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A decomposition of strongly unimodular matrices into incidence matrices of digraphs

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Abstract

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A new decomposition of strongly unimodular matrices is given.

1. Introduction—Building up a strongly unimodular matrix

A matrix A is said to be *totally unimodular* (TU) if the determinant of each of its square submatrices is $0, \pm 1$. A well-known theorem by Ghouila-Houri [3] asserts that a matrix A is TU if and only if each submatrix A' of A has the following property: the rows of A' can be split into two parts so that, for each column, the sums of the entries in each part differ at most by one. A matrix A is *strongly unimodular* (SU) if (i) A is TU and (ii) every matrix obtained from A by setting a ± 1 entry to 0 is also TU. These matrices are also called 1-TU matrices in [5]. The notation of SU matrices was introduced in [2], where the following equivalent condition (for hypergraphs, i.e. $0, 1$-matrices only) was also given: a matrix is SU iff all its bases are triangular. This characterization has been extended to $0, \pm 1$-matrices in [5].

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It is convenient to describe a (0, ±1)-matrix A by means of an associated signed graph. The nodes r_i, $i=1, \ldots, m$, and c_j, $j=1, \ldots, n$, correspond to rows and columns of A respectively. A pair r_ic_j forms an edge if the entry a_{ij} is nonzero, and the weight (=the sign) of the edge r_ic_j is a_{ij}. The edges weighted by $+1$, resp. -1, will be called *positive*, resp. *negative*. The weight of a cycle in the graph is defined as the sum of the weights of its edges. Notice that the weight of a cycle must be either 0 (mod 4) or 2 (mod 4) since the graph is bipartite. As usual, we will often not distinguish between a matrix and its graph. In particular, we say that a graph has a certain property if the matrix has it. An important subclass of SU matrices, introduced earlier in [7], are restricted unimodular matrices.

A (0, ±1)-matrix A is called *restricted unimodular* (RU) if the weight of each cycle in the associated graph is 0 (mod 4). It has been shown in [7] that every RU matrix can be built up by a series of certain operations starting from so-called basic RU matrices (see below). Further, it has been shown in [1] that SU matrices can be built up from RU matrices by means of one additional operation. Let us now recall these decompositions.

First, it is obvious that the membership of a matrix to RU and SU classes is invariant under multiplying a row or column by -1. We stress this obvious fact, since a suitable multiplication of a set of rows and columns by -1 can simplify the description of the other operations.

Operaton 0. Multiplying a row or a column by -1.

Lemma 1.1. *Both RU and SU properties are invariant under operation* 0.

Operation 1. Let G_1 and G_2 be two graphs (on disjoint sets of nodes), and x_i be a node of G_i of degree at least 2 and such that all edges incident to x_i are positive, $i=1, 2$. Construct G as follows: Connect each neighbour of x_1 to each neighbour of x_2 by a positive edge, and delete nodes x_1 and x_2.

Lemma 1.2 [1]. (i) *If G_1 and G_2 are SU, then G constructed by Operation* 1 *is SU as well.*

(ii) *If G is a SU graph which is not RU, then G can be obtained by Operations* 0 *and* 1 *from some SU graphs G_1 and G_2.*

Operation 2. Let G_1 and G_2 be two graphs, and let x and y be nodes of G_1 and G_2 of degree 2. Assume that xx_2, yy_1 and yy_2 are positive edges, and xx_1 is a negative edge. Construct G by deleting x and y, and adding two positive edges x_1y_1 and x_2y_2.

Operation 3 (*1-sum*). Construct G from G_1 and G_2 by identifying a node x_1 of G_1 with a node x_2 of G_2.

A $(0, \pm 1)$-matrix is called *basic* RU if it is TU and if either the matrix or its transpose has exactly two nonzero entries per column. Let us recall a result by Heller and Tompkins [4].

Lemma 1.3 [4]. *Let A be a $(0, \pm 1)$-matrix with exactly two nonzero entries per column. Then the following are equivalent:*

(i) *A is TU,*

(ii) *the rows of A can be split into two classes so that the nonzero entries of each column lie in the same class if and only if they have opposite sign.*

It follows immediately from Lemma 1.3 that the notions of RU, SU and TU coincide for matrices with exactly two nonzero entries per column. Up to multiplication of some rows by -1, these matrices are exactly the node–arc incidence matrices of digraphs.

Lemma 1.4 [7]. (i) *If G_1 and G_2 are RU, then a graph G constructed by Operation 2 or Operation 3 is RU as well.*

(ii) *Let G be* RU *but not basic* RU. *Then G can be obtained by means of Operations 0 and 2 or Operation 3 from some* RU *graphs G_1 and G_2.*

Lemmas 1.1, 1.2 and 1.4 together yield a decomposition of SU matrices into basic RU matrices. In this paper, we give another type of decomposition (Theorem 2.2). As a by-product of this result, we prove in the last section that each SU matrix has the so-called 'on-line balancing property'.

2. Main result

We first establish a lemma, which was proved in [2] for 0,1-matrices only.

Lemma 2.1. *If A is an SU matrix, then there exists a non-empty subset S of rows of A such that every column of A with at least 2 nonzero entries has either 0 or 2 nonzero entries in S.*

Proof. For simplicity, we refer to the property stated in the conclusion of the lemma as 'Property 2.1.' Let A be an SU matrix of size $m \times n$. We proceed by induction on $m \times n$. According to the results summarized in the previous section, it is sufficient to prove that Property 2.1 holds for basic RU matrices, and that, if the property holds for two graphs G_1, G_2 and all their induced subgraphs, then it also holds for the graph obtained from G_1, G_2 by Operations 0, 1, 2 or 3.

(i) *Basic RU matrices.* Property 2.1 trivially holds if A has two nonzero entries per column. So, assume that A has two nonzero entries per row. Let us define an auxiliary graph H as follows: let A_1 be a matrix obtained from A by replacing each '-1' by '$+1$'. Then, H is the graph such that A_1 is its $E \times V$ incidence matrix (thus rows of A_1 correspond to edges of H). If H is not connected, then we conclude by induction. Else, if H has a cycle, then the rows of A corresponding to the edges of the cycle define a suitable set S. If H is a tree, then choose a path between two arbitrary leaves of H. Then, the set S formed by the rows of A corresponding to the edges of this path satisfies property 2.1.

(ii) *Operation* 0 *preserves Property* 2.1. Trivial.

(iii) *Operation* 2 *preserves Property* 2.1. Let G_1, G_2 be as in the definition of operation 2 and A_1, A_2 be the corresponding matrices, with S_1 and S_2 the subsets of rows satisfying Property 2.1. Assume without loss of generality that x is a column vertex and y is a row vertex. Now if S_1 (resp. S_2) does not contain both x_1 and x_2 (resp. y) then S_1 (resp. S_2) satisfies Property 2.1 with respect to A. If S_1 contains both x_1 and x_2 and S_2 contains y then $S = S_1 \cup S_2$ satisfies Property 2.1 for A.

(iv) *Operation* 3 *preserves Property* 2.1. With the same notation as above, assume first that both x_1 and x_2 are column-vertices.

· If x_1 (resp. x_2) contains either 0 or 2 nonzero entries from S_1 (resp. S_2), then S_1 (resp. S_2) satisfies Property 2.1 with respect to A.

· If x_1 contains exactly one nonzero entry from S_1, and x_2 contains exactly one nonzero entry from S_2, then $S = S_1 \cup S_2$ satisfies the required property with respect to A.

A similar argument works when both x_1 and x_2 are row-vertices.

(v) *Operation* 1 *preserves Property* 2.1. With the same notation as before, assume without loss of generality that x_2 is a column of A_2. Then, S_2 satisfies the required property with respect to A (this is because x_2 has degree at least 2 in G_2). This concludes the proof. □

Theorem 2.2. *If A is an SU matrix, then there exists a partition $(S_1, \ldots, S_k)$ of the rows of A with the following properties*:

(i) *every column of A has* 0, 1 *or* 2 *nonzero entries in each S_i, for $i = 1, \ldots, k$*;

(ii) *if a column has exactly one nonzero entry in some S_i, then all its entries in $S_{i+1}, \ldots, S_k$ are zeroes.*

Proof. The result follows directly from an iterated application of Lemma 2.1. □

Remark. This gives another decomposition of strongly unimodular matrices into (almost) basic RU matrices.

3. Application: On-line balancing

Lovász asked in [6] whether the theorem of Ghouila-Houri mentioned in the first paragraph of this paper can be strengthened as follows.

Let us say that a matrix A is *on-line balanced,* if there are row-multipliers $s_1, \ldots, s_m \in \{-1, 1\}$ and a permutation π of the rows such that

$$\left|\sum_{i=1}^{k} s_{\pi(i)} a_{\pi(i)j}\right| \leqslant 1 \quad \text{for every } k = 1, \ldots, m \text{ and } j = 1, \ldots, n$$

(where m and n are the number of rows and columns of A respectively).

Clearly, if each submatrix of a matrix A can be on-line balanced, then A is TU. Lovász asked whether the converse is also true, i.e., whether every TU matrix can be on-line balanced. We here give a positive answer for SU matrices. The question is still open for arbitrary TU matrices.

Theorem 3.1. *Every strongly unimodular matrix can be on-line balanced.*

Proof. This follows directly from Theorem 2.2. To get an on-line balancing of A, take its rows in the order induced by the partition given in Theorem 2.2, and use the Heller–Tompkins characterization (Lemma 1.3) for each partition class S_i, $i = 1, \ldots, k$. □

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