure. The paper presents a solution that is based on 2 aspects. The first one is the development and the validation of a good separable approximation. The perimeter is a coupled, non monotonous function of the design variables, so we propose to use a quadratic separable approximation in which the estimation of the curvature coefficients are selected with an heuristic procedure. The second ingredient is to create an internal optimization loop where the optimization algorithm is called several times with updated approximations of the geometrical constraints, until they are known with a high precision in each improved design. The update of the approximations of geometrical constraints is possible since they are inexpensive to evaluate, whilst the approximations of the structural responses, which are expensive to evaluate, are kept unchanged.

EXAMPLES OF SOME FORMULATIONS OF TOPOLOGY PROBLEMS WITH SEVERAL CONSTRAINTS

In standard implementation of homogenization method with optimality criteria, topology design is mainly considered as a minimum problem with a single constraint. Here, several more complicated formulations of the design problem are considered. These kinds of problems enlarge the scope of topology applications. These optimization problems involve several constraints. As it will be shown later, one can handle efficiently their solution with the mathematical programming approach.

If we denote the generalized displacements of the finite element approximation by $\boldsymbol{q}$, the stiffness matrix by $\boldsymbol{K}$, the discretized load vector $\boldsymbol{g}$, the discretized equilibrium of the structure is $\boldsymbol{K}\boldsymbol{q} = \boldsymbol{g}$. Denoting the volume of material by $\boldsymbol{V}$, the vector of microstructural variables by $\boldsymbol{\mu}$ and the vector of orthotropy angles by $\boldsymbol{\theta}$, then the classic minimum compliance problem of Bendsøe and Kikuchi (1988) writes:

$$\begin{aligned} \min_{\mu,\theta} \quad & \boldsymbol{g}^T \boldsymbol{q} \\ s.t. \quad & . \; V \le \bar{V} \end{aligned} \tag{1}$$

Topology design problems can easily be extended to manage designs with multiple load cases. If we are able to manage several constraints, Krog and Olhoff (1996) and Duysinx (1996) showed that it is easy to consider the design for the "worst case" with a min-max technique. In this case, one looks for the design that minimizes the largest compliance of all the load cases (of index *k*) for a given volume of material.

$$\begin{aligned} \min_{\mu,\theta} \quad & \max_{k=1,nc} \quad \boldsymbol{g}^{kT} \boldsymbol{q}^k \\ s.t. \quad & . \; V \le \bar{V} \end{aligned} \tag{2}$$

This non-differentiable problem can be transformed into a differentiable one by using the standard procedure in which an auxiliary variable β is introduced:

$$\begin{aligned} \min_{\mu,\theta} \quad & \beta \\ s.t. \quad & . \; V \le \bar{V} \\ & . \; \boldsymbol{g}^{kT} \boldsymbol{q}^k \le \beta \quad k = 1, nc \end{aligned} \tag{3}$$

Problems with several constraints happen also when considering multiple eigenfrequency problems.

If one want to push the eigenfrequency domain towards higher frequencies, one can solve the problem:

$$\begin{aligned} \max_{\mu,\theta} \quad & \min_{k=1,nf} \quad \omega_k^2 \\ s.t. \quad & . \; V \le \bar{V} \end{aligned} \tag{4}$$

where ω_k^2 are the lowest eigenvalues of the eigenproblem $(\boldsymbol{K} - \omega^2 \boldsymbol{M})\boldsymbol{q} = 0$ and where $\boldsymbol{M}$ is the mass matrix of the system.

Another very interesting problem with several constraints arises if we look for an optimal distribution with a bounded perimeter. As demonstrated by Ambrosio and Buttazzo (1993), the design problem is regular if the perimeter is penalized. Following the adaptation of this concept to structural design made by Haber *et al.* (1996), we solve the problem with two geometrical constraints:

$$\begin{aligned} \min_{\mu,\theta} \quad & \boldsymbol{g}^T \boldsymbol{q} \\ s.t. \quad & . \; V \le \bar{V} \\ & . \; P \le \bar{P} \end{aligned} \tag{5}$$

where $\boldsymbol{P}$ is the perimeter of the distribution.

THE SEQUENTIAL CONVEX PROGRAMMING METHOD

Optimality criteria algorithms for structural optimization are heuristic update algorithms that are well suited to minimization problems with a single constraints. They are easy to implement and they are nearly independent of the size of the design variable vector. But when the optimization problem becomes more complex, for example when there are several constraints, a more general algorithm must be used and optimality criteria should be replaced by sequential convex programming (SCP). Fleury (1982) showed that this SCP method is a rigorous generalization of the optimality criteria method.

Generally speaking, structural optimization problems with several inequality constraints can be stated mathematically as follows:

$$\begin{aligned} \min_{\substack{x_i \le x_i \le \bar{x}_i \\ i=1,\ldots,n}} \quad & g_0(\boldsymbol{x}) \\ s.t. \quad & g_j(\boldsymbol{x}) \le 0 \quad j = 1,\ldots,m \end{aligned} \tag{6}$$

where the n variables x_i are the design variables of the problem. Their value are bounded between their minimum value $\underline{x}_i$ and their maximum value $\bar{x}_i$. The objective function is denoted by $g_0(\boldsymbol{x})$ whilst the m constraints are represented by the functions $g_j(\boldsymbol{x})$ ($j>0$).

The sequential programming approach is based on two concepts:

- replacing the implicit optimization problem by a sequence of approximated convex sub-problems by expanding the constraints $g_j(x)$ with a convex approximation $\tilde{g}_j(x)$ around the current design point x^0;

$$\begin{aligned} \min_{\substack{x_i \le x_i \le \bar{x}_i \\ i=1,\dots,n}} \quad & \tilde{g}_0(x) \\ s.t. \quad & \tilde{g}_j(x) \le 0 \quad j=1,\dots,m \end{aligned} \tag{7}$$

- solving these convex sub-problems with efficient mathematical programming algorithms as dual methods (Fleury, 1979) or as a SQP algorithm (*i.e.* Schitttkowski, 1985).

The sequential convex programming approach has several advantages.

- This approach is open to various formulations of design. Indeed, the concept of approximation and the solution procedure of the convex sub-problem is independent of the number and the nature of the constraints. So, one can address problems with one or several constraints independently of their nature: volume, perimeter, compliances of different load cases, eigenfrequencies or, in the future, local criteria like stress criteria.

- Dual methods seem to be very well adapted to layout problems, since they transpose the optimization problem into the dual space whose dimension is equal to the number of active constraints. As this dimension is usually much smaller than the huge size of the original problem, the solution accommodates easily of a very large number of design variables. Furthermore, by its nature, dual methods select automatically the set of active constraints.
- Efficient and precise structural approximations can be selected or developed to reduce the number of iterations necessary to reach an optimal solution.

DUAL METHODS

Theory and application of dual methods to structural problem are now well established (*e.g.* Fleury, 1979, 1993a). So we will provide only a short summary of these methods and we will focus on the aspects related to the treatment of large problems.

Consider a convex optimization problem with constraints such as sub-problem (7). If x^* is a regular point of the problem, then x^* is a solution of the primal problem *if and only if* there is a vector of m non negative Lagrange multipliers λ^* such that the Karush-Kuhn-Tucker

conditions are satisfied:

$$\nabla g_0(\boldsymbol{x}) + \sum_{k=1}^{m} \lambda_k^* \nabla g_k(\boldsymbol{x}) = 0$$
$$g_j(\boldsymbol{x}) \le 0 \;\; ; \;\; \lambda_j^* \, g_j(\boldsymbol{x}) = 0 \;\; ; \;\; \lambda_j^* \ge 0 \qquad (j = 1,...,m) \tag{8}$$

We are used to introduce the Lagrangian function of the problem:

$$L(\boldsymbol{x}, \boldsymbol{\lambda}) = g_0(\boldsymbol{x}) + \sum_{j=1}^{m} \lambda_j g_j(\boldsymbol{x}) \tag{9}$$

Then, one can define the dual function of the problem as:

$$l(\boldsymbol{\lambda}) = \min_{\underline{\boldsymbol{x}} \le \boldsymbol{x} < \bar{\boldsymbol{x}}} L(\boldsymbol{x}, \boldsymbol{\lambda}) \tag{10}$$

If the primal problem is convex, solving primal constrained problem is equivalent to solving the quasi unconstrained dual problem in the space of Lagrange multipliers:

$$\max_{\lambda_j \ge 0} \quad l(\boldsymbol{\lambda}) \tag{11}$$

When the problem is convex, definition (10) of the dual function introduces a unique relation between the primal variables $\boldsymbol{x}$ and the dual variables $\boldsymbol{\lambda}$:

$$\boldsymbol{x}(\boldsymbol{\lambda}) = \arg \min_{\underline{\boldsymbol{x}} \le \boldsymbol{x} < \bar{\boldsymbol{x}}} L(\boldsymbol{x}, \boldsymbol{\lambda}) \tag{12}$$

Practically, if all the functions are separable, the problem is separable too and the primal-dual relation (12) can be solved separately for each variable. The primal-dual relationship is, thus, given by developing the optimality condition of (12) for any given dual variables $\boldsymbol{\lambda}$:

$$\frac{\partial}{\partial x_i} L_i(x_i, \boldsymbol{\lambda}) = 0 \tag{13}$$

Generally speaking, this equation can be solved numerically, but if the objective and constraint functions have particular forms, this equation can be solved analytically and the primal-dual relations admit closed-forms. When the primal-dual relations are easy to compute, then dual methods are very attractive since one works in the dual space of smaller dimension.

In this research, the second order dual solver of the CONLIN optimizer (Fleury, 1989a, 1993b) was used. As explained in details by Fleury (1989a, 1993b), the maximization of the dual function is performed by solving a sequence of quadratic problems in the space of non zero Lagrange multiplier (*i.e.* attached to the active constraints). The advantage of this CONLIN optimizer relies in its ability to solve sub-problems made of "pure" Convex

Linearization (CONLIN) (Fleury and Braibant, 1986) problems as well as problems including different kinds of convex approximations like quadratic separable approximations (Fleury, 1989b), generalized power expansions (Fleury, 1989b) or generalized method of moving asymptotes. The solution of these convex separable problems is done by resorting to a sequence of quadratic sub-problems. In their turn, these quadratic problems can be solved efficiently by duality technique.

The dual maximization algorithm of CONLIN has shown itself a being very efficient and robust even if the number of design variable is very high and even if the convex problem is formulated with a mix of sophisticated approximations. The computational effort required by the CONLIN solver always remain reasonable.

Thus, dual methods present numerous advantages for the solution of convex optimization sub-problems related to topology optimization and problems with a huge number of design variables and a restricted number of constraints. At first, primal problem with constraints is replaced by a quasi unconstrained dual problem. The dimension of the dual optimization problem is strongly reduced, since the maximization is performed in the subspace of non-zero Lagrange multipliers. Because of the small dimension of the subspace of active constraints, the huge number of design variables is not a difficulty if the relations (12-13) between primal design variables and Lagrange multipliers are easy to compute. To fulfil that condition, approximations must be convex and separable. Furthermore, for CONLIN and MMA, an analytic and explicit relation can be found. Finally, dual maximization is a rigorous automatic selection of active constraints and no guess of active constraints is necessary.

STRUCTURAL APPROXIMATIONS

Developing efficient techniques for approximating the structural responses is another important feature of the convex sequential programming. The approximations must be convex and it must be separable to manage huge problems. The impact of selecting accurate structural approximation is of primary importance to have good performances and to arrive quickly to the solution.

Topology problems with homogenization method formally looks like a sizing optimization problem. So, we take approximations that are successful for sizing and try them for solving topology problems. At first, we compare first order approximations such as CONLIN or MMA. Then, we try also to introduce second order schemes. Despite their high quality, these approximations need a second order sensitivity that is prohibitive. The challenge is to determine second order information with the minimum expense. This is achieved by using Quasi-Newton techniques preserving diagonal structure.

First order approximations

The simplest first order approximation is the first order Taylor expansion. This linear approximation is efficient for the volume constraint, but its lack of convexity makes it too unreliable for structural constraints. For structural responses it is better to turn to convex approximations.

For structural optimization, the key role of reciprocal variables is well known. Expanding the constraint in terms of reciprocal variables leads to an accurate approximation that is furthermore convex if all the derivatives are negative (as it is for compliance). If the derivatives have mixed signs, the Convex Linearization (or CONLIN) (Fleury and Braibant, 1986) generalizes the reciprocal variables approximation while leading to the desired strictly convex approximations. To this end, CONLIN combines an expansion in terms of the reciprocal variable if the derivative is negative and in terms of the direct variable if the derivative is positive:

$$\tilde{g}(x) = g(x^0) + \sum_{\frac{\partial g}{\partial x_i} \geq 0} \frac{\partial g(x^0)}{\partial x_i} (x_i - x_i^0) - \sum_{\frac{\partial g}{\partial x_i} \leq 0} (x_i^0)^2 \frac{\partial g(x^0)}{\partial x_i} \left(\frac{1}{x_i} - \frac{1}{x_i^0}\right) \qquad \textbf{(14)}$$

where $\mathbf{x}^0$ is the current design point.

For practical applications of topology design, it is now usual to use a stiffness-density relationship with a power penalization of the intermediate densities (also called Simply Isotropic Material with Penalization or SIMP law): $E = \mu^p E^0$ and $\rho = \mu \rho^0$, where the index "0" is relative to the Young's modulus E and the Poisson's ratio ρ of the solids, μ is a density-like distributed parameter and p is an exponent chosen superior to 1. The choice p=3 reveals generally interesting to generate void and solid designs. Especially in this case, the material distribution problem is similar to an optimization problem with bending elements of plates. So, by analogy, it is sometimes more interesting to try to use an expansion in terms of a power *p* of the reciprocal variables (Fleury and Sander, 1983) as this approximation is more conservative:

$$\tilde{g}(x) = g(x^0) + \sum_{i=1}^{n} \frac{-1}{p} (x_i^0)^{p+1} \frac{\partial g(x^0)}{\partial x_i} \left(\frac{1}{(x_i)^p} - \frac{1}{(x_i^0)^p}\right) \qquad \textbf{(15)}$$

The drawbacks of the previous approximations is that their curvature is fixed. To fit better to the problem characteristics, one can select the MMA approximation (Svanberg, 1987):

$$\tilde{g}(x) = r^0 + \sum_{i=1}^{n} \frac{p_i}{U_i - x_i} + \sum_{i=1}^{n} \frac{q_i}{x_i - L_i} \qquad \textbf{(16)}$$

with:

$$p_i = \max\{0, (U_i - x_i)^2 \frac{\partial g_j(x^0)}{\partial x_i}\} \quad q_i = \max\{0, -(x_i - L_i)^2 \frac{\partial g_j(x^0)}{\partial x_i}\} \qquad \textbf{(17)}$$

The parameters U_i and L_i are moving asymptotes. They can be updated with a heuristic rule based on the oscillations of the design variables (Svanberg, 1987).

We tested these first order approximations in solving material distribution problems and we observed good results. These approximations give rise to quick descent rate during the first 10 iterations, but the convergence is slower around the optimum, so that 20 to 40 iterations more are generally necessary to arrive to a stationary solution. Nevertheless, if these performances (in terms of the number of iterations) are compared to results obtained with standard implementations of optimality criteria, CONLIN or MMA give equal or better performances. Among the approximations, the expansion in terms of power p of the design variables is often too conservative and the convergence rate is the slowest. Performance of CONLIN is generally very satisfactory. Due to the weaker curvatures of CONLIN and MMA in the first iterations, it is worth using move-limits to have a smooth convergence history.

From our experiences, the CONLIN (Fleury and Braibant, 1986) approximations give rise to good results in topology design. For the compliances that are self-adjoint, all the derivatives are negative and CONLIN restores the reciprocal design variables expansion that is well known to reduce the non-linearity of the structural responses. But convexity and conservativity properties of the approximation in CONLIN are important when treating eigenfrequencies or constraints whose first derivatives have mixed signs. The main disadvantage of CONLIN is that the approximations introduce fixed curvatures, so that the approximation may be too much or too little conservative. This can give rise to a slow or unstable convergence towards the optimum. To remedy this, we select the MMA (Svanberg, 1987) approximation scheme which generalises and improves the CONLIN scheme by introducing two sets of asymptotes. The choice of the moving asymptotes provides the way to modify the curvature and to fit better to the characteristics of the problem.

Because MMA fits better to the convexity of the problem, MMA is often a bit quicker than CONLIN. Because of the flexibility introduced by the moving asymptotes, MMA is often a bit quicker than CONLIN.

Nevertheless, we can conclude that both the CONLIN and the MMA lead to satisfactory results for topology design and improve often greatly the performance of the solution procedure. In many problems we observed that a solution is often achieved in 30 to 50 iterations depending on the difficulty of the problem and the precision of the stopping criterion. One strong advantage of CONLIN and MMA arises from the very reliable dual solvers that are used to solve the associated convex sub-problems. On another hand, one major drawback of first order approximations is that we can observe a deceleration of the progression towards the optimum once the algorithm arrives in the neighbourhood of the optimum. To accelerate the convergence rate in the final stage, one needs better approximations based on curvature information (Fleury, 1989).

We should also briefly discuss the selection of a termination criteria. We prefer not using termination criteria based on limited improvement of the objective function since the optimum is very flat and the value of the objective function decreases very slowly during more than half of the iterations. So, such a method will lead termination of the optimization process at a too early stage. When all is said and done, one looks for the optimal material distribution, so we think that it is better to adopt a termination criteria based on the design variable motion. This can be based on any norm of the difference of the design variable vectors between two iterations. We often use arithmetic mean modification (order 1 norm) or the maximum modification (infinite norm). As the problem under consideration is a constrained

optimization problem, one can also use Kuhn-Tucker conditions. Satisfaction of any one of these last termination criteria avoids premature stopping.

Second order approximations

The main objective is now to reduce the number of iteration to reach optimum design when the algorithm is in the flat part of the objective function and when the design point is in the neighbourhood of the optimum. To accelerate the process at this stage, second order schemes can be very helpful as demonstrated by Fleury (1989). Unfortunately, the computation of the second derivatives is generally very expensive, especially for topology problems in which the number of design variables is very large. As we will see in the next section, a remedy to this problem is a new procedure for estimating curvatures. We then use this curvature information in two high quality approximations based on second order information.

The first second order approximation we consider is the MMA second order approximation proposed by Smaoui *et al.* (1988):

$$\tilde{g}(x) \simeq c_0 + \sum_{i=1}^{n} \frac{a_i}{x_i - b_i} \qquad (18)$$

The automatic selection of the parameters is made to match first and second derivatives:

$$a_i = -(x_i^0 - b_i)^2 \frac{\partial g(x^0)}{\partial x_i} \quad b_i = x_i^0 + 2\frac{\partial g(x^0)}{\partial x_i} / \max(\epsilon, \frac{\partial^2 g(x^0)}{\partial x_i^2}) \quad (0 < \epsilon \ll 1) \qquad (19)$$

We use also the separable quadratic approximation proposed by Fleury (1989) that is a second order Taylor expansion where the coupling terms of the Hessian matrix are dropped:

$$\tilde{g}(x) = g(x^0) + \sum_{i=1}^{n} b_i (x_i - x_i^0) + \frac{1}{2}\sum_{i=1}^{n} a_i (x_i - x_i^0)^2 \qquad (20)$$

The coefficients of the linear terms are determined by the first order derivatives. and second derivatives. The curvature terms of the approximation are given by the second order, which might be increased with some additional terms to play the role of move-limits:

$$a_i = \max(\epsilon, \frac{\partial^2 g(x^0)}{\partial x_i^2}) + \delta_{ii} \qquad b_i = \frac{\partial g(x^0)}{\partial x_i} \qquad (0 < \epsilon \ll 1) \qquad (21)$$

Curvature estimation with a BFGS preserving diagonal structure of the updates

To avoid the direct second order sensitivity analysis, the idea is to use the available first order information and to build an approximation of the Hessian with a Quasi-Newton update procedure. Nevertheless, the "full" Quasi-Newton becomes also expensive when the number of design variables is large. On the other hand, only diagonal terms are useful since separable

approximations are used for problems with a large number of design variables. Thus, we use a modified BFGS update scheme able to generate a sequence of diagonal Hessian estimates (Duysinx *et al.*, 1985, Duysinx and Nguyen, 1997). The algorithm is the adaptation, to diagonal matrices, of more general results for Quasi-Newton updates preserving the sparse structure of the Hessian estimates (Thapa, 1981).

If we go from a design point $\boldsymbol{x}$ to another one $\boldsymbol{x}^+$, we seek to enrich the diagonal approximation of the Hessian matrix $\boldsymbol{B}$ of a given structural response with a Quasi-Newton update procedure. The update formula must satisfy the Quasi-Newton equation.

$$\boldsymbol{B}^+ \boldsymbol{s} = \boldsymbol{y} \quad \textit{where} \quad \boldsymbol{s} = \boldsymbol{x}^+ - \boldsymbol{x} \quad \textit{and} \quad \boldsymbol{y} = \nabla g(\boldsymbol{x}^+) - \nabla g(\boldsymbol{x}) \tag{22}$$

One of the most famous formula is the Broyden-Fletcher-Goldfard-Shanno (BFGS) update:

$$\boldsymbol{B}^+ = \boldsymbol{B} + \boldsymbol{U}_{BFGS} \quad \textit{with} \quad \boldsymbol{U}_{BFGS} = \frac{\boldsymbol{y}\boldsymbol{y}^T}{\boldsymbol{s}^T\boldsymbol{y}} - \frac{\boldsymbol{B}\boldsymbol{s}\,\boldsymbol{s}^T\boldsymbol{B}}{\boldsymbol{s}^T\boldsymbol{B}\boldsymbol{s}} \tag{23}$$

Nevertheless, although this powerful update satisfies simultaneously the symmetry, the positive definiteness of the update as well as the Quasi-Newton condition, it doesn't preserve sparse or diagonal structure of the previous estimate. Now, we denote by $\boldsymbol{B}_D^+$ and $\boldsymbol{B}_{ND}^+$ the matrices which are formed respectively with the diagonal and the off-diagonal terms of the "full" update while $\hat{\boldsymbol{B}}^+$is the diagonal Quasi-Newton update we look for. According to the theorems of Thapa (1981), the diagonal update can be found by defining a diagonal correction matrix $\boldsymbol{E}$, so that the diagonal updated matrix $\hat{\boldsymbol{B}}^+ = \boldsymbol{B}_D^+ + \boldsymbol{E}$ be the closest to the classic updated matrix $\boldsymbol{B}_D^+$ in the Frobenius norm and the diagonal update continues to satisfy the Quasi-Newton condition $\hat{\boldsymbol{B}}^+ \boldsymbol{s} = \boldsymbol{y}$. The correction matrix $\boldsymbol{E}$ that we look for is the solution of the minimum problem:

$$\begin{aligned} &\min \ \|\boldsymbol{E}\|_F \\ &\text{s.t. } \boldsymbol{E}\boldsymbol{s} = \boldsymbol{B}_{ND}^+ \boldsymbol{s} \\ &\quad E_{ij} = 0 \quad (i \neq j) \end{aligned} \tag{24}$$

The solution writes:

$$\boldsymbol{E} = \operatorname{diag}\{2\lambda_i s_i\}$$

where the vector $\boldsymbol{\lambda}$ is the trivial solution of the linear diagonal system:

$$\boldsymbol{Q}\boldsymbol{\lambda} = \boldsymbol{B}_{ND}^+ \boldsymbol{s} = \boldsymbol{y} - \boldsymbol{B}\boldsymbol{s} - \boldsymbol{U}_D \boldsymbol{s} \quad \textit{with} \quad \boldsymbol{Q} = \operatorname{diag}\{s_i^2\} \tag{26}$$

The computation of the diagonal BFGS update requires solving this last diagonal system. Nonetheless, it is not necessary to compute the off-diagonal terms of the classical update correction, since only the diagonal terms are used. So, this diagonal update is very inexpensive

and requires a much lower storage capacity than a "full" BFGS update since the diagonal update needs only some vector manipulations.

This general algorithm was adapted to yield quickly convergent Hessian estimates (Duysinx and Nguyen, 1997). One of the ingredients of the adaptation is to take advantage of the reduction of the non-linearity of the constraints by working in the space of reciprocal design variables. The curvature is expressed in terms of the reciprocal variables by a change of variables, after which the update of the diagonal Hessian estimate is performed in terms of those intermediate variables. The new estimate is then converted back into the direct variables to be used in the approximation. As noted by Duysinx and Nguyen (1997), it is very important to start from a good initial estimate of curvatures. In the reciprocal design space, a uniform very small curvature is often successful, since it is equivalent to the curvature of a reciprocal design expansion. Finally, too large updates can be damped to avoid oscillations.

We remark that the cost of the procedure of second order estimation is small. In a benchmark reported by Duysinx (1996), it was observed that for a given topology problem, the time spent in the diagonal BFGS update is only 3 % of the time spent in the optimizer CONLIN (Fleury, 1989a and 1993a) and only 0.01 % of the time needed for sensitivity analysis of the compliance with a commercial finite element package.

Comparison of first and second order approximations in topology problems.

Combining diagonal BFGS update with second order approximation schemes such as the MMA second order or the quadratic separable schemes gives very interesting results that gives rise to important savings in terms of number of iterations and of computational time. This can be explained as follow. Firstly, the estimation of the curvature improves greatly the quality of the approximation with only the help of the accumulated first order information. Secondly, instead of ignoring the second order coupling terms, diagonal BFGS provides a way to take them into account by correction terms on the diagonal coming from the diagonal update. Due to our initial guess of the Hessian, one can observe, in the first iterations, a convergence history that is very similar to first order approximations. But after some iterations, the update procedure improves the estimation of the Hessian and one can see a real advantage in the convergence speed. Around an accumulation point satisfying the optimality conditions, we could observe a convergence speed superior to first order methods, sometimes closed to super-linear behaviour.

As a conclusion, second order approximations can advantageously be used for large scale optimization problems like topology design. Starting from an initial choice of curvatures which is close to the reciprocal design variable expansion results in similar characteristics as first order approximations, during the first iterations. This choice generally yields a good descent rate of the objective function in the beginning. Then, since the Hessian estimate is improved and the approximation is enriched by this curvature information, it leads to a better convergence rate during final convergence and the number of iterations to reach a stationary solution is reduced.

Attention must nevertheless be paid to the well-known fact that second order algorithms are more sensitive to local optima. This fact was observed also with our procedure and particularly with the quadratic separable approximation. This drawback can be attenuated by

adding move-limits or by adding additional convex terms in the quadratic approximation.

An example

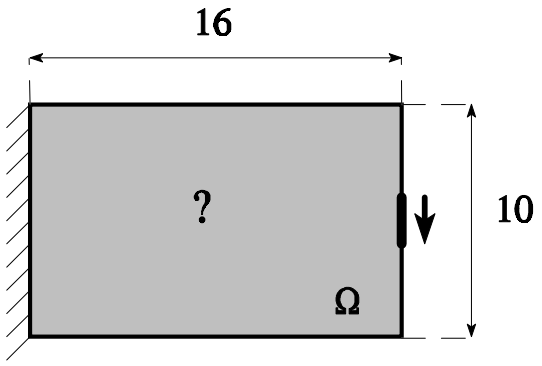

Figure 1 Short cantilever beam problem

The four approximations -CONLIN, MMA, MMA second order with diagonal BFGS (DQNMMA), and quadratic separable approximation with diagonal BFGS (DQNQUA)-, are compared on the short cantilever beam problem whose geometry of the problem is given at figure 1. This design is a classical bench-mark of topology optimization. For the sake of simplicity, the material law is simply given by a cubic relation between the rigidity and the relative density (Bendsøe, 1989) : $E = \mu^3 E^0$ and $\rho = \mu \rho^0$. The Poisson's ratio and the Young's modules of the solid are: ν^0=0.3 and E^0=100 GPa. The compliance under the given load case is minimized while the volume is bounded to 37.5% of the volume of the design domain. The design domain is discretized by a regular mesh of 1040 finite elements of degree 2. The finite element analysis and the sensitivity analysis are performed with the SAMCEF package.

Histories of compliance are given at figure 3. At first glance, the different convergence curves are very similar and the different algorithms tend towards local optima with nearly the same compliances. Nevertheless, when material distributions are visualized, the optimal distributions reveal a very similar topology, except in the quadratic approximation (Figure 2 presents the material distribution obtained with CONLIN). Thus several optima exist when intermediate densities are highly penalized and attention must be paid to local configurations. Also, second order approximations are partly more sensitive to local optima.

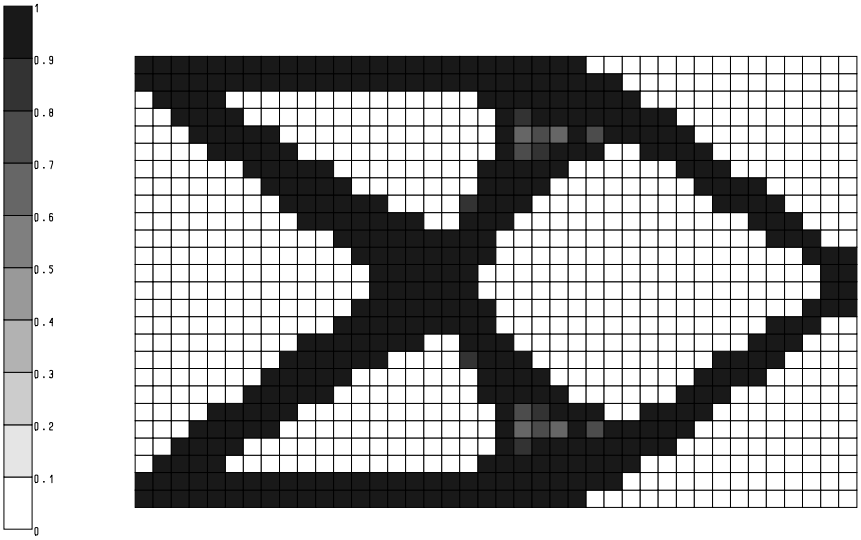

Figure 2 : Distribution of density (CONLIN)

First order approximations give smooth and monotone history curves. At the beginning, the descent rate is good but, after more or less 10 iterations, compliance reduction is seriously slowed down when close to the optimum. Progression becomes much slower. On figure 3, one can note that the mean modification of the design variables between two iterations diminishes slowly, even with small oscillations.

Second order approximations also results in convergence curves with a very good descent rate. The progression towards the optimum is not handicapped too much by the non monotone behaviour of the first iterations (iterations 3 and 4). That fact may be due to an uncertain value of the estimation of the curvatures by the diagonal BFGS. After this phase which is necessary to stabilize the estimations of the curvatures, second order information gives rise to a very good descent rate. When in the flat part of the compliance curve, second order information preserves a good convergence speed and continues to accelerate the progression towards a stationary solution. This can be clearly noted on the mean modifications of the design variables. Stationarity is reached much faster with second order schemes and diagonal BFGS than with first order approximations. We recover in fact the asymptotic superlinear descent rate of quasi-Newton methods. This characteristic saves a high number of iterations. The MMA second order method arrives to stationarity in less than 30 iterations while CONLIN and MMA needs more than 40 iterations to satisfy a weaker termination criteria.

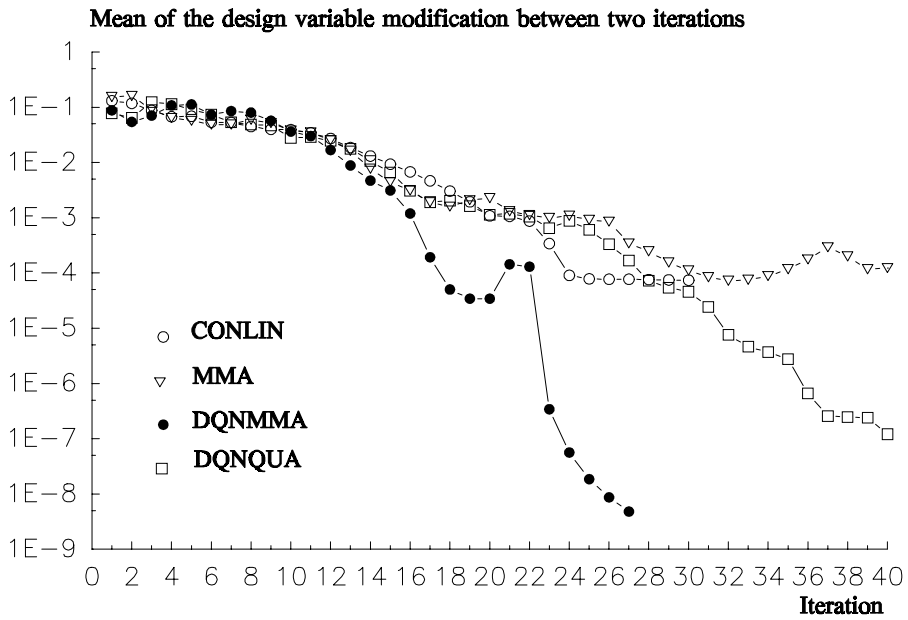

Figure 3 : Mean modification of the design variables between two iterations

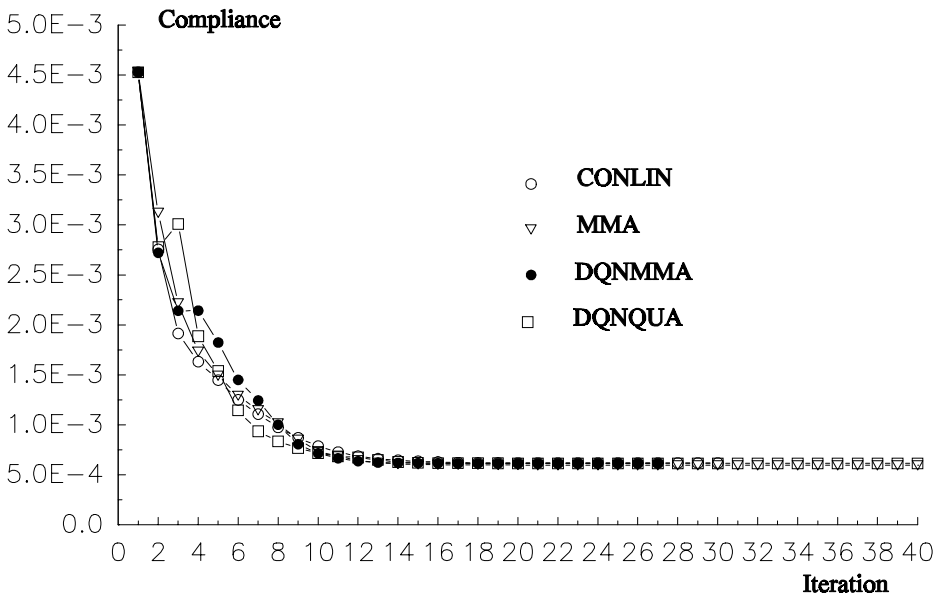

Figure 4 : Compliance history curves

OPTIMAL DISTRIBUTION WITH PERIMETER CONSTRAINT

Perimeter constraint has a major place in topology design, because it is a very interesting alternative to the use of optimal microstructures as in the homogenization method. Perimeter control also seems further attractive than the rigorous homogenization method in engineering applications since it allows to generate clear density distributions with well separated voids and solids zones, so that an unambiguous macroscopic topology often appear. Ambrosio and Buttazzo (1993) demonstrated that the design problem with perimeter penalization is well-posed. Application of perimeter control to topology design of structures was presented by Haber *et al.* (1996). Our work continues this line of thought with the goal to remedy some difficulties that occur in the numerical implementations of the perimeter control. We focus on providing an efficient numerical strategy to take perimeter constraint into account in order to use perimeter control as a real practical design tool. The perimeter is a rather difficult constraint to satisfy and to approximate, as it will be seen.

We based our numerical experiences on SIMP materials to provide in the same time a continuous approximation of the design distribution problem as well as a penalization of the intermediate densities. SIMP material is also easy to implement in any industrial code. Finally, SIMP material introduces only one variable per element, so that the size of the problem is kept at a minimum.

The main original aspect of this study is related to the solution aspects. Although in the original work of Haber *et al.* (1996) the perimeter control was treated with a interior penalty function, we propose to generalize the solution by solving the problem as a constrained problem in which perimeter is one of the inequality constraints. If one wants to control the

perimeter by prescribing a target value with a penalization, one can use a relaxation technique and introduce an additional variable δ and which is quadratically penalized in the objective function (the ***pds*** factor is a tuning parameter to control the relative weight of the penalization compared to the magnitude of the objective function).

$$\begin{aligned} \min_{\substack{\mu \le \mu \le 1 \\ 1 \le \delta \le 2}} \quad & g^T q + pds\, \delta^2 \\ s.t. \quad & . \; V \le \bar{V} \\ & . \; P \le \bar{P} + (\delta - 1)\, \Delta P \end{aligned} \tag{27}$$

ΔP is an allowable maximum violation of the target bound $\bar{P}$. Standard versions of solvers like CONLIN (Fleury, 1993b) or MMA (Svanberg, 1987) are able to handle the solution of this kind of optimization problems with relaxation.

When the density varies continuously, Haber, *et al.* (1996) propose to replace the elementary measure of the perimeter by the total variation of the density ρ. For a density field which is element by element constant, we can write:

$$P = \sum_k l_k \left(\sqrt{\langle \rho \rangle_k^2 + \epsilon^2} - \epsilon \right) \tag{28}$$

where $\langle \rho \rangle_k$ is the jump of material density trough the element interface k of length l_k. The parameter ϵ is a small positive number to guarantee the differentiability of the measure. Values of ϵ are generally taken between 10^{-2} to 10^{-4}.

Figure 5 sketches the perimeter measure of a square element of density μ surrounded by four elements of density μ_1, μ_2, μ_3, μ_4. Perimeter is nearly a piecewise linear function even if it is globally non monotonous. It turns out that perimeter constraint is not easy to approximate with classical schemes. Monotonous approximations like CONLIN or MMA give rise to oscillatory behaviours. Furthermore, the perimeter is globally non linear and there are important couplings between neighbouring finite element (FE) densities. Thus, the trust region of separable approximations is narrow.
Nevertheless, in order to treat problems with a large number of variables and to use dual solvers, we need a convex and separable approximation. From our numerical experience, good results are expected with a quadratic separable approximation of the general form:

$$\tilde{P}(\mu) = P(\mu^0) + \sum_{i=1}^{n} \frac{\partial P}{\partial \mu_i} (\mu_i - \mu_i^0) + \frac{1}{2} \sum_{i=1}^{n} a_i (\mu_i - \mu_i^0)^2 \tag{29}$$

The main problem is now to choose the curvatures carefully. Their values must be a compromise between precision and conservativity. Too small values of the curvatures would imply important constraint violations while too large curvatures would lead to a freezing of the motion of the variables. On one hand, curvatures will try to fit the true shape of the constraint while on the other hand, curvatures will play the role of move-limits that restrict the validity of the approximation. Also, the analytic second order derivatives are not useful

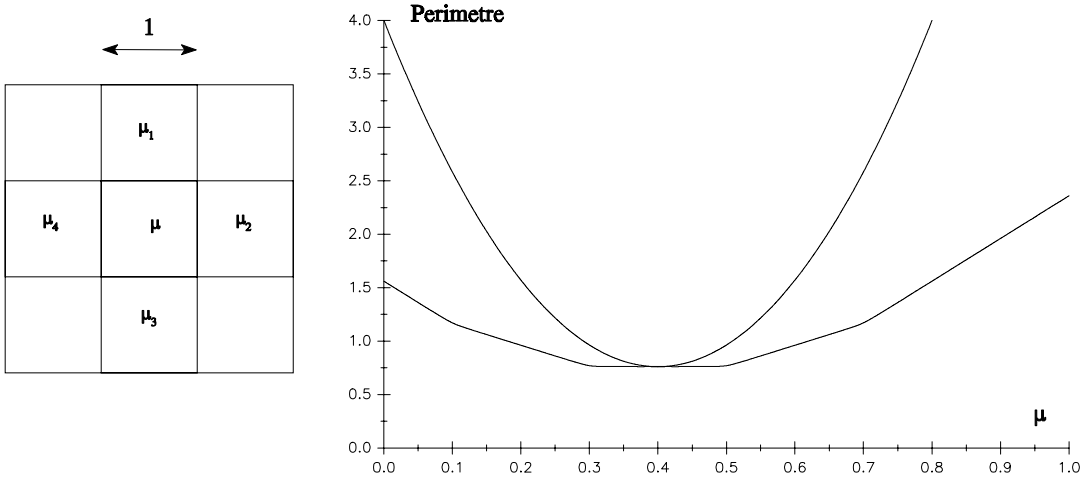

Figure 5 : Example of a perimeter measure and its quadratic approximation

since they are zero, except in angular points where they are very large (or do not exist). So the choice of the artificial curvatures is based on a heuristic rule, which is explained in the next section.

A heuristic estimation of curvatures for perimeter approximation

In the following, we develop a heuristic estimation of curvatures for approximating the perimeter when there is one density variable μ_i per element as it is for SIMP materials. These estimates of the curvatures are based on a bound over individual contributions of each element. According to the quadratic approximation, the contribution of element "*i*" to global perimeter is:

$$\tilde{P}_i(\mu_i) = P_i(\mu_i^0) + \frac{\partial P}{\partial \mu_i}(\mu_i - \mu_i^0) + \frac{1}{2} a_i (\mu_i - \mu_i^0)^2 \qquad \textbf{(30)}$$

$P_i^0(\mu_i^0)$ is the contribution of element "*i*" to the perimeter with the current distribution of density. This contribution of element *i* is maximum when the density jump across the element interfaces becomes equal to unity. Suppose now that this situation happens at point μ_i^*. Then one can write the curvature in term of the new parameter μ_i^*:

$$a_i = 2 \frac{\sum_{k \in K_i} l_k \left(\sqrt{1+\epsilon^2} - \sqrt{\langle \rho \rangle_k^2 + \epsilon^2} \right) - \frac{\partial P}{\partial \mu_i}(\mu_i^* - \mu_i^0)}{(\mu_i^* - \mu_i^0)^2} \qquad \textbf{(31)}$$

where the sum is realized over the set K_i of the interfaces of element "*i*".

The question is now to find the point μ_i^* where this situation could probably happen. If the separability hypothesis were true, the point μ_i^* could be chosen at the boundary of the

admissible set of μ_i. But, because of the neglected coupling effects, this situation happens sooner and this choice leads to approximations that are not conservative enough. Instead, we propose to play with the point μ_i^* as a move limit. From our numerical experience, we propose to take:

$$\mu_i^* = \mu_i^0 \pm \alpha(\bar{\mu}_i - \underline{\mu}_i) \qquad \alpha = [0.33, 0.4] \tag{32}$$

This choice prevents an oscillation of the neighbouring design variables. The proposed approximation as applied is illustrated in figure 5.

An internal loop procedure for perimeter approximation

Even if the approximation procedure of the perimeter is efficient, the perimeter constraint remains difficult to approximate and we generally note that the number of iterations increases drastically when a perimeter constraint is considered. It often takes more than 100 iterations. This effect can be imparted to the fact that we need to take a convex approximation with a high curvature to have a sufficiently conservative approximation. Unfortunately, this has the drawback of slowing down the optimization process. On the other hand, perimeter constraint is a geometrical constraint, and, thus, contrary to structural responses, it is easy and inexpensive to evaluate.

The idea is thus to create an internal loop over the optimization algorithm with several updates of geometrical constraints as the perimeter. The strategy is given at figure 6. The outer loop is usual: it includes the finite element analysis, the sensitivity analysis and the optimization procedure. The optimization procedure encloses an inner loop that is repeated with updated approximations of the perimeter until the perimeter approximations coincide with its real value within a sufficiently high precision at the proposed new optimum. Since structural constraints are expensive to evaluate (it requires one finite element run and a sensitivity analysis) and since they are sufficiently well approximated by high quality schemes, the structural approximations are kept unchanged during this inner loop. As the inner loop is repeated until the perimeter has a given precision, the perimeter constraint does not slow the optimization process and we noted a spectacular acceleration to reach optimal solutions. The number of iterations is often divided by a factor 3 or 4.

Let denote by l the sub-iteration index. To implement the update procedure of the approximations for geometrical constraints, we need writing, around the reference design point $\boldsymbol{x}^0$ where the structural approximations are expressed, an approximation of the perimeter that matches the true perimeter value and its derivatives in an other sub-iteration point $\boldsymbol{x}^l$. This is possible by defining fictitious parameters.

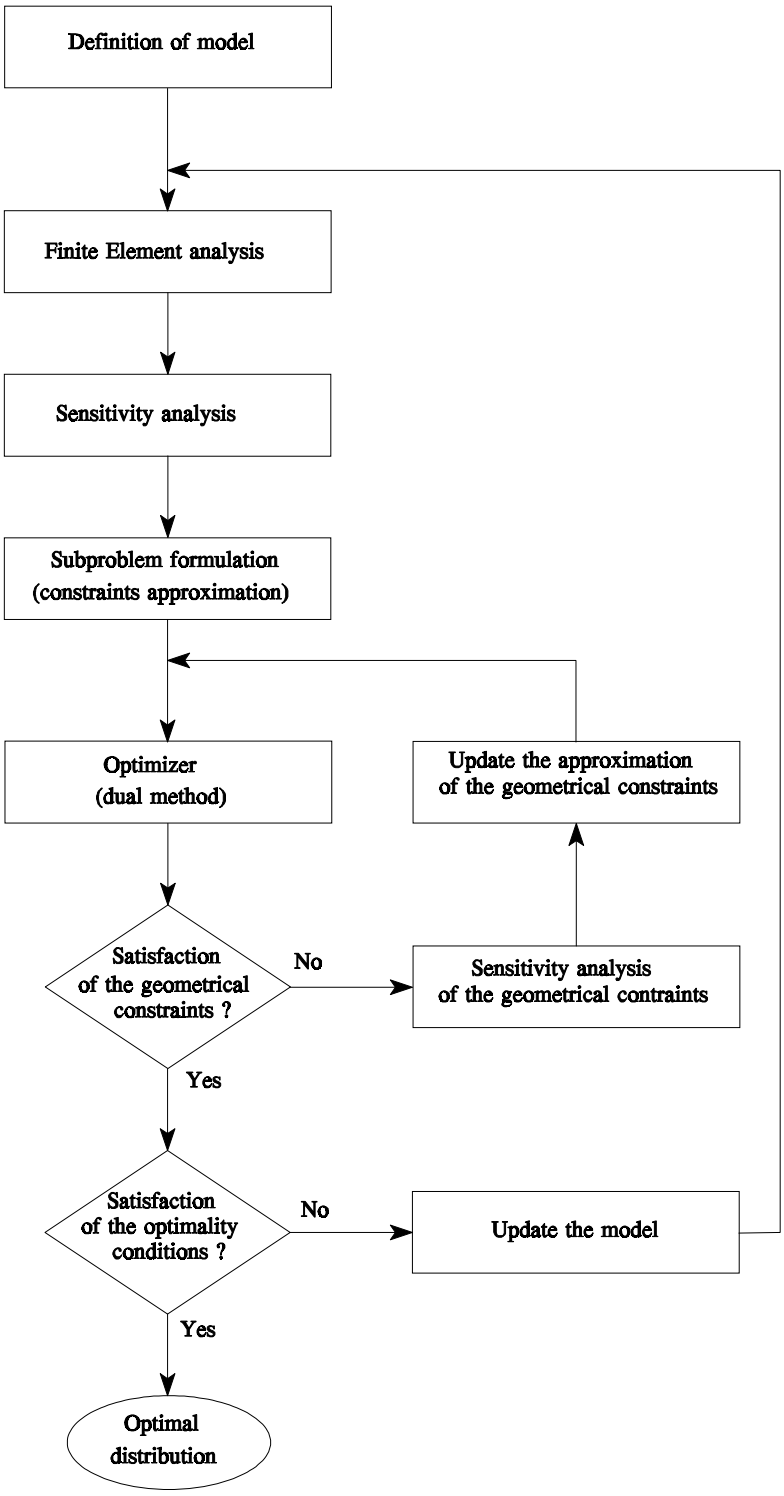

Figure 6 : Optimization process with outer and inner loops

For a quadratic approximation given in (8), one can define:

$$\breve{a}_i = a_{ii}(x^l) \qquad \breve{b}_i = \frac{\partial g(x^l)}{\partial x_i} + a_{ii}(x_i^0 - x_i^l)$$
$$\breve{g}(x^0) = g(x^l) + \sum_{i=1}^{n} \frac{\partial g(x^l)}{\partial x_i}(x_i^0 - x_i^l) + \frac{1}{2}\sum_{i=1}^{n} a_i (x_i^0 - x_i^l)^2 \qquad \textbf{(33)}$$

This approximation is exact in x^l, but it is written around x^0.

The quality of the approximation curvatures is also improved with the information collected during the inner loop. If the variable *i* tends to oscillate, the curvatures a_i can be multiplied by a factor to increase the curvature and the conservativity. If the process is monotone in this variable, the convexity is decreased. We adopt a similar procedure to the update strategy of the moving asymptotes of Svanberg (1987).

For the two first iterations l = 1 and 2, the default heuristic curvatures are adopted. After two iterations, if the process oscillate *i.e.* $(x_i^{l-2} - x_i^{l-1})(x_i^{l-1} - x_i^l) \le 0$, one has to increase the curvature of the approximation :

$$a_i^l = a_i^{l-1} * s_1 \qquad s_1 > 1 \qquad \textbf{(34)}$$

If the process is stable and monotone *i.e.* $(x_i^{l-2} - x_i^{l-1})(x_i^{l-1} - x_i^l) > 0$, the approximation curvature is decreased:

$$a_i^l = a_i^{l-1} * s_2 \qquad s_2 < 1 \qquad \textbf{(35)}$$

Parameter s_2 is generally chosen equal to s_1 or better to $\sqrt{s_1}$ to stabilize the process.

The use of an accurate approximation of perimeter constraints combined with the inner loop strategy proved its great efficiency. This procedure generally leads to optimal distributions in less than 50 iterations and the optimization process is stable and reliable.

APPLICATIONS

MBB beam

The so-called MBB beam is probably one of the most famous applications of topology design. The detailed presentation of this problems and the geometrical and material data have been presented by Olhoff *et al.* (1992), who solved this problem with a fully relaxed material distribution optimization problem using rank 2 materials. The solution procedure adopted by Olhoff *et al.* (1992) is an optimality criteria technique. Here the application is revisited with a perimeter constraint and the proposed mathematical programming approach. The design problem with a perimeter constraint is regular even with a power (p=3) penalization of

intermediate densities. Half of the structure is covered by a 75 x 25 F.E. mesh. Compliance and volume are approximated with CONLIN approximation. The first solution is an optimum with no perimeter bound. With stringent termination criteria, the optimization process is stopped after 75 iterations. The density chart of the unbound distribution is given at figure 7.

We examine now how it is possible to control the perimeter when a reliable and conservative approximation is available. The perimeter measure is made with a quite high value of the parameter ϵ: 0.5 10^{-1}. Figure 8 show different solutions when the allowable perimeter is progressively reduced. The number of holes comes from 17 to 5 when the perimeter bound decreases from 7,5 to 4,5. These results were obtained rather automatically.

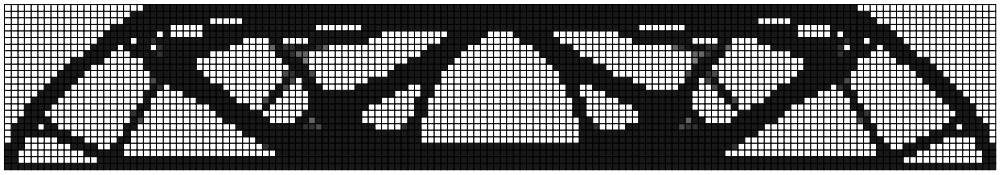

Figure 7 : MBB beam with no perimeter bound

Despite the fact that perimeter is actually controlled with our quadratic approximation, the optimization process is very slow: about 150 analyses are necessary to arrive to these solutions. This phenomenon seems to be linked to the presence of intermediate density zones that disappear progressively during optimization. These intermediate density regions seem to be a way for the algorithm to reduce the perimeter and satisfy the bound during intermediate solutions.

We are going to demonstrate now the acceleration that results from the use of our inner loop strategy over perimeter approximation. The problem is still solved with a SIMP material, but the penalization of the intermediate densities is stronger: p=4. The measure of the perimeter is also improved by fixing the parameter ϵ to 10^{-2}. This makes the perimeter control more difficult to approximate. For this value of ϵ, a perimeter bound of 6,5 gives rise to a topology with 5 holes. The first optimization is realized without the inner loop. The decrease of weight is slow and optimization process continues during a large number of iterations. The perimeter is even sometimes slightly violated despite the conservative approximation. Optimization process is stopped after 80 iterations and the corresponding material distribution is given at figure 9. At this moment, the compliance is 88.45 Nm. As shown by figure 9, some wide mild zones of intermediate densities remains in the density map. These ones disappear slowly and more or less 80 iterations will be necessary to arrive to the stopping criteria previously used.

For the second numerical experience, we activate the inner loop strategy over the optimization algorithm and the perimeter approximation is updated. The result is spectacular. After 35 iterations, we get the material distribution of figure 10. The mild zones are nearly completely eliminated and the compliance is already 83.11 Nm. Since the inner loop is performed, the perimeter value is known precisely and there is nearly no constraint violation of the perimeter bound at each iterations.

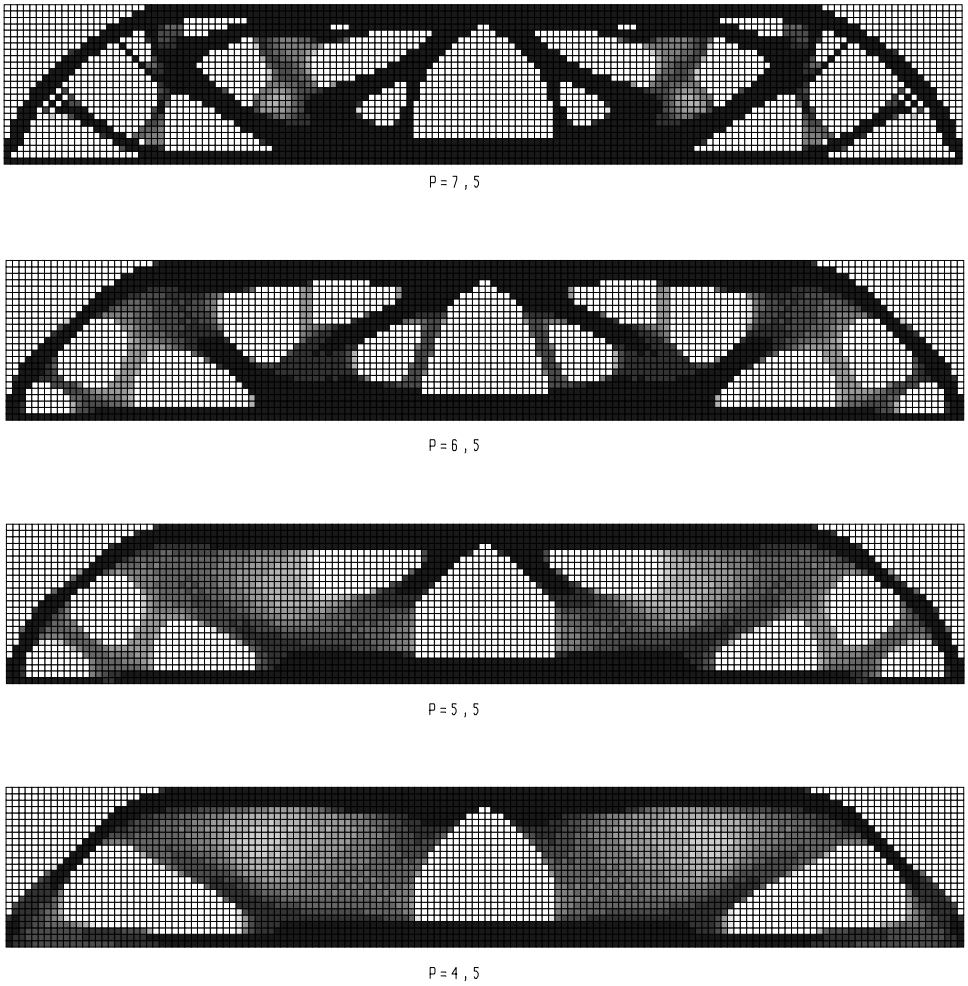

Figure 8 : MBB problem with different perimeter bounds

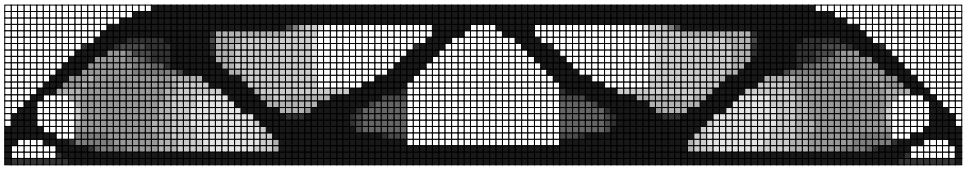

Figure 9 : Density map after 80 iterations without inner loop strategy

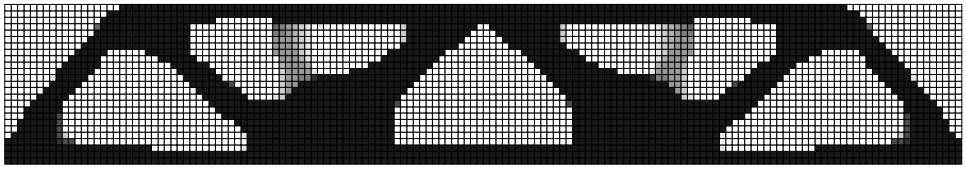

Figure 10 : Density map after 35 iterations with an inner loop strategy

Finally, we compare the computational times. Since, the overall computational time spent for solving one optimization convex sub-problem is at most 1% of the cost of the sensitivity analysis, the inner loop computational time is largely balanced by the reduction of the number of iterations. The update strategy for the perimeter constraint saves a lot of computation time.

A multiple load case problem: the 3-bar truss design

The last example illustrates how multiple load cases are taken into account with the mathematical programming approach. Like Fukushima, Suzuki and Kikuchi (1991), we will revisit the problem of the 3-bar truss under 3 load cases.

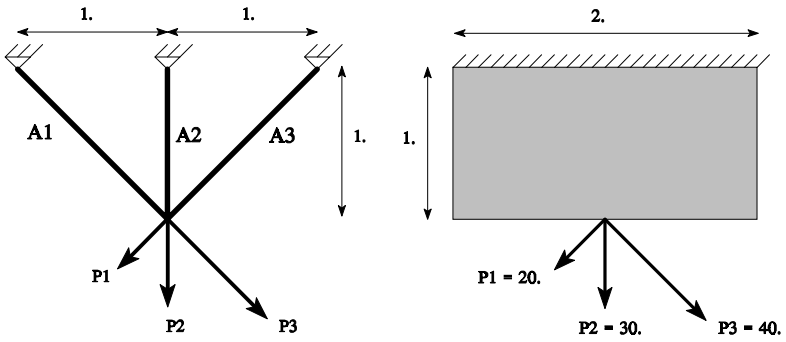

Figure 11 : 3-bar truss problem

The design domain is a rectangle of dimensions 2 by 1 m. The boundary conditions and the three load cases are given at figure 11. Three different load cases have the following magnitudes: P_1=20 N, P_2=30 N and P_3=40 N. The stiffness-density law is still a cubic penalization of intermediate densities: $E = \mu^3 E^0$ and $\rho = \mu \rho^0$, with the standard isotropic solid properties: E^0=100. GPa and ν=0.3.

We look for the configuration that minimizes the maximum of the 3 compliances with a prescribed relative volume of material of 20 %. The mesh is regular and is made of 800 finite elements of degree 2.

The curves giving the history of the compliances and of the modifications of the design variables are presented at figures 12 and 13 while figure 14 shows the density map at different optimization stages. As it can seen from modification of the design variables between two iterations, the stationarity of the material distribution is nearly achieved after 30 iterations, while it should have been forecast sooner with a criterion based on the compliances. The optimization process is going on until the modification of the design variables is under a given precision criterion over the design variable modification.

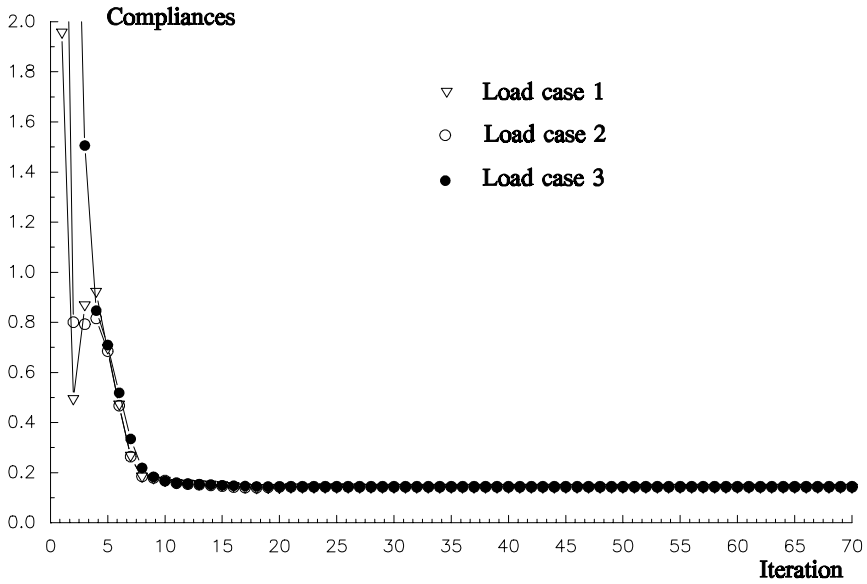

Figure 12 : Compliance curves

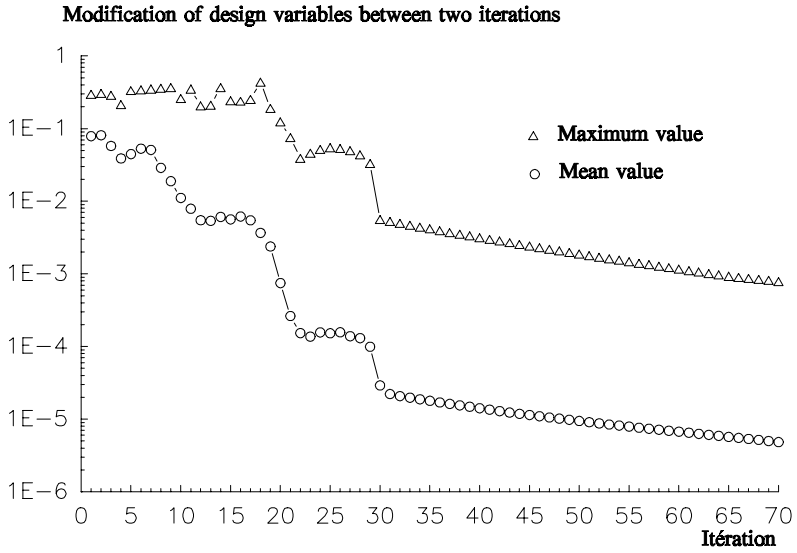

Figure 13 : Modifications of the design variables between two iterations

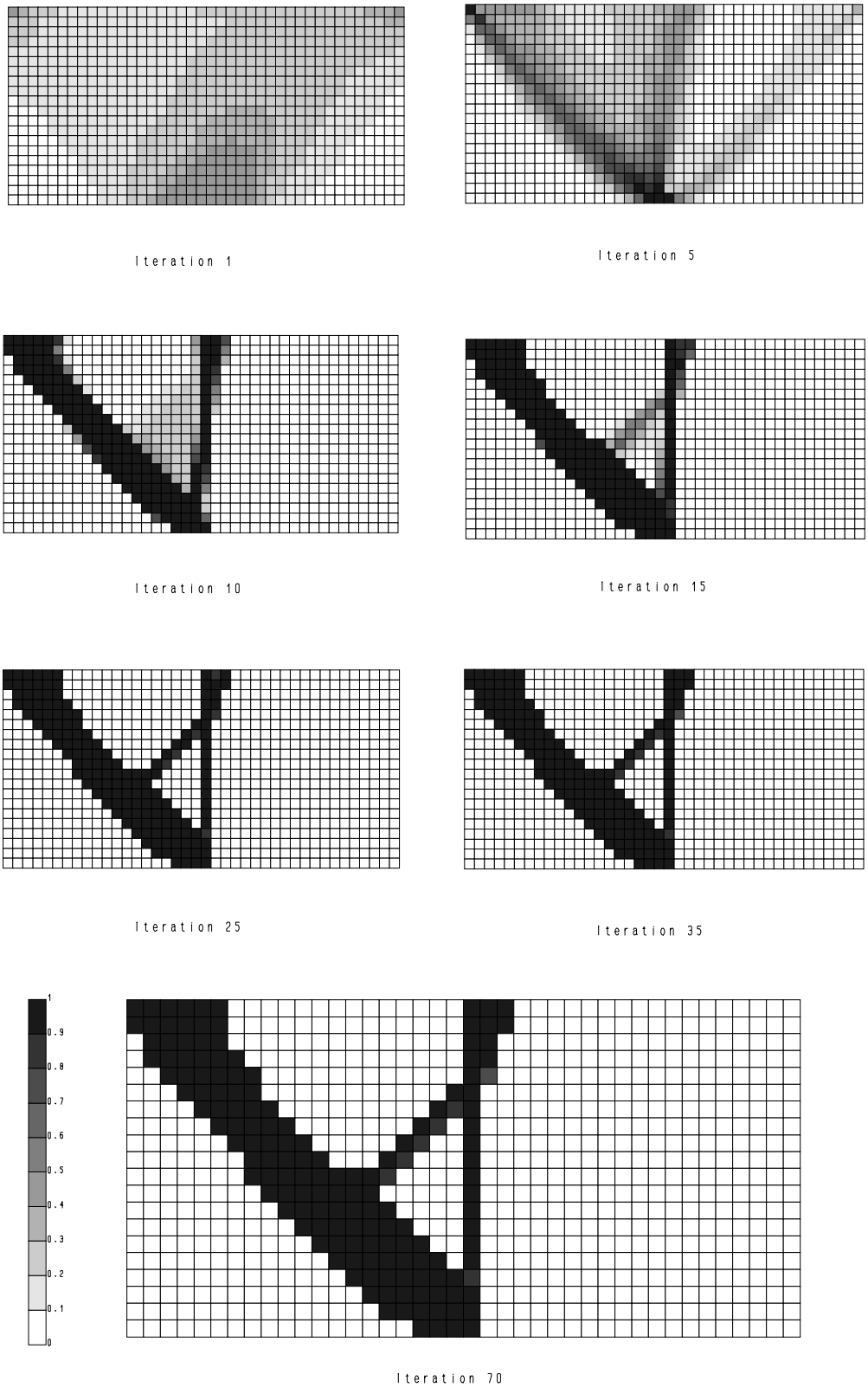

Figure 14 : Genesis of the topology

Despite the individual compliances look erratic, the hull of the maximum of the compliances decreases monotonously. From the beginning, the load case 3 influences the optimum since this compliance matches the maximum value at nearly each iteration.

Attention must be paid to the analysis of figure 12 and to the interpretation of the topology suggested at figure 14. The scale of the history of the compliance could be confusing and we must precise that, at the optimum, the compliances of load cases 2 and 3 are equal (to a given tolerance) while load case 1 has a smaller compliance.

Compliance of load case 1 intervenes only in the design of iterations 3 and 4 and then, it is deactivated after this first convergence stage. The compliance of load case 1 remains the smallest and this load case doesn't influence any more the material distribution. Load case 2 is progressively activated and its compliance becomes equal to the compliance of load case 3. Finally, the optimum distribution is modelled by both load cases 2 and 3. Nevertheless, as the Lagrange multiplier of load case 3 is larger than the load case 2, the third load case is more influent in the final design.

As one can observe in many others examples, only a subset of the load cases is activated in the optimal configuration. The automatic selection of the critical compliances in the "min max" optimization is a fundamental characteristic of dual solvers. This result is, in general, very difficult to guess a priori.

The optimal distribution looks like 2-bar truss with a small reinforcement. The bar A3 which is aligned with the inactive load case 1 has disappeared. Load case 3 is the most stringent and bar A1 is more important than bar A2 aligned with load case 2.

CONCLUSION

In structural optimization, a well-known implementation of mathematical programming is the sequential convex programming method and it has successfully been applied to sizing and shape optimization. We demonstrated here that this mathematical programming approach is also valid for topology and layout problems. Sequential convex programming extends the scope of topology design. Due to its general character, the treatment of problems with several constraints of any nature is obvious. Dual methods are well adapted to solve easily problems with a large number of design variables since the number of active constraints is small and since separable convex approximations are used. The selection of appropriate approximation schemes is also important to reduce the number of iterations to arrive to the optimum. First order approximations, CONLIN and MMA reveal to be reliable and precise. But, to further improve the performances, we developed a procedure based on the combination of a diagonal BFGS technique to estimate the curvatures and on second order approximations.

We have also designed an efficient procedure to control automatically the perimeter. The perimeter is difficult to approximate accurately. To realize practical applications, we have proposed and validated a quadratic separable approximation in which we use a heuristic estimation of the curvatures. To improve the convergence speed of the procedure, an inner loop in the optimization algorithm has been proposed, where the perimeter approximation is updated until the perimeter value is known with sufficient precision.

Our experience shows that the sequential convex programming method presents numerous advantages which should lead to further developments of layout problems. One major advantage is the possibility to take into account several constraints of various nature into the design process. Thus we believe that sequential programming methods are able to improve greatly the capabilities of topology optimization as a real-life design tool.

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