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Bimatroidal Independence Systems

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Zusammenfassung: Ein Unabhängigkeitssystem $\Sigma = (X, F)$ heißt bimatroidal, wenn zwei Matroide $M = (X, F_M)$ und $N = (X, F_N)$ existieren, so daß $F = F_M \cup F_N$. In diesem Falle heißt $\{M, N\}$ eine bimatroidale Zerlegung von Σ. In dieser Arbeit werden erstmals bimatroidale Systeme untersucht. Ist die Klasse aller Kreise eines beliebigen Unabhängigkeitssystems Σ (oder sind äquivalent damit die Restriktionen eines Mengenüberdeckunsproblems) gegeben, so stellt sich folgende Frage: erlaubt Σ eine bimatroidale Zerlegung (M, N), und wenn, wie können die Kreise von M und N erzeugt werden? Für dieses Problem werden eine Reihe von Resultaten gezeigt. Ferner wird ein zeitpolynomialer Algorithmus für dieses Problem in dem Fall angegeben, daß je zwei verschiedene Kreise von Σ höchstens ein Element gemeinsam haben. Außerdem schlagen wir verschiedene zeitpolynomiale Algorithmen für Mengenüberdeckungsprobleme vor, die über der Klasse der Kreise bimatroidaler Systeme derfiniert sind.

Abstract: An independence system $\Sigma = (X, F)$ is called bimatroidal if there exist two matroids $M = (X, F_M)$ and $N = (X, F_N)$ such that $F = F_M \cup F_N$. When this is the case, $\{M, N\}$ is called a bimatroidal decomposition of Σ. This paper initiates the study of bimatroidal systems. Given the collection of circuits of an arbitrary independence system Σ (or, equivalently, the constraints of a set-covering problem), we address the following question: does Σ admit a bimatroidal decomposition $\{M, N\}$ and, if so, how can we actually produce the circuits of M and N? We derive a number of results concerning this problem, and we present a polynomial time algorithm to solve it when every two circuits of Σ have at most one common element. We also propose different polynomial time algorithms for set covering problems defined on the circuit-set of bimatroidal systems.

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1 Introduction

An *independence system* is a pair $\Sigma = (X, F)$, where X is a finite set, and F is a collection of subsets of X closed under inclusion:

$$\text{if } A \in F \text{ and } B \subseteq A, \quad \text{then } B \in F.$$

A subset of X is called *independent* if it is in F, and *dependent* otherwise. A *basis* of Σ is a maximal independent subset of X, and a *circuit* of Σ is a minimal dependent subset of X (where "maximal" and "minimal" are understood inclusion-wise). The set of circuits of Σ is denoted $C(\Sigma)$.

The union of two independence systems $\Sigma_1 = (X, F_1)$ and $\Sigma_2 = (X, F_2)$ is the independence system $\Sigma_1 \cup \Sigma_2 = (X, F_1 \cup F_2)$. We say that Σ_1 is *included* in Σ_2, and we write $\Sigma_1 \subseteq \Sigma_2$, if $F_1 \subseteq F_2$.

The *set covering problem* has as input the complete list of circuits of an independence system $\Sigma = (X, F)$, and a nonnegative *weight function* $w : X \to \mathbf{R}$. It is formulated as:

$$\max \ w(B) = \sum_{j \in B} w(j)$$

$$\text{subject to } B \in F. \tag{1}$$

Notice that, even though the formulation given here is slightly unusual, problem (1) is precisely equivalent to the "classical" set-covering problem (since the set-theoretic complements of the independent sets are exactly the covers of the collection of circuits). Therefore, and despite the fact that the length of the input can be exponentially large in the size of X, the set covering problem is NP-hard.

By contrast, the set covering problem turns out to be especially easy to solve over *matroids,* i.e. over independence systems satisfying the following circuit-axiom: if C_1, C_2 are distinct circuits, and $x \in C_1 \cap C_2$, then there exists a circuit C_3 such that $C_3 \subseteq C_1 \cup C_2 - x$. Indeed, the straightforward *greedy algorithm* always yields an optimal solution to the set covering problem over a matroid (see e.g. Lawler 1976, Welsh 1976, and Section 3).

Now, if Σ is not a matroid, but is expressed as a union of matroids $M_1, M_2, \ldots, M_k$, then a maximum weight independent set of Σ can be obtained by successively solving the set covering problem over each of $M_1, M_2, \ldots, M_k$ (observe that, with the definition we adopted for the union of independence systems, the union of two matroids is generally not a matroid).

Motivated by this observation, Benzaken and Hammer (1985) define the *matroidal number* of an independence system Σ as the smallest number of matroids whose union is Σ (this number is always well-defined). They call *matroidal component* of Σ any matroid included in Σ, and a matroidal component is said to be *prime* if it is maximal with respect to inclusion. It is easy to see that Σ is equal to the union of its prime components. A general procedure, called *obstruction removal technique,* is proposed by Benzaken and Hammer (1985) for finding all prime matroidal components of an independence system (described by a list of its circuits).

Clearly, the decomposition approach outlined above is likely to be successful only when applied to independence systems with small matroidal number. In particular, let us call an independence system *bimatroidal* if its matroidal number is two. The purpose of this paper is to initiate the study of bimatroidal systems.

If Σ is bimatroidal, and M, N are two matroids satisfying $\Sigma = M \cup N$, then $\{M, N\}$ is a *bimatroidal decomposition* of Σ. This decomposition is called *prime* if both M and N are prime matroidal components of Σ. We do not know the complexity of recognizing or decomposing general bimatroidal systems. In Section 2, we suggest that this problem might be difficult, by exhibiting a class of independence systems having a unique, exponentially long, prime bimatroidal decomposition. We also present a polynomial-time recognition and decomposition procedure for *linear* bimatroidal systems, i.e. bimatroidal systems in which every two circuits have at most one common element (see Berge 1987).

In Section 3, we establish that bimatroidal systems belong to a "good" class of independence systems, over which the set covering problem is solvable in polynomial time. We also present a specialized polynomial algorithm, which, we conjecture, is guaranteed to find an optimal solution of the set covering problem if and only if it is applied to a bimatroidal system.

2 Bimatroidal Decomposition

2.1 The Length of a Bimatroidal Decomposition

Consider the following question: is there a polynomial $P(u, v)$ such that every bimatroidal system $\Sigma = (X, F)$ admits a bimatroidal decomposition $\{M, N\}$ for which the number of circuits of M and N is bounded by $P(|X|, |C(\Sigma)|)$?

Proposition 1 below provides a negative answer to this question if we restrict our attention to prime bimatroidal decompositions. For $k \leqslant p \leqslant n$, denote by $U_{n,p,k}$ the matroid on $\{1, \ldots, n\}$ whose circuits are exactly the subsets of $\{1, \ldots, p\}$ of cardinality k. Then:

Proposition 1: For $p \geqslant 2$, if $M = U_{2p,2p,p}$ and $N = U_{2p,p,p-1}$, then:

(a) the independence system $\Sigma(p) = M \cup N$ on $\{1, \ldots, 2p\}$ admits the unique prime bimatroidal decomposition $\{M, N\}$;

(b) M has $\binom{2p}{p}$ circuits, and $\Sigma(p)$ has $p^2 + 1$ circuits.

Proof: (a) For $p \geqslant 2$, the rank of M (i.e., the cardinality of a basis of M) is $p - 1$, and the rank of N is $2p - 2$. So, $\Sigma(p)$ has bases of two distinct cardinalities, and it follows easily that N must be present in any bimatroidal decomposition of $\Sigma(p)$. In particular, N is prime. Moreover, the second component used in such a decomposition must have rank $p - 1$. But M includes all matroids of rank $p - 1$ on $\{1, \ldots, 2p\}$. So, M is prime, and $\{M, N\}$ is the unique prime bimatroidal decomposition of $\Sigma(p)$.

(b) We leave it to the reader to check that the circuits of $\Sigma(p)$ are exactly those subsets of $\{1, \ldots, 2p\}$ of size p containing zero or one element from $\{p + 1, \ldots, 2p\}$. Part (b) of the statement easily follows from this fact. Q.E.D.

Proposition 1 implies that the independence systems $\Sigma(p)$ have no polynomially bounded prime bimatroidal decomposition. By contrast, it should be noticed that $\Sigma(p)$ can be alternatively written as $N \cup Q$, where N is as in Proposition 1, and the circuits of Q are $\{1, \ldots, 2p\}$, $\{p + 1\}$, $\{p + 2\}$ ..., $\{2p\}$. Since Q is a matroid, $\{N, Q\}$ is a (non-prime) bimatroidal decomposition of $\Sigma(p)$, involving only a total number of $2p + 1$ circuits.

2.2 Admissible Decompositions and Strong Obstructions

Assume the independence system Σ is not a matroid. Let C_1, C_2 denote distinct circuits of Σ, and $x \in C_1 \cap C_2$. The triple $(C_1, C_2; x)$ is called an *obstruction* of Σ if $C_1 \cup C_2 - x$ is independent in Σ (i.e., if C_1, C_2 and x do not satisfy the circuit axiom for matroids). By abuse of language, we shall sometimes say that $(C_1, C_2; x)$ is an obstruction of $C(\Sigma)$, or that (C_1, C_2) itself is an obstruction. The set $V = \{v \in C_1 \cap C_2 : (C_1, C_2; v)$ is an obstruction of $\Sigma\}$ is then called the *obstructing set* of (C_1, C_2).

Proposition 2: Let Σ be an independence system, $(C_1, C_2; x)$ be an obstruction of Σ with obstructing set V, and M be any matroid included in Σ. Then, either $C_1 \cup C_2 - x$ or V is dependent in M.

Proof: Assume by contradiction that $C_1 \cup C_2 - x$ and V are both independent in M. Since $|C_1 \cup C_2 - x| > |V|$, and M is matroid, there exists a subset Z of $C_1 \cup C_2$ such that:

(a) $|V \cup Z| = |C_1 \cup C_2 - x|$;

(b) $V \cup Z$ is independent in M;

(see Welsh 1976, Thm. 1.5.1). By (a), $V \cup Z$ either contains C_1, or contains C_2, or is of the form $C_1 \cup C_2 - y$, where $y \in (C_1 \cap C_2) - V$. Hence, in either case, $V \cup Z$ is dependent in Σ. But this contradicts (b), since M is included in Σ. Q.E.D.

If G_1, G_2 are two collections of sets, then $G_1 \vee G_2$ is by definition the collection of minimal members of $G_1 \cup G_2$. We use the shorthand $G \vee C$ for $G \vee \{C\}$.

With an obstruction $(C_1, C_2; x)$, we associate two independence systems Σ_1 and Σ_2, defined as follows:

$$C(\Sigma_1) = C(\Sigma) \vee V,$$

$$C(\Sigma_2) = C(\Sigma) \vee (C_1 \cup C_2 - x).$$

We say that Σ_1 and Σ_2 are the independence systems *obtained by removing the obstruction* $(C_1, C_2; x)$ *from* Σ (clearly, $(C_1, C_2; x)$ is an obstruction of neither Σ_1 nor Σ_2). The next result was proved by Benzaken and Hammer (1985).

Proposition 3: If Σ is not a matroid, and if Σ_1, Σ_2 are obtained by removing any obstruction from Σ, then $\Sigma_1 \subset \Sigma$, $\Sigma_2 \subset \Sigma$ and $\Sigma = \Sigma_1 \cup \Sigma_2$. Moreover, every matroidal component of Σ is a matroidal component of either Σ_1 or Σ_2.

Proof: Follows easily from Proposition 2. Q.E.D.

From this, it follows that the successive determination of the independence systems $\Sigma_1, \Sigma_2, \Sigma_{11}, \Sigma_{12}, \Sigma_{21}, \Sigma_{22}, \ldots$, results in the determination of all prime matroidal components of Σ (plus possibly some non-prime matroidal components). This procedure goes by the name of *obstruction removal technique*, or *ORT*.

The ORT immediately suggests a simple-minded procedure to solve the bimatroidal decomposition problem: list all prime matroidal components of the given independence system Σ, and check if any pair of them makes up a bimatroidal decomposition of Σ.

Clearly, this method becomes rapidly prohibitive, even for very small problems. On the other hand, it can in some cases be streamlined considerably, by dropping early in the ORT those independence systems which cannot possibly be used in any bimatroidal decomposition of Σ. This idea can be expressed more formally as follows.

A pair $\{\Gamma, \Lambda\}$ of independence systems such that $\Sigma = \Gamma \cup \Lambda$ constitutes an *admissible decomposition* of Σ if: either Σ is not bimatroidal, or there exists a bimatroidal decomposition $\{M, N\}$ of Σ such that $M \subseteq \Gamma$ and $N \subseteq \Lambda$. Notice that, if Σ_1, Σ_2 are obtained by removing some obstruction from Σ, then $\{\Sigma_1, \Sigma_2\}$ is an admissible decomposition of Σ.

Suppose now that $\{\Gamma, \Lambda\}$ is an admissible decomposition of Σ such that Λ is not a matroid, and suppose that $(C_1, C_2; x)$ is an obstruction of Λ. Denote by Λ_1, Λ_2 the independence systems obtained by removing $(C_1, C_2; x)$ from Λ. A simple way of generating another admissible decomposition of Σ is sometimes provided by the following observation. Say that $(C_1, C_2; x)$ is a *strong obstruction* of Λ if either $\Gamma \cup \Lambda_1 \subset \Sigma$ or $\Gamma \cup \Lambda_2 \subset \Sigma$. Then:

Proposition 4: If Σ, Γ, Λ, Λ_1, Λ_2 are as described above, and $\Gamma \cup \Lambda_1 \subset \Sigma$, then $\{\Gamma, \Lambda_2\}$ is an admissible decomposition of Σ.

Proof: Assume that Σ is bimatroidal. Then, there exists a bimatroidal decomposition $\{M, N\}$ of Σ such that $M \subseteq \Gamma$ and $N \subseteq \Lambda$. By Proposition 3, $N \subseteq \Lambda_1$ or $N \subseteq \Lambda_2$. But if $N \subseteq \Lambda_1$, then $\Sigma = M \cup N \subseteq \Gamma \cup \Lambda_1 \subset \Sigma$: contradiction. So, $N \subseteq \Lambda_2$. Q.E.D.

It is not true in general that, in every admissible decomposition of a bimatroidal system, one of the components presents a strong obstruction. But we present in the next Section a large class of independence systems, for which the concepts introduced above have proved useful.

2.3 Linear Independence Systems

The independence system Σ is *linear* if, for every pair of circuits $\{C_1, C_2\}$ of Σ, $|C_1 \cap C_2| \leq 1$ (i.e. if the hypergraph $H(\Sigma) = (X, C(\Sigma))$ is linear in the sense of Berge (1987)).

Remark: The collection of stable sets of a graph G defines an independence system Σ_G, whose circuits are the edges of the graph. Hence, Σ_G is linear, provided that G is simple. Benzaken and Hammer (1985) characterized the class of graphs G for which Σ_G is bimatroidal.

Consider now the following procedure.

Decomposition algorithm

Input: the circuit-set $C(\Sigma)$ of a non-matroidal linear independence system Σ.

Output: the circuit-sets $C(\Sigma_1)$ and $C(\Sigma_2)$ of two independence systems Σ_1 and Σ_2, such that Σ is bimatroidal if and only if $\{\Sigma_1, \Sigma_2\}$ is a bimatroidal decomposition of Σ.

Step 1: Find an obstruction $(K_1 \cup x, K_2 \cup x; x)$ of Σ. Let:

$$C(\Sigma_1) \leftarrow C(\Sigma) \vee \{x\}, \quad C(\Sigma_2) \leftarrow C(\Sigma) \vee (K_1 \cup K_2), \quad x_2 \leftarrow x, \; k \leftarrow 2.$$

Step 2: If $C(\Sigma_k)$ has an obstruction of type $(H_1 \cup y, H_2 \cup y; y)$ with $x_k \in H_1 - H_2$, let:

$$C(\Sigma_k) \leftarrow C(\Sigma_k) \vee \{y\}.$$

Repeat Step 2 until $C(\Sigma_k)$ has no more such obstructions.

Step 3: If $C(\Sigma_k)$ has an obstruction of type $(H_1 \cup x_k, H_2 \cup x_k; x_k)$, let:

$$C(\Sigma_k) \leftarrow C(\Sigma_k) \vee (H_1 \cup H_2).$$

Repeat Step 3 until $C(\Sigma_k)$ has no more such obstructions.

Step 4: If $C(\Sigma_k)$ has an obstruction of type $(\{y, v\}, \{y, u\}; y)$ and $\{x_k, u, v\}$ is a circuit in $C(\Sigma_k)$, let:

$$C(\Sigma_k) \leftarrow C(\Sigma_k) \vee \{u, v\}.$$

Repeat Step 4 until $C(\Sigma_k)$ has no more such obstructions.

Step 5: If $k = 1$, output $C(\Sigma_1)$ and stop. If $k = 2$, output $C(\Sigma_2)$ and continue.

Step 6: If, for some z, $\{z\}$ is a circuit of Σ_2 but not of Σ (i.e., if $C(\Sigma_2)$ has been updated at least once in Step 2), then let $x_1 \leftarrow z$, $k \leftarrow 1$, and go to Step 2. Else, continue.

Step 7: If $C(\Sigma_1)$ has an obstruction of type $(H_1 \cup y, H_2 \cup y; y)$ let:

$$C(\Sigma_1) \leftarrow C(\Sigma_1) \vee \{y\}.$$

Repeat Step 7 until $C(\Sigma_1)$ has no more such obstructions.

Step 8: Output $C(\Sigma_1)$ and stop.

The correctness of this algorithm follows from our next Proposition:

Proposition 5: If Σ is a bimatroidal linear independence system, and the decomposition algorithm outputs the circuit-sets $C(\Sigma_1)$ and $C(\Sigma_2)$, then $\{\Sigma_1, \Sigma_2\}$ is the unique prime bimatroidal decomposition of Σ.

Proof: Since the proof of this result is rather long and tedious, we only sketch here its main steps, and we refer the interested reader to Crama (1987) for a more detailed description.

Assume that Σ is bimatroidal, and consider the outputs Σ_1, Σ_2 of the algorithm. The proof is in three steps: first, we show that all the obstructions removed from Σ_2 in the course of the algorithm are strong; next, we show that Σ_2 is a matroid; finally, we show that the same conclusions hold for Σ_1. The result follows then easily, by repeated applications of Proposition 4.

1. So, we claim first that the obstructions of Σ_2 considered in Steps 2, 3, 4 of the algorithm are strong.

Case 1: If $(H_1 \cup y, H_2 \cup y; y)$ is as described in Step 2 of the algorithm, then its removal from Σ_2 produces Σ_{21} and Σ_{22}, with:

$$C(\Sigma_{21}) = C(\Sigma_2) \vee \{y\},$$

$$C(\Sigma_{22}) = C(\Sigma_2) \vee (H_1 \cup H_2).$$

Notice that $H_1 \cup H_2$ is independent in Σ_2, and hence in Σ. But $H_1 \cup H_2$ is dependent in Σ_1 and in Σ_{22}, and so: $\Sigma_1 \cup \Sigma_{22} \subset \Sigma$. This shows that $(H_1 \cup y, H_2 \cup y; y)$ is a strong obstruction.

Case 2: A similar reasoning shows that the obstructions of Σ_2 considered in Step 3 of the algorithm are strong.

Case 3: Let $(\{y, v\}, \{y, u\}; y)$ be an obstruction and $\{x, u, v\}$ a circuit of Σ_2. Then, $\{x, y\}$ is independent in Σ_2, else the obstruction $(\{x, y\}, \{y, v\}; y)$ would have been removed in Step 2, thus making $\{y\}$ a circuit of Σ_2. Define:

$$C(\Sigma_{21}) = C(\Sigma_2) \vee \{y\},$$

$$C(\Sigma_{22}) = C(\Sigma_2) \vee \{v, u\}.$$

Then, $\{x, y\}$ is independent in Σ, but dependent in Σ_{21} and Σ_1. It follows again that the obstruction is strong.

2. A lengthy case-by-case analysis of the potential remaining obstructions shows that the output Σ_2 is a matroid. We omit here this part of the proof.

3. (a) Consider now Σ_1. If, for some z, $\{z\}$ is a circuit of Σ_2, but not of Σ, then one checks as in part 1 that the obstructions removed from Σ_1 are strong, and that the final output Σ_1 is a matroid (see Step 6 of the algorithm).

(b) If there is no such z, then let us write:

$$C(\Sigma_1) = G_1 \cup \{x\},$$

$$C(\Sigma_2) = G_2 \cup D_2$$

where G_1 is the set of circuits of Σ (or, equivalently, of Σ_1) not containing x, and G_2 is the set of circuits of Σ_2 not containing x.

Let us now define the independence system M by:

$$C(M) = G_2 \cup \{x\}.$$

Because Σ_2 is a matroid, it is obvious that M is a matroid too. Moreover, it is very easy to check that $M \subseteq \Sigma_1$, since $\Sigma_2 \subseteq \Sigma$. So, by definition, M is a matroidal component of Σ_1.

Now, let $(H_1 \cup y, H_2 \cup y; y)$ be an obstruction of Σ_1 (as in Step 7 of the algorithm). The independence systems obtained by removing this obstruction from Σ_1 are defined by:

$$C(\Sigma_{11}) = G_1 \vee \{y\} \vee \{x\},$$

$$C(\Sigma_{12}) = G_1 \vee (H_1 \cup H_2) \vee \{x\}.$$

By Proposition 3, either $M \subseteq \Sigma_{11}$ or $M \subseteq \Sigma_{12}$. If $M \subseteq \Sigma_{11}$, then $\{y\}$ is dependent in M, and hence (by definition of M), $\{y\}$ is dependent in Σ_2. But this contradicts our previous assumption (b).

So, $M \subseteq \Sigma_{12}$. It follows that $(H_1 \cup H_2)$ is dependent in M, and hence in Σ_2. Since $(H_1 \cup H_2)$ is dependent in Σ_{12} and Σ_2, but not in Σ, we have: $\Sigma_{12} \cup \Sigma_2 \neq \Sigma$, and $(H_1 \cup y, H_2 \cup y; y)$ is a strong obstruction of Σ_1. Therefore, by Proposition 4, $\{\Sigma_{11}, \Sigma_2\}$ is an admissible decomposition of Σ.

This reasoning can be repeated until Σ_1 has no more obstructions, i.e. is a matroid. Q.E.D.

We can now easily conclude:

Proposition 6: Linear bimatroidal independence systems can be recognized and decomposed in polynomial time.

Proof: Indeed, the decomposition algorithm described above runs in time polynomial in $|C(\Sigma)|$ and $|X|$ (the number of elements occurring in $C(\Sigma)$), since no step is executed more than $|C(\Sigma)|^2$ times. Q.E.D.

Remark: For a linear independence system with circuit-set $C(\Sigma) = \{C_1, \ldots C_m\}$, the following inequality is easily proved to hold (see Berge 1987):

$$\sum_{j=1}^{m} |C_j|(|C_j| - 1) \leqslant |X| \ (|X| - 1),$$

and so $m = |C(\Sigma)| = 0(|X|^2)$. Hence, our decomposition algorithm for linear independence systems actually runs in time polynomial in $|X|$ (the size of the ground-set).

3 Optimization Over Bimatroidal Independence Systems

Even when Σ is a bimatroidal independence system, the result described in Section 2.1 suggests that a decomposition approach may not yield an optimal solution of the set covering problem over Σ in polynomial time. On the other hand, we are going to prove in this Section, by a more direct approach, that the set covering problem *can* be solved in polynomial time over bimatroidal independence systems.

We first establish a result that will be repeatedly used:

Proposition 7: Let Σ be a bimatroidal independence system, and $\{M, N\}$ be a matroidal decomposition of Σ. If (C_1, C_2) is an obstruction of Σ with obstructing set V, then (up to the names of M and N):

(a) V is independent in N and is a circuit of M;

(b) for all $x \in V$, $C_1 \cup C_2 - x$ is independent in M and is dependent in N.

Proof: Since V is independent in Σ, we may assume without loss of generality that V is independent in N. Then, it follows from Proposition 2 that, for every x in V, $C_1 \cup C_2 - x$ is dependent in N; thus, $C_1 \cup C_2 - x$ is independent in M, and V must be dependent in M.

Moreover, for every x in V, $V - x$ is a subset of $C_1 \cup C_2 - x$, and hence is independent in M. So, V is a circuit of M. Q.E.D.

Remark: Proposition 7 provides a simple solution to the problem of finding a maximum cardinality independent set of a bimatroidal system Σ. Indeed, if B_1 is any basis of Σ containing V, and B_2 is any basis of Σ containing $C_1 \cup C_2 - x$, then B_1 is a basis of N and B_2 is a basis of M. Hence, either B_1 or B_2 has maximum cardinality in Σ.

Consider an independence system $\Sigma = (X, F)$, a nonnegative weight function w on X, and an independent subset A of X. We assume, for simplicity, that the elements of X are labeled $x_1, x_2, \ldots, x_n$, in such a way that $w(x_i) \geqslant w(x_j)$ if $i < j$. We call Greedy (Σ, w, A) the following procedure:

Greedy (Σ, w, A)

Step 1: Let: $G \leftarrow A$;

Step 2: Repeat for $i = 1, \ldots, n$: if $G \cup x_i \in F$, then $G \leftarrow G \cup x_i$;

Step 3: Return G, and stop.

The set G returned by this procedure will be called the *greedy extension of* A *in* Σ. If $A = \phi$, then G is simply called *greedy solution.* Observe that Greedy (Σ, w, ϕ) (or, for short, Greedy) is the classical greedy heuristic, and hence always yields an optimal solution of the set covering problem over Σ when Σ is a matroid. The next Proposition describes more precisely what happens when the greedy solution is not optimal.

Proposition 8: If the greedy solution G is not an optimal solution of the set covering problem over Σ, then there exists an optimal solution B, and an obstruction $(C_1, C_2; x)$ of Σ, such that:

(a) $x \in G$;

(b) $C_1 \cup C_2 - x \subseteq B$.

Proof: Let $G = \{g_1, \ldots, g_p\}$, and suppose that G is not an optimal solution of the set covering problem over Σ. Let $B = \{b_1, \ldots, b_s\}$ denote an optimal solution of this problem. We assume that the elements of G and B are ordered as those of X. Define the index k by: $k = \min \{j : g_j \neq b_j\}$, and assume that B has been chosen among the optimal solutions of the set covering problem so that k be maximum.

One easily checks that $w(g_k) \geqslant w(b_k)$, and $g_k \notin B$ (else b_k would have been picked in G before g_k).

Since $g_k \notin B$ and $w(g_k) \geqslant 0$, $B \cup g_k$ includes at least one circuit (else, $B \cup g_k$ is an optimal solution of the set covering problem, contradicting the choice of B). Assume first that $B \cup g_k$ includes exactly one circuit, say C. Since G is independent, there is at least one element b_j in $C - G$. By definition of $k, j \geqslant k$. Hence, $w(b_j) \leqslant w(b_k) \leqslant w(g_k)$. Now, $B \cup g_k - b_j$ is independent in Σ, and:

$$w(B \cup g_k - b_j) = w(B) + w(g_k) - w(b_j) \geqslant w(B).$$

But this contradicts the choice of B.

So, $B \cup g_k$ includes at least two circuits, say C_1 and C_2. Since $C_1 \cup C_2 - g_k$ is included in B (and hence is independent), $(C_1, C_2; g_k)$ is an obstruction of Σ satisfying the required conditions (a) and (b). Q.E.D.

We introduce now the following definition: if $\Sigma = (X, F)$ is an arbitrary independence system, and $A \subseteq X$, then Σ/A is the independence system on $X - A$ whose circuits are exactly the minimal members of $\{C - A : C \text{ is a circuit of } \Sigma\}$. Equivalently, it is easy to check that $\Sigma/A = (X - A, F/A)$, where:

$$F/A = \{J : J \subseteq X - A \text{ and } J \cup A \in F\}.$$

In the special case where Σ is a matroid and A is independent in Σ, Σ/A is the *contraction* of Σ to $X - A$, and is well-known to be a matroid (see Welsh 1976).

Now, fix $\Sigma = (X, F)$, and let:

$$\{O_1, \ldots, O_t\} = \{C_1 \cup C_2 - x : (C_1, C_2 : x) \text{ is an obstruction of } \Sigma\}.$$

For a given weight function w on X, we denote by G_i the greedy extension of O_i in Σ (for some linear ordering of X compatible with w) ($i = 1, \ldots, t$). We also let G denote a greedy solution of the set covering problem over Σ. With these notations, we can now state:

Proposition 9: If Σ/O_i is a matroid, for $i = 1, \ldots, t$, then there is an optimal solution of the set covering problem over Σ among $G, G_1, \ldots, G_t$.

Proof: By Proposition 8, all we have to prove is that, for $i = 1, \ldots, t$, G_i is an optimal solution of:

$$\max \; w(A) \qquad (2)$$

$$\text{subject to } A \in F, \; O_i \subseteq A.$$

Fix i. It is clear that $G_i - O_i$ is produced by a correct application of the greedy algorithm to the matroid Σ/O_i and hence $G_i - O_i$ is an optimal solution of:

$$\max \; w(A)$$

$$\text{subject to } A \in F/O_i.$$

It follows then easily that G_i is optimal for (2). Q.E.D.

As a consequence of Proposition 9, the set covering problem is solvable in polynomial time over those independence systems Σ such that Σ/O_i is a matroid, for $i = 1, \ldots, t$. Notice that this sufficient condition can also be tested in polynomial time, since the number of circuits of Σ/A is bounded by the number of circuits of Σ, for all $A \subseteq X$.

Returning now to the bimatroidal case, we prove:

Proposition 10: The set covering problem over bimatroidal independence systems can be solved in polynomial time.

Proof: Let $\{M, N\}$ be a bimatroidal decomposition of Σ, with $\Sigma = (X, F)$ and $M = (X, F_M)$. Let $(C_1, C_2; x)$ be an obstruction of Σ, and $O = C_1 \cup C_2 - x$. By Proposition 7, we can assume that O is dependent in N. So,

$$F/O = \{J : J \subseteq X - O, J \cup O \in F\}$$

$$= \{J : J \subseteq X - O, J \cup O \in F_M\}$$

$$= F_M/O,$$

or, equivalently, $\Sigma/O = M/O$. But M is a matroid, and hence M/O is a matroid too. The claim follows now directly from Proposition 9. Q.E.D.

It might be interesting to point out that if Σ/O_i is a matroid for $i = 1, \ldots, t$, then Σ is not necessarily bimatroidal. More precisely, one can show that, for every integer $m \geqslant 1$, there is an independence system Σ_m with matroidal number m, and such that Σ_m/O_i is a matroid, for $i = 1, \ldots, t$ (this is for instance the case if the bases of Σ_m are the edges of a perfect matching on $2m$ vertices; see Crama 1987).

This remark shows that we did not yet exploit much of the structure of bimatroidal independence systems, in our quest for an efficient optimization algorithm over such a system. In particular, the procedure suggested by Proposition 10 may a priori require a large number of applications of the greedy algorithm, whereas we know that two applications of Greedy are in principle enough if a bimatroidal decomposition of the independence system is available. Our goal in the remainder of this Section is to sharpen some of the general results we have obtained so far, and to present a more specialized algorithm for the set covering problem over bimatroidal systems.

To this effect, we first state a refined version of Proposition 8, valid for bimatroidal independence systems (in this statement, the term "last" is meant with respect to the ordering of the ground set used for the greedy algorithm):

Proposition 11: If Σ is bimatroidal, and the greedy solution G is not optimal for the set covering problem over Σ, then there exists an optimal solution B, and an obstruction $(C_1, C_2; g_k)$ of Σ with obstructing set V, such that:

(a) $V \subseteq G$;

(b) $C_1 \cup C_2 - g_k \subseteq B$;

(c) g_k is the last element of V;

(d) if W is another obstructing set included in G, and g_t is the last element of W, then $k < t$.

Proof: 1. Repeat the proof of Proposition 8, but in the last-but-one sentence, let $C_1, C_2, \ldots, C_s$ denote the circuits of Σ contained in $B \cup g_k$. For $1 \leqslant i < j \leqslant s$, $(C_i, C_j; g_k)$ is an obstruction of Σ, with obstructing set V_{ij}. We claim that $V_{ij} = V_{12}$, for $1 \leqslant i < j \leqslant s$.

2. It is enough to show that $V_{12} = V_{13}$. Let $\{M, N\}$ denote a bimatroidal decomposition of Σ, and assume without loss of generality that B is independent in M. Then, $C_1 \cup C_2 - g_k$ and $C_1 \cup C_3 - g_k$ are independent in M, and it follows from Proposition 7 that V_{12} and V_{13} must be circuits of M. If V_{12} and V_{13} are distinct, then, M being a matroid, $V_{12} \cup V_{13} - g_k$ is dependent in M. But $V_{12} \cup V_{13} - g_k$ is contained in B, and hence this contradicts the assumption that B is independent in M. Therefore, $V_{12} = V_{13}$.

3. Now, if $x \in V_{12} - G$, then $x \in B - G$, and hence $w(x) \leqslant w(g_k)$. Since $x \in C_1 \cap C_2 \cap \ldots \cap C_s$, $B \cup g_k - x$ is independent in Σ, and is optimal for the set covering problem. But this contradicts the choice of B. So, $V_{12} \subseteq G$, and $(C_1, C_2; g_k)$ satisfies (a), (b).

4. Assume $g_i \in V_{12}$, where $i > k$. Then again, $w(g_i) \leqslant w(g_k)$, and $B \cup g_k - g_i$ is independent in Σ : contradiction. This proves (c).

5. Suppose now that W and g_t are as described under (d). From the assumptions in part 2 of the proof, it follows that W is a circuit of M. If $t < k$, then $W \subseteq G \cap B$, and this contradicts the assumption that B is independent in M. If $t = k$, then $V_{12} \cup W - g_k \subseteq B$. But, since V_{12} and W are circuits of the matroid M, this leads again to a contradiction, and (d) is proved. Q.E.D.

Consider now the following algorithm for the set covering problem (where, as usual, we assume that $w(x_i) \geqslant w(x_j)$ if $i < j$):

Bigreedy (Σ, w)

Step 1: Let: $G \leftarrow \phi; B \leftarrow \phi$;

Step 2: Repeat for $i = 1, \ldots, n$: if $G \cup x_i \in F$, then $G \leftarrow G \cup x_i$;

Step 3: If G contains no obstructing set of Σ, then return G and stop: else, let V be the obstructing set uniquely defined by (a), (c) and (d) in Proposition 11; let x be the last element of V, and let:

$$\{O_1, \ldots, O_r\}$$
$$= \{C_1 \cup C_2 - x : (C_1, C_2; x) \text{ is an obstruction of } \Sigma \text{ with obstructing set } V\};$$

Step 4: Repeat for $i = 1, \ldots, n$: if there exists some $j \in \{1, \ldots, r\}$ such that $B \cup x_i \cup O_j \in F$, then $B \leftarrow B \cup x_i$;

Step 5: If $w(G) \geqslant w(B)$, then return G and stop; else, return B and stop.

The idea in Step 4 is to build, in a greedy fashion, a basis B of Σ including at least one of the sets $O_1, \ldots, O_r$. The validity of this procedure is asserted by the next Proposition:

Proposition 12: If $\Sigma = (X, F)$ is bimatroidal, then Bigreedy (Σ, w) returns an optimal solution of the set covering problem over Σ.

Proof: We already know that, if Bigreedy stops (at Step 3) because G contains no obstructing set, then G is an optimal solution of the set covering problem over Σ. So, assume that G contains some obstructing set. Let $\{M, N\}$ denote a bimatroidal decomposition of Σ. By Proposition 7, we may as well assume that G is a basis of N and B is a basis of M.

The basis G is an optimal solution of the set covering problem over N. Hence, if G is not optimal for the set covering problem over Σ, it must be the case that some basis of M is optimal for this problem: say A is such a basis. By Proposition 11, we can assume without loss of generality that one of the sets $O_1, \ldots, O_r$ is included in A.

For the simplicity of the argument, it is convenient to assume, at this point of the proof, that the weight function w is injective on X, i.e.: for all $x, y \in X$, if $x \neq y$, then $w(x) \neq w(y)$ (if this is not the case, then $w(x_i)$ can be "perturbed" by a small quantity, say ϵ^i; details are left to the reader).

Let $A = \{a_1, \ldots, a_p\}$ and $B = \{b_1, \ldots, b_p\}$, where $i < j$ implies $w(a_i) > w(a_j)$ and $w(b_i) > w(b_j)$. Clearly, A is an optimal solution of the set covering problem over M. Hence, $w(a_i) \geqslant w(b_i)$ for $i = 1, \ldots, p$ (see e.g. Lawler 1976, Thm. 7.6.2). If $A \neq B$, then there is a smallest index $j \in \{1, \ldots, p\}$ such that $w(a_j) > w(b_j)$. But then, by injectivity: $\{a_1, \ldots, a_{j-1}\} = \{b_1, \ldots, b_{j-1}\}$ and a_j should have been placed in B before b_j, in Step 4 of Bigreedy: contradiction.

So $A = B$ and B is an optimal solution of the set covering problem over Σ. This proves the Proposition. Q.E.D.

From the proof of Proposition 12 one sees that Bigreedy comes very close to the "ideal" algorithm for the bimatroidal set covering problem, as sketched in the Introduction of the paper: in a greedy fashion, and without requiring the explicit knowledge of the numerical weights, or of a bimatroidal decomposition $\{M, N\}$ of Σ, Bigreedy produces optimal solutions of the MWI problems over M and N, respectively. This observation leads us to formulate the following conjecture:

Conjecture: Let Σ be an independence system on X. If, for all nonnegative weight functions w on X, Bigreedy (Σ, w) returns an optiaml solution of the set covering problem over Σ, then Σ is bimatroidal.

References

Benzaken C, Hammer PL (1985) Boolean techniques for matroidal decomposition of independence systems and applications to graphs. Discrete Mathematics 56:7–34

Berge C (1987) Hypergraphes. Gauthier-Villars, Paris

Crama Y (1987) Recognition and solution of structured discrete optimization problems. PhD thesis, RUTCOR, Rutgers University, New Brunswick, NJ. Published by University Microfilms International, Ann Arbor, MI

Lawler EL (1976) Combinatorial optimization: networks and matroids. Holt, Rinehart and Winston, New York

Welsh DJA (1976) Matroid theory. Academic Press, London New York San Francisco

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