A Characterization of a Cone of Pseudo-Boolean Functions via Supermodularity-Type Inequalities

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Abstract

A pseudo-Boolean function is a real valued function defined on the vertices of the unit n-dimensional hypercube. It has a unique expression as a multilinear polynomial in n variables. It is called almost-positive if all the coefficients in that expression, except maybe those in the linear part, are nonnegative. The almost-positive functions form a convex cone, given explicitly by its extreme rays. Here we describe this cone by a system of linear inequalities, which can be viewed as a natural generalization of supermodularity to higher orders. We also point out a characterization in terms of the sign of partial derivatives.

A *pseudo-Boolean function* is a function $f: B^n \to \mathbb{R}$, where n is a positive integer, $B^n = \{0, 1\}^n$ and $\mathbb{R}$ is the set of real numbers. By denoting $N = \{1, \dots, n\}$ and identifying subsets of N with their characteristic vectors, f can be considered a real valued function defined on the Boolean algebra of all subsets of N. The arguments of f will be written in either form (vector or subset), as convenient.

It is well-known that every pseudo-Boolean function f has a unique polynomial expression of the form

$$f(x) = \sum_{T \subseteq N} \left[a_T \prod_{k \in T} x_k \right], \tag{1}$$

where $x = (x_1, \dots, x_n) \in B^n$ and a_T are real coefficients. These coefficients can be derived explicitly from f using the formula

$$a_T = \sum_{S \subseteq T} (-1)^{t-s} f(S), \qquad T \subseteq N, \tag{2}$$

where t and s are the cardinalities of T and S, respectively. By letting x vary over $\mathbb{R}^n$, we may regard formula (1) as defining a multilinear function which agrees with f on B^n. No confusion will arise from referring to this function as f as well. We let deg(f) denote the degree of the polynomial (1).

A pseudo-Boolean function f is *almost-positive* if $a_T \geq 0$ for all $T \subseteq N$ such that $|T| \geq 2$. This class of functions has received attention in the optimization literature, as its members can be maximized using network flow methods (see for example [3]). In a game theoretic context, this is precisely the class of characteristic function games (with side-payments) for which the core coincides with the selectope (see [2]).

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Clearly, the almost-positive functions form a convex cone in $\mathbb{R}^{B^n}$, namely the one generated by the monomials $\Pi_{k \in T} x_k$ for $T \subseteq N$, $|T| \geq 2$, together with $\pm x_k$ for $k \in N$ and ± 1. In terms of finding interesting valid inequalities for this cone, it has been observed that almost-positive functions are *supermodular,* i.e., they satisfy

$$f(S \cup T) \geq f(S) + f(T) - f(S \cap T), \qquad \forall\, S, T \subseteq N. \tag{3}$$

The purpose of this note is to observe that this property can be generalized to "supermodularity of higher orders," and the resulting system of inequalities characterizes the cone of almost-positive functions. We also point out a characterization in terms of the sign of partial derivatives, which gives a geometric intuition for this class of functions.

To understand the generalization of (3), it is useful to interpret it as stating that an attempt to evaluate f at the union of two sets via an inclusion-exclusion formula yields an underestimate of the actual value. The same idea for the union of m sets, where m is an integer ≥ 2, suggests the inequalities

$$\begin{aligned} f\left(\bigcup_{i=1}^{m} S_i\right) \geq \sum_{i=1}^{m} f(S_i) - \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} f(S_i \cap S_j) + \ldots \\ + (-1)^{m+1} f\left(\bigcap_{i=1}^{m} S_i\right), \qquad \forall\, S_1, \ldots, S_m \subseteq N. \end{aligned} \tag{4}$$

A pseudo-Boolean function which satisfies (4) for a given m will be called *supermodular of order* m. In our terminology, the classical supermodularity becomes supermodularity of order 2. It should be noted that supermodularity of order m implies supermodularity of order m′ whenever $2 \leq m' < m$; to see this, put $S_{m'+1} = \ldots = S_m = \emptyset$.

Theorem *Let* f *be a pseudo-Boolean function. The following statements are equivalent:*

(a) f *is almost-positive.*
(b) f *is supermodular of order* m *for all integers* $m \geq 2$.
(c) *if* $\deg(f) \geq 2$ *then* f *is supermodular of order* deg(f).
(d) *all partial derivatives of* f *of orders* ≥ 2 *are nonnegative on the nonnegative orthant* $\mathbb{R}^n_+$.

Proof

(a)⇒(b): It suffices to show that the generators of the cone of almost-positive functions mentioned above satisfy the inequalities (4). Let $S_1, \ldots, S_m \subseteq N$ and consider first a monomial of the form $f(x) = \Pi_{k \in T} x_k$, where $T \subseteq N$ and $|T| \geq 2$. Then $f(S) = 1$ or 0 according as $S \supseteq T$ or not. Assume that $S_i \supseteq T$ holds true for exactly p of the indices $i \in \{1, \ldots, m\}$. If $p = 0$ then the right hand side of (4) is zero, so we may assume $p \geq 1$. In this case (4) reduces to

$$1 \geq \sum_{\ell=1}^{p} (-1)^{\ell+1} \binom{p}{\ell},$$

which actually holds with equality (consider the expansion of $(1-1)^p$). Next, consider a monomial of the form $f(x) = \pm x_k$, where $k \in N$. Assume that $k \in S_i$ holds true for exactly p of the indices $i \in \{1, \ldots, m\}$. If $p = 0$ then all terms in (4) are zero, while for $p \geq 1$ the above counting argument still works (a minus sign is no obstacle, since the inequality holds with equality). Finally, for a constant function $f = \pm 1$ the same argument with $p = m$ works.

(b) ⇒ (c): This is obvious.

(c) ⇒ (a): Let $T \subseteq N$ with $|T| \geq 2$. If $|T| > \deg(f)$ then $a_T = 0$, so we may assume $|T| \leq \deg(f)$. By supposition f is supermodular of order deg(f), so it is also supermodular of order $|T|$. Using this fact for the collection $\{S_i = T \setminus \{i\}\}_{i \in T}$ and formula (2) we obtain $a_T \geq 0$.

(a) ⇒ (d): This is obvious.

(d) ⇒ (a): Let $T \subseteq N$ with $|T| \geq 2$. The partial derivative of f of order $|T|$ with respect to the variables x_k, $k \in T$, evaluated at any point x with $x_\ell = 0$ for $\ell \notin T$, equals a_T. Hence $a_T \geq 0$.

This completes the proof of the theorem. Although our proof does not directly relate supermodularity of higher orders to the sign of partial derivatives of higher orders, the following is true: for a given integer $m \geq 2$, f is supermodular of order m if and only if all partial derivatives of f of orders $2, \ldots, m$ are nonnegative on B^n.

Remark We have recently realized that the Theorem of this paper can also be derived from Choquet's results on the theory of capacities (see p. 171 and p. 217 of [1]).

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