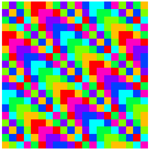
From combinatorial games to shape-symmetric morphisms (3)

Michel Rigo

http://www.discmath.ulg.ac.be/
http://orbi.ulg.ac.be/handle/2268/216249

Jean-Morlet chair: Shigeki Akiyama
Research school: Tiling Dynamical System, 24 November 2017



For this third talk

- Answering a question I've been asked
- Quick summary and conclusions on shape-symmetry
- A bit more about games (and connections with words)
- Link between morphisms/substitutions and automata
- Games with a finite set of moves

Recall the following:

Proposition

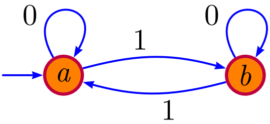
A sequence is k-automatic IFF its k-kernel is finite.

Definition

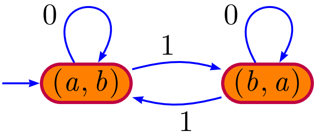
The k-kernel of a sequence $\mathbf{x} = (x(n))_{n\geq 0}$ is a set of subsequences:

$$\mathrm{Ker}_k(\mathbf{x}) = \left\{(x(k^i n + s))_{n\geq 0} \mid i \geq 0,\ 0 \leq s < k^i\right\}.$$

Thue–Morse word is 2-automatic.
What about the 3-kernel of Thue–Morse?



Transition monoid



$\Rightarrow \#\mathrm{Ker}_2(\mathbf{t}) = 2$

$$abbabaabbaababba\cdots$$

$$baababbaabbabaab\cdots$$

There is an important theorem of Cobham:

Theorem [Cobham 1969]

Let $k, \ell \geq 2$ be two multiplicatively independent integers.
If a sequence is both k-automatic and ℓ-automatic,
then it is ultimately periodic.

Thue–Morse word has no cube, it is not ultimately periodic.

As a consequence of Cobham's theorem, Thue–Morse is not
3-automatic, $\mathrm{Ker}_3(\mathbf{t})$ is infinite.

Generalizations exist (F. Durand, multidimensional versions, ...)
Perron–Frobenius eigenvalue is replacing the base.

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Let us conclude with shape-symmetric morphisms

Let's try something...

$$\varphi_W : a \mapsto \begin{array}{|c|c|}\hline c & d \\ \hline a & b \\ \hline\end{array} \quad b \mapsto \begin{array}{|c|}\hline e \\ \hline i \\ \hline\end{array} \quad c \mapsto \begin{array}{|c|c|}\hline i & j \\ \hline\end{array} \quad d \mapsto \begin{array}{|c|}\hline i \\ \hline\end{array} \quad e \mapsto \begin{array}{|c|c|}\hline f & b \\ \hline\end{array}$$

$$f \mapsto \begin{array}{|c|c|}\hline h & d \\ \hline g & b \\ \hline\end{array} \quad g \mapsto \begin{array}{|c|c|}\hline h & d \\ \hline f & b \\ \hline\end{array} \quad h \mapsto \begin{array}{|c|c|}\hline i & m \\ \hline\end{array} \quad i \mapsto \begin{array}{|c|c|}\hline h & d \\ \hline i & m \\ \hline\end{array}$$

$$j \mapsto \begin{array}{|c|}\hline c \\ \hline k \\ \hline\end{array} \quad k \mapsto \begin{array}{|c|c|}\hline c & d \\ \hline l & m \\ \hline\end{array} \quad l \mapsto \begin{array}{|c|c|}\hline c & d \\ \hline k & m \\ \hline\end{array} \quad m \mapsto \begin{array}{|c|}\hline h \\ \hline i \\ \hline\end{array}$$

and the coding

$$\mu_W : a, e, g, j, l \mapsto 1, \quad b, c, d, f, h, i, k, m \mapsto 0$$

THEOREM ♕ [8]

The morphism φ_W and the coding μ_W give the 2-dimensional infinite word coding the $\mathcal{P}$-positions of Wythoff.

Definition

Let $\gamma\colon \mathcal{B}_d(A) \to \mathcal{B}_d(A)$ be a d-dimensional morphism having the d-dimensional infinite word x as a fixed point.

This word is shape-symmetric with respect to γ if, for all permutations ν of $[\![1, d]\!]$, we have, for all $n_1, \ldots, n_d \geq 0$,

$$|\gamma(x(n_1, \ldots, n_d))| = (s_1, \ldots, s_d)$$

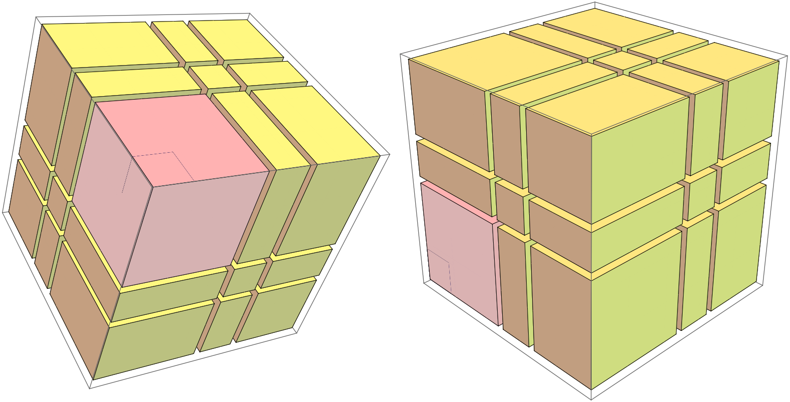
$$\Downarrow$$

$$|\gamma(x(n_{\nu(1)}, \ldots, n_{\nu(d)}))| = (s_{\nu(1)}, \ldots, s_{\nu(d)}).$$

Reconsider our map φ (one can indeed prove that it is a d-dimensional morphism having a shape-symmetric fixed point).

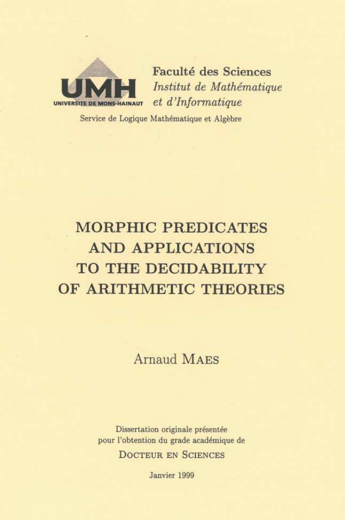
$$a \mapsto \begin{array}{|c|c|}\hline c & d \\ \hline \mathbf{a} & b \\ \hline\end{array} \mapsto \begin{array}{|c|c|c|}\hline i & \mathbf{j} & i \\ \hline c & d & \mathbf{e} \\ \hline \mathbf{a} & b & i \\ \hline\end{array} \mapsto \begin{array}{|c|c|c|c|c|}\hline h & d & c & h & d \\ \hline i & m & k & i & m \\ \hline i & \mathbf{j} & i & f & b \\ \hline c & d & \mathbf{e} & h & d \\ \hline \mathbf{a} & b & i & i & m \\ \hline\end{array}$$

sizes : 1, 2, 3, 5



Initial blocks of some 3-dimensional shape-symmetric picture [12, Maes' thesis p. 107].

Associated decision problems



UMH
UNIVERSITÉ DE MONS-HAINAUT

Faculté des Sciences
Institut de Mathématique et d'Informatique

Service de Logique Mathématique et Algèbre

MORPHIC PREDICATES AND APPLICATIONS TO THE DECIDABILITY OF ARITHMETIC THEORIES

Arnaud Maes

Dissertation originale présentée pour l'obtention du grade académique de Docteur en Sciences

Janvier 1999

Some Results from Maes' papers:

- Determining whether or not a map $\mu\colon \mathcal{B}_d(A) \to \mathcal{B}_d(A)$ is a d-dimensional morphism is a decidable problem.
- If μ is prolongable on a letter a, then it is decidable whether or not the fixed point $\mu^\omega(a)$ is shape-symmetric.

[10, 11, 12]

A bit more about games

We have seen that the Fibonacci word is "coding" the $\mathcal{P}$-positions of Wythoff's game.

$$(1,2),\ (3,5),\ (4,7),\ (6,10),\ldots$$

$$abaababaabaababaababa\cdots$$

- Several games coded by the same word?
- Given a word, find a game?

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Adding or removing moves

$\rightsquigarrow$ How can we alter the set of moves ♕ to keep the same set of $\mathcal{P}$-positions?

Remark

This means that several rule-sets (different "games") could lead to the same set of $\mathcal{P}$-positions.

In a subtraction game, observe that a move can be adjoined (*without altering the set of $\mathcal{P}$-positions*) if and only if

$$\text{it does not belong to } \mathcal{P} - \mathcal{P}.$$

Corollary

We can adjoined the move $(i,j)_{i<j}$ to Wythoff's rule-set iff

$$(i,j) \neq (\lfloor n\,\varphi \rfloor - \lfloor m\,\varphi \rfloor, \lfloor n\,\varphi^2 \rfloor - \lfloor m\,\varphi^2 \rfloor)\ \forall n > m \geq 0$$

$$\textit{and } (i,j) \neq (\lfloor n\,\varphi \rfloor - \lfloor m\,\varphi^2 \rfloor, \lfloor n\,\varphi^2 \rfloor - \lfloor m\,\varphi \rfloor)\ \forall n > m \geq 0.$$

Theorem (E. Duchêne et al. 2010 [8])

$(i,j)_{i<j}$ may be adjoined iff there exist valid F-representations u, u' such that one the three properties is satisfied :

- $(\mathrm{rep}_F(i-1), \mathrm{rep}_F(j-1)) = (u0, u01)$
- $(\mathrm{rep}_F(i-2), \mathrm{rep}_F(j-2)) = (u0, u01)$
- $(\mathrm{rep}_F(j-\lfloor i\varphi \rfloor - 2), \mathrm{rep}_F(j-\lfloor i\varphi \rfloor - 2 + i)) = (u1, u'0);$

$$
\begin{array}{|cccccccccccc}
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \\
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0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \\
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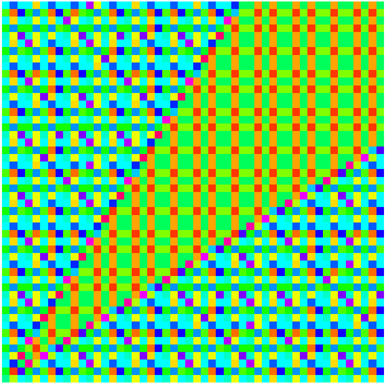
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\hline
\end{array}
$$

- There is no redundant move in Wythoff's game.

Question: Does the infinite 2-dimensional word on the previous slide have a shape-symmetric morphic structure?



We conjectured a morphism over 26 letters.

Given a word, e.g. Tribonacci word: $abacabaabacab\cdots$ [9]

I. Any positive number of tokens from up to two piles can be removed.

II. Let α, β, γ be three positive integers such that

$$2\max\{\alpha, \beta, \gamma\} \leq \alpha + \beta + \gamma.$$

Then one can remove α (resp. β, γ) from the first (resp. second, third) pile.

III. Let $\beta > 2\alpha > 0$. From position (a, b, c) one can remove the same number α of tokens from any two piles and β tokens from the unchosen one with the following condition. If a' (resp. b', c') denotes the number of tokens in the pile which contained a (resp. b, c) tokens before the move, then the configuration $a' < c' < b'$ is not allowed.

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Another direction leads to the concept of *invariant games* [13, 14].

A game $G : \mathbb{N}^n \to 2^{\mathbb{N}^n}$ (assigning each position to a set of available moves) is invariant if there exists a set $I \subseteq \mathbb{N}^n$ such that, for all positions $\mathbf{p}$, we have

$$G(\mathbf{p}) = I \cap \{\mathbf{m} \in \mathbb{N}^n \mid \mathbf{m} \le \mathbf{p}\}.$$

Otherwise stated, we may apply exactly the same moves to every position, with the only restriction that there are enough tokens left.

Example

The game of Nim is invariant:

$$I_{\text{NIM}} = \{(i,0) \mid i \ge 1\} \cup \{(0,j) \mid j \ge 1\}.$$

Wythoff's game is invariant:

$$I_{\text{WYTHOFF}} = I_{\text{NIM}} \cup \{(k,k) \mid k \ge 1\}.$$

For an example of non-invariant game, consider the following map,

$$G_{\text{EVEN}} : \mathbb{N}^2 \to 2^{\mathbb{N}^2},$$

$$(x, y) \mapsto \begin{cases} \{(i, 0) \mid i \in [\![1, x]\!]\}, & \text{if } x + y \text{ is even;} \\ \{(i, i) \mid i \in [\![1, \min\{x, y\}]\!]\}, & \text{otherwise.} \end{cases}$$

Theorem (E. Duchêne, A. Parreau, M.R.) [14]

Under some conditions, one can decide whether or not there exists an invariant game for a given morphic word.

Making use of Presburger arithmetic...

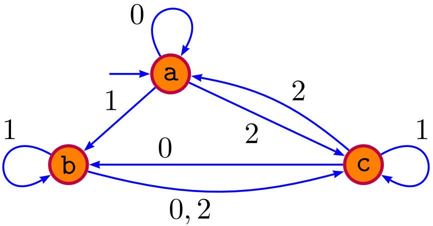
For instance, one can decide if there exists an invariant set of rules for the Tribonacci game.

Back to the link between morphisms and automata

From Lecture 1:

- Image under a coding of a fixed point of a k-uniform morphism;
- Sequence of outputs of a DFAO fed with base-k expansions;

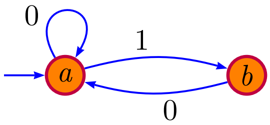
$$f : \begin{cases} \mathtt{a} & \mapsto & \mathtt{abc} \\ \mathtt{b} & \mapsto & \mathtt{cbc} \\ \mathtt{c} & \mapsto & \mathtt{bca} \end{cases}$$



$$f^{\omega}(\mathtt{a}) = \mathtt{abccbcbcabcacbcbcacbcbcaabc}\cdots$$

$$g(f^{\omega}(\mathtt{a})) = \mathtt{100000001001000001000001100}\cdots$$

From Lecture 2:

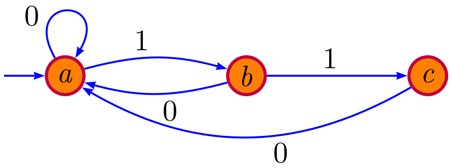


$$\varepsilon, 1, 10, 100, 101, 1000, 1001, 1010, 10000, 10001, 10010, 10100, \ldots$$

This is exactly the language of (greedy) Fibonacci representations $\mathrm{rep}_F(\mathbb{N})$. We have a "Fibonacci-automatic sequence".

$$f_n = a \cdot \mathrm{rep}_F(n).$$

An extra example: Tribonacci $a \mapsto ab$, $b \mapsto ac$, $c \mapsto a$



$$\varepsilon, 1, 10, 11, 100, 101, 110, 1000, 1001, 1010, 1011, 1100, 1101, \ldots$$

This is exactly the language of (greedy) Tribonacci representations $\mathrm{rep}_T(\mathbb{N})$. We have a "Tribonacci-automatic sequence".

$$t_n = a \cdot \mathrm{rep}_T(n).$$

There is a difference between the first example and the last two ones...

$$f : a \mapsto ab,\ b \mapsto c,\ c \mapsto cd,\ d \mapsto a$$

$$g : a, d \mapsto 1,\ b, c \mapsto 0$$

$$f^\omega(a) = abccdcdacdaabcdaab \cdots$$

$$g(f^\omega(a)) = 100010110111001110 \cdots$$

Can you relate this sequence to Fibonacci system?

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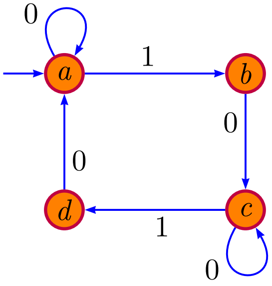

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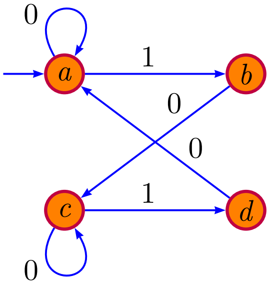

We see that the accepted language is again Fibonacci!

$$x_n = a \cdot \mathrm{rep}_F(n).$$

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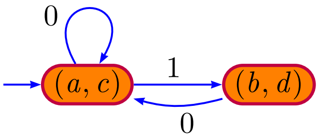

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General theorem "morphic $\Rightarrow$ automatic" [5]

Let A be an ordered alphabet. Let $\mathbf{w} \in A^{\mathbb{N}}$ be an infinite word, fixed point $f^\omega(a)$ of a morphism $f : A^* \to A^*$.

- associate with f a DFA $\mathcal{M}$ over the alphabet $\{0, \ldots, \max |f(b)| - 1\}$;
- A is the set of states;
- the initial state is a, all states are final;
- if $f(b) = c_0 \cdots c_m$, then $b \xrightarrow{j} c_j$, $j \leq m$;
- consider the language L accepted by $\mathcal{M}$ except words starting with 0;
- genealogically order L: $L = \{w_0 < w_1 < w_2 < \cdots\}$.

The nth symbol of $\mathbf{w}$, $n \geq 0$, is $\boxed{\mathcal{M} \cdot w_n}$.

As a summary, we have two ingredients to re-obtain a morphic/substitutive word:

- A regular language (accepted by $\mathcal{M}$);

  $$\text{base-}k\text{, Fibonacci, Tribonacci, } \ldots$$

  One can have the same underlying language for various morphisms.
- An automaton fed (in a prescribed order) with words from this language. The automaton is given by the morphism.

Link with ANS

Actually, we have already seen *Abstract Numeration Systems*...

Definition [15]

An abstract numeration system $\mathcal{S} = (L, A, <)$ is a regular language L over a totally ordered finite alphabet $(A, <)$.

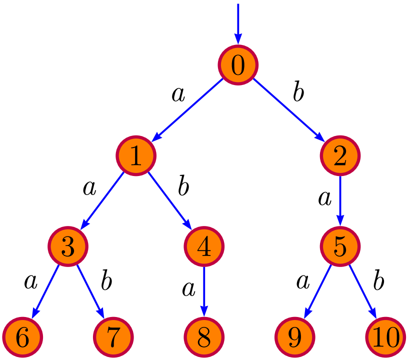
- Enumerating the words in L using genealogical ordering provides a one-to-one correspondence between $\mathbb{N}$ and L :

$$\mathrm{rep}_{\mathcal{S}} : \mathbb{N} \to L, \qquad \mathrm{val}_{\mathcal{S}} : L \to \mathbb{N}.$$

- This generalizes any positional system U for which $\mathrm{rep}_U(\mathbb{N})$ is regular.

P. Lecomte, M.R., Numeration systems on a regular language, Theory Comput. Syst. **34** (2001), 27–44.

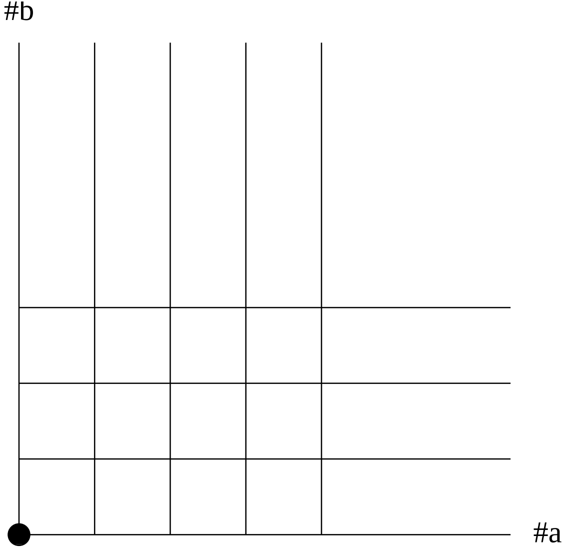
Example : consider a prefix-closed language $L = \{b, \varepsilon\}\{a, ab\}^*$



n	$\mathrm{rep}_{\mathcal{S}}(n)$
0	ε
1	a
2	b
3	aa
4	ab
5	ba
6	aaa
7	aab
8	aba
9	baa
10	bab

Abstract numeration systems

A non-positional ANS $L = a^* b^*$



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7	aab
8	abb
9	bbb
10	$aaaa$

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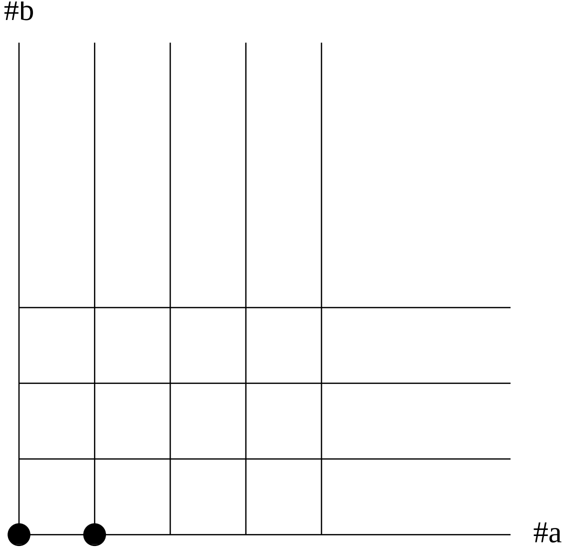

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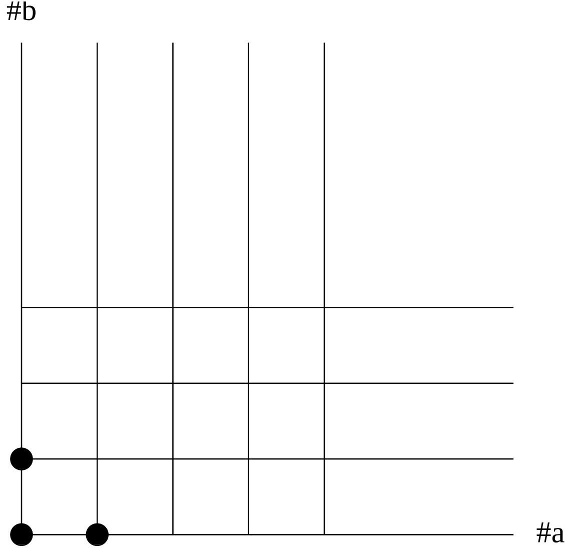

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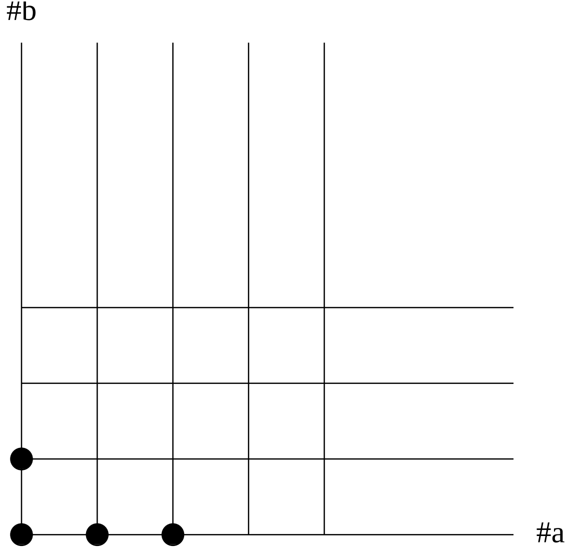

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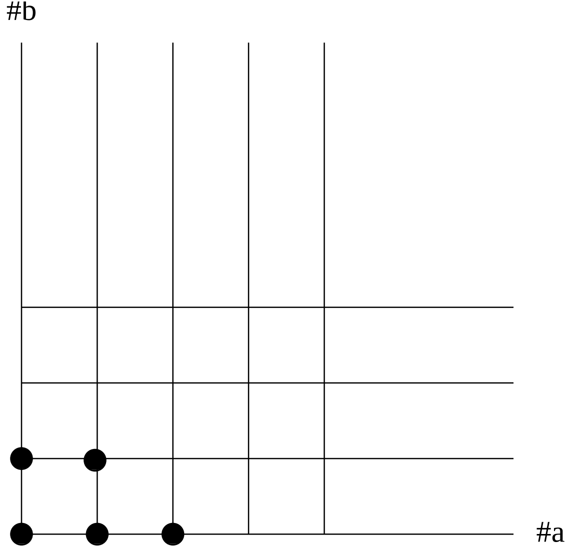

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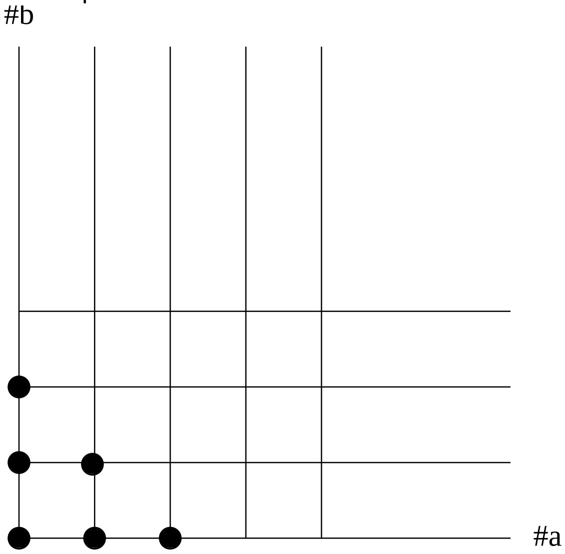

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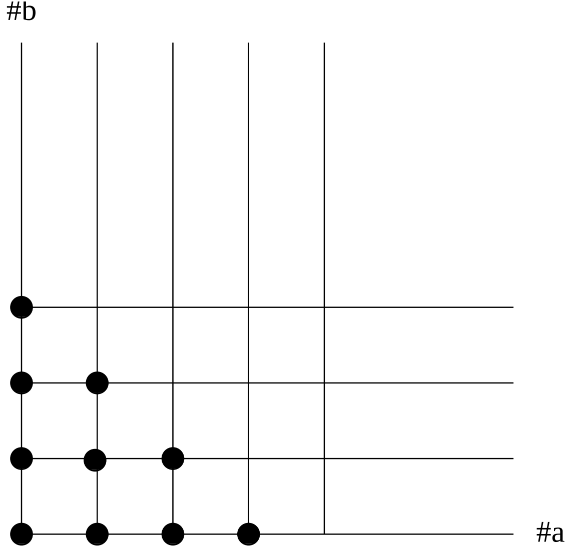

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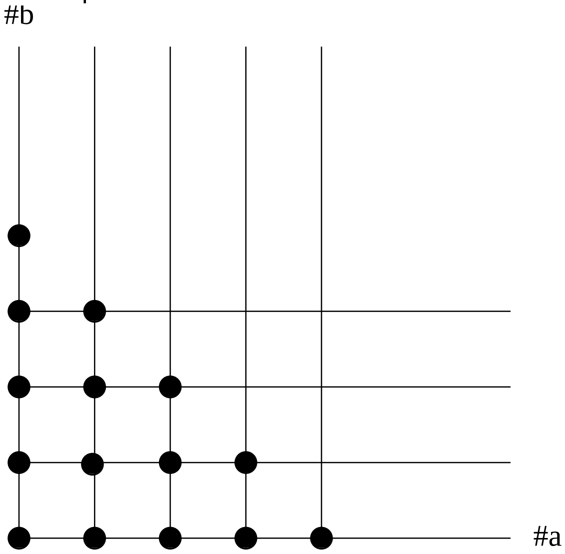

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6	aaa
7	aab
8	abb
9	bbb
10	$aaaa$

Abstract numeration systems

A non-positional ANS $L = a^*b^*$

$$\mathrm{val}_{\mathcal{S}}(a^p b^q) = \frac{1}{2}(p+q)(p+q+1) + q = \binom{p+q+1}{2} + \binom{q}{1}$$

ε	a	b	aa	ab	bb	aaa	$\cdots$
0	1	2	3	4	5	6	$\cdots$

$$U_0 = 1,\ U_1 = 2,\ w(a) = 1, w(b) = 2$$

Generalization : $\mathrm{val}_\ell(a_1^{n_1} \cdots a_\ell^{n_\ell}) = \sum_{i=1}^{\ell} \binom{n_i + \cdots + n_\ell + \ell - i}{\ell - i + 1}.$

$$\boxed{\forall n \in \mathbb{N}, \exists z_1, \ldots, z_\ell : n = \binom{z_\ell}{\ell} + \binom{z_{\ell-1}}{\ell-1} + \cdots + \binom{z_1}{1}}$$

with the condition $z_\ell > z_{\ell-1} > \cdots > z_1 \geq 0$

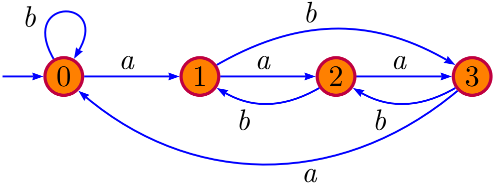
Katona, Gel'fand, Lehmer, Fraenkel, Lew, Morales, . . . [16, 17, 18]

There was already some form of abstract numeration system in Maes' Ph.D. thesis (1999) [12].

Rem. 6.9, p. 134, "*The set of codes of* $\mathbb{N}$ *given by the above automaton is of course a regular language ... The language read by* A *is* 0^*L. *However, the above coding is not a numeration system in the sense of [6]. Indeed, the representation of a natural number is not obtained using a 'Euclidian division' algorithm.*"

$\mathcal{S}$-AUTOMATIC SEQUENCES

Two ingredients: an ANS $S = (a^*b^*, a < b)$ and a DFAO $\mathcal{M}$ [4]



$$\boxed{x_n = \mathcal{M} \cdot \mathrm{rep}_S(n)}$$

$$\mathbf{x} = 01023031200231010123023031203120231002310123010123\cdots$$

This can be extended to a multidimensional setting.

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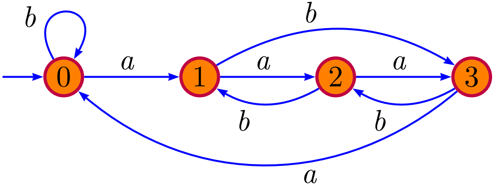
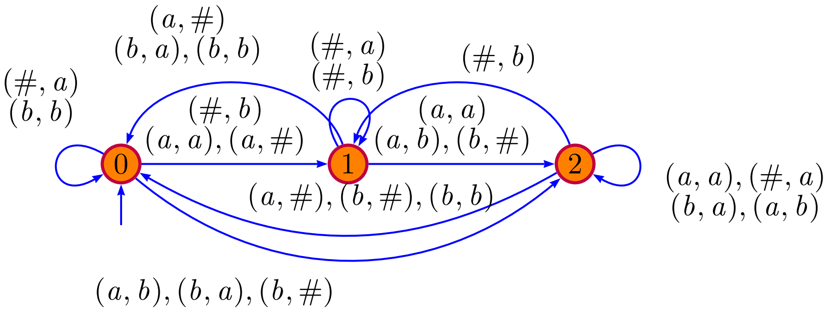


$$\boxed{x_n = \mathcal{M} \cdot \mathrm{rep}_S(n)}$$

$\mathbf{x} = 0102303120023101012302303120312023100231012301 0123\cdots$

This can be extended to a multidimensional setting.

$S = (a^*b^*, a < b)$ padding symbol: $\#$



$\vdots$

220200200020000
112222222222222
020200200020000
200000000000000
022222222222222
220200200020000
112122122212222
$001011011101111\cdots$

Theorem (A. Maes, M.R. [5])

An infinite word is morphic if and only if it is S-automatic for some abstract numeration system S.

$\Rightarrow$ Already proven.
$\Leftarrow$ We need to get rid off erasing morphisms.
Simulate the product of the two automata.

Theorem (É. Charlier, T. Kärki, M.R. [19])

Let $d \geq 1$. The d-dimensional infinite word x is S-automatic, for some abstract numeration system $S = (L, \Sigma, <)$ where $\varepsilon \in L$, if and only if x is the image by a coding of a shape-symmetric infinite d-dimensional word.

SUMMARY

k-automatic sequence $\Updownarrow$ k-uniform morphism $+$ coding [A. Cobham'72] [2]	$\mathcal{S}$-automatic sequence $\Updownarrow$ non-erasing morphism $+$ coding [A. Maes, M.R.'02] [4, 5]
multidimensional setup $x : \mathbb{N}^d \to A$ k-automatic sequence $\Updownarrow$ morphism $g : A \to (A^k)^d$ $+$ coding [O. Salon'87] [?]	$\mathcal{S}$-automatic sequence $\Updownarrow$ "shape-symmetric" morphism $+$ coding [É. Charlier, T. Kärki, M.R.'09] [19]

Some work in progress

It permits me a few words about Presburger arithmetic...

Games with a finite set of moves

Games like Nim or Wythoff have an infinite set of moves.

In one dimension [1]

Every (invariant) finite subtraction game on one pile, *i.e.*, $I \subset \mathbb{N}$ is finite, has an ultimately periodic Grundy function.

Proof:

Let $m = \#I$ (max. number of options), then $\mathcal{G}(n) \leq m$ for all n.

Let $k = \max I$, there are $(m+1)^k$ possible k-tuples taking values in $\{0, \ldots, m\}$. $\mathcal{G}(n)$ depends only on $\mathcal{G}(n-i)$ for $1 \leq i \leq k$.

Hence, there exist $i < j$

$$\mathcal{G}(i+n) = \mathcal{G}(j+n) \text{ for all } n \in \{0, \ldots, k-1\}.$$

Thus $j - i$ is a period of $\mathcal{G}$ with preperiod i. $\square$

Another similar result

[1, A. Siegel, p. 188]

Consider an (invariant) finite subtraction game on one pile, with $I \subset \mathbb{N}$ as set of moves. If there exist $N \geq 0$ and $p \geq 1$ such that

$$\mathcal{G}(n+p) = \mathcal{G}(n), \quad \forall N \leq n < N + \max I$$

then $\mathcal{G}(n+p) = \mathcal{G}(n)$ for all $n \geq N$.

If we may optionally split a pile...

Definition from Wikipedia:

An octal game is played with tokens divided into heaps.
Two players take turns moving until no moves are possible.

Every move consists of selecting just one of the heaps, and either

- removing all of the tokens in the heap, leaving no heap,
- removing some but not all of the tokens, leaving one smaller heap, or
- removing some of the tokens and dividing the remaining tokens into two nonempty heaps.

Heaps other than the selected heap remain unchanged.
The last player to move wins in normal play.

Coding of an octal game (Conway code)

$$d_0 \bullet d_1\, d_2\, d_3 \cdots \qquad d_i \in \{0, \ldots, 7\}$$

d_i written in base 2: $e_2^{(i)}\, e_1^{(i)}\, e_0^{(i)}$ gives the conditions under which i token may be removed.

- $e_0^{(i)} = 1$, then a (full) heap with i token can be suppressed
- $e_1^{(i)} = 1$, then a heap with $n > i$ token can be replaced with a heap with $n - i$ token left
- $e_2^{(i)} = 1$, then a heap with n token can be replaced with two heaps containing respectively a and b token, $a, b \geq 1$, $a + b = n - i$.

Example

The game of NIM is coded by $0 \bullet 3333\cdots$, $\mathrm{rep}_8(3) = 011$.

A finite subtraction game $I = \{3, 5, 6\}$ is of the form $0 \bullet 003033$.

Theorem (Octal game periodicity [1])

Consider a finite octal game $d_0 \bullet d_1 d_2 \cdots d_k$, $d_k \neq 0$. If there exist $N \geq 0$ and $p \geq 1$ such that

$$\mathcal{G}(n+p) = \mathcal{G}(n), \quad \forall N \leq n < 2N + p + k$$

then $\mathcal{G}(n+p) = \mathcal{G}(n)$ for all $n \geq N$.

Are all finite octal games ultimately periodic? [20, R. Guy]

$0 \bullet 07$ has period 34 and preperiod 53 [Guy, Smith 1956]

$0 \bullet 007$ no known periodicity...

$0 \bullet 165$ has period 1550 and preperiod 5181 [21, Austin, 1976]

$0 \bullet 106$ has period $\simeq 3.10^{11}$ and preperiod $\simeq 4.10^{11}$ [Flammenkamp, 2002]

http://wwwhomes.uni-bielefeld.de/achim/octal.html

With more than one pile (let's say two piles).

Natural question

What is the structure of the $\mathcal{G}$-values for a finite subtraction game over k piles? Do we get a Presburger definable set, i.e., each value determines a semi-linear set?

Cobham–Semenov' theorem: Let $p, q \geq 2$ be multiplicatively independent integers. If $X \subset \mathbb{N}^k$ is both p-recognizable and q-recognizable, then it is definable by a first-order formula in the Presburger arithmetic $\langle \mathbb{N}, + \rangle$ [3].

Work in progress: X. Badin De Montjoye, V. Gledel, V. Marsault, A. Massuir, M.R.

A FEW WORDS ABOUT AN EXTENSION OF $\langle \mathbb{N}, + \rangle$

https://www.youtube.com/watch?v=U9t10GAsn1k

$V_k(n)$ is the largest power of k dividing $n > 0$.

Reformulation by Charlier, Rampersad, Shallit

BÜCHI–BRUYÈRE THEOREM

Let $k \geq 2$.

If one can express a property of a k-automatic sequence $\mathbf{x}$ using:

(first-order) quantifiers, logical operations, integer variables, addition, subtraction, indexing into $\mathbf{x}$ and comparison of integers or elements of $\mathbf{x}$,

then this property is decidable.

Honkala (1986) vs. $(\exists p)(\exists N)(\forall i \geq N)\, \mathbf{x}(i) = \mathbf{x}(i+p)$.

A few words about an extension of $\langle \mathbb{N}, + \rangle$

https://www.youtube.com/watch?v=U9t10GAsn1k

$V_k(n)$ is the largest power of k dividing $n > 0$.

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Büchi–Bruyère Theorem

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A. Thue (1906)

The Thue–Morse word $\mathbf{t}$ is overlap-free.

$$\neg(\exists i)(\exists \ell \geq 1)[(\forall j < \ell)(\mathbf{t}(i+j) = \mathbf{t}(i+\ell+j)) \wedge \mathbf{t}(i) = \mathbf{t}(i+2\ell)]$$

Quite a few properties that can be checked for k-automatic sequences:

- (arbitrarily large) unbordered factors
- reccurrent word
- linearly recurrent word
- $\mathrm{Fac}(\mathbf{x}) \subset \mathrm{Fac}(\mathbf{y})$
- $\mathrm{Fac}(\mathbf{x}) = \mathrm{Fac}(\mathbf{y})$
- existence of an unbordered factor of length n

  $\vdots$

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$\vdots$

Computer Science > Formal Languages and Automata Theory

Automatic Theorem Proving in Walnut

Hamoon Mousavi

(Submitted on 18 Mar 2016)

Walnut is a software package that implements a mechanical decision procedure for deciding certain combinatorial properties of some special words referred to as automatic words or automatic sequences. Walnut is written in Java and is open source. It is licensed under GNU General Public License.

Subjects: **Formal Languages and Automata Theory (cs.FL)**; Logic in Computer Science (cs.LO); Mathematical Software (cs.MS); Combinatorics (math.CO)

Cite as: **arXiv:1603.06017 [cs.FL]**

(or **arXiv:1603.06017v1 [cs.FL]** for this version)

Submission history

From: Seyyed Hamoon Mousavi Haji [view email]

[v1] Fri, 18 Mar 2016 23:53:10 GMT (684kb,D)

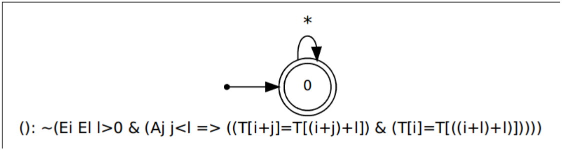
Automatic proof that the Thue–Morse word is overlap-free

$$\neg(\exists i)(\exists \ell \geq 1)[(\forall j < \ell)(\mathbf{t}(i+j) = \mathbf{t}(i+\ell+j)) \wedge \mathbf{t}(i) = \mathbf{t}(i+2\ell)]$$

Up to 97 states in an intermidiate step

```
rigo@X1:~/Walnut/Walnut/Walnut/bin$ java Main.prover
eval test "~(Ei El l>0 & (Aj j<l => ((T[i+j]=T[(i+j)+l]) & (T[i]=T[((i+l)+l)]))))":
l>0 has 2 states: 14ms
 j<l has 2 states: 0ms
  T[(i+j)]=T[((i+j)+l)] has 12 states: 136ms
   T[i]=T[((i+l)+l)] has 6 states: 39ms
    (T[(i+j)]=T[((i+j)+l)]&T[i]=T[((i+l)+l)]) has 72 states: 41ms
     (j<l=>(T[(i+j)]=T[((i+j)+l)]&T[i]=T[((i+l)+l)])) has 97 states: 62ms
      (A j (j<l=>(T[(i+j)]=T[((i+j)+l)]&T[i]=T[((i+l)+l)]))) has 1 states: 186ms
       (l>0&(A j (j<l=>(T[(i+j)]=T[((i+j)+l)]&T[i]=T[((i+l)+l)])))) has 1 states: 1ms
        (E l (l>0&(A j (j<l=>(T[(i+j)]=T[((i+j)+l)]&T[i]=T[((i+l)+l)]))))) has 1 states: 0ms
         (E i (E l (l>0&(A j (j<l=>(T[(i+j)]=T[((i+j)+l)]&T[i]=T[((i+l)+l)])))))) has 1 states: 1ms
          ~(E i (E l (l>0&(A j (j<l=>(T[(i+j)]=T[((i+j)+l)]&T[i]=T[((i+l)+l)])))))) has 1 states: 0ms
total computation time: 506ms
```

GraphViz / `xdot ../Result/test.gv`



Frobenius' problem

Chicken McNuggets can be purchased only in 6, 9, or 20 pieces. *The largest number of nuggets that cannot be purchased is* 43.

$$(\forall n)(n > 43 \to (\exists x, y, z \geq 0)(n = 6x + 9y + 20z))$$
$$\wedge \neg((\exists x, y, z \geq 0)(43 = 6x + 9y + 20z))\,.$$

What about Pisot numbers?

$$\langle \mathbb{N}, +, V_U \rangle$$

We need addition to be computable by an automaton...
Frougny's paper from 1992: addition is normalization.

For instance, you can answer automatically many questions about Fibonacci word, Tribonacci word, ...

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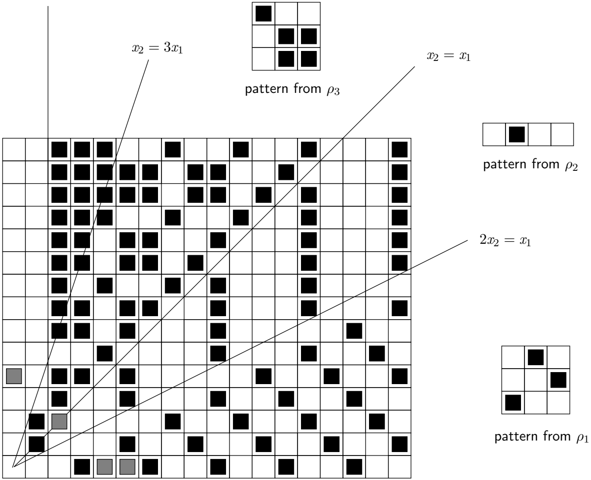
For instance, you can answer automatically many questions about Fibonacci word, Tribonacci word, ...

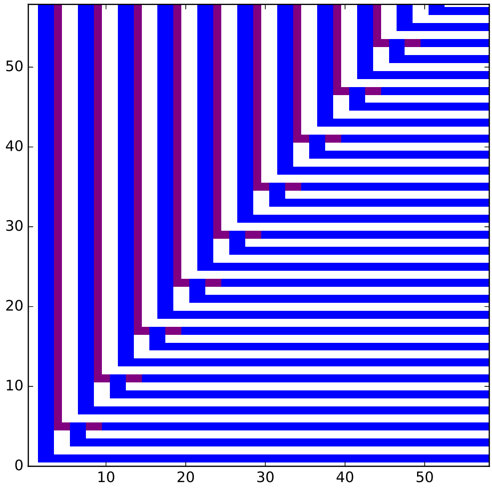
What does a semi-linear sets look like?

The subsets of $\mathbb{N}$ definable in $\langle \mathbb{N}, + \rangle$ are exactly the ultimately periodic sets.



What about Grundy values for games with a finite set of rules?

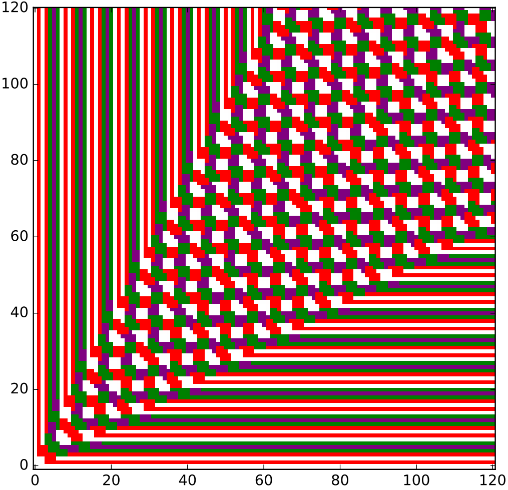
If we let $I = \{(2, 1),\ (3, 5)\}$, we get



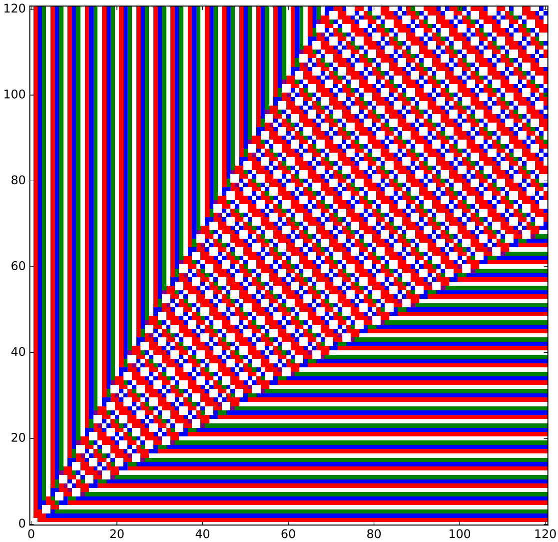
Proposition

If $\#I = 2$, then the set of $\mathcal{G}$-values is Presburger definable.

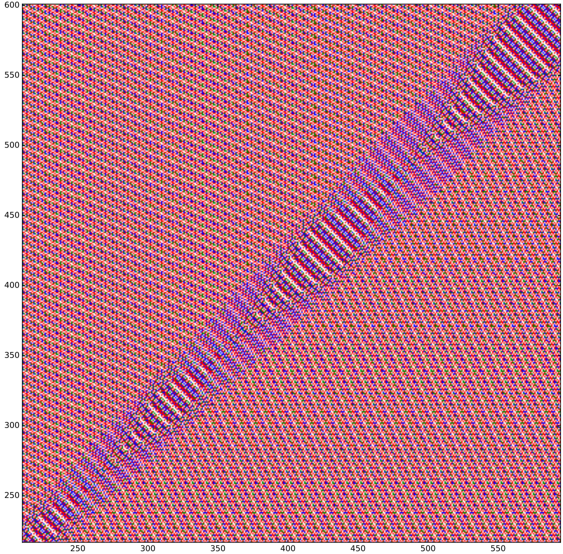
If we let $I = \{(1,3),\ (3,1),\ (4,4)\}$, we get



If we let $I = \{(1,2),\ (2,1),\ (3,5),\ (5,3)\}$, we get

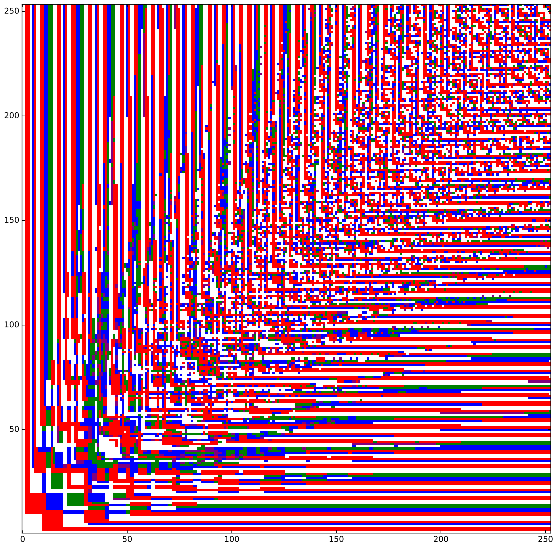


If we let $I = \{(1,2),\ (2,1),\ (3,5),\ (5,3),\ (2,2)\}$, we get

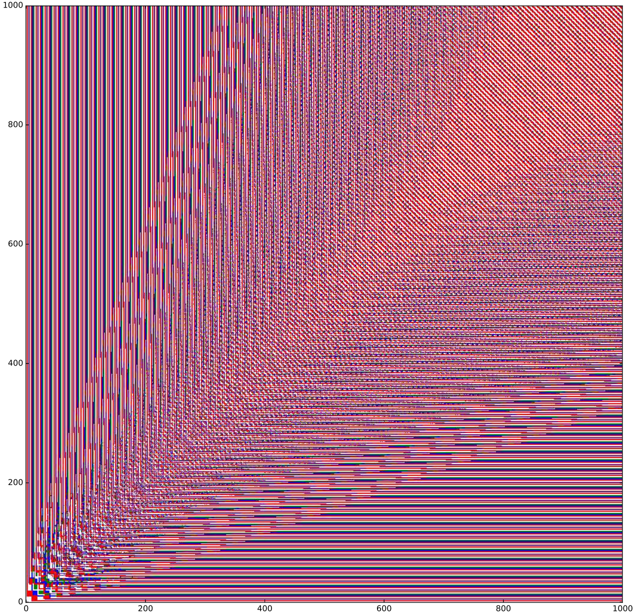


We think that this one is NOT Presburger definable.

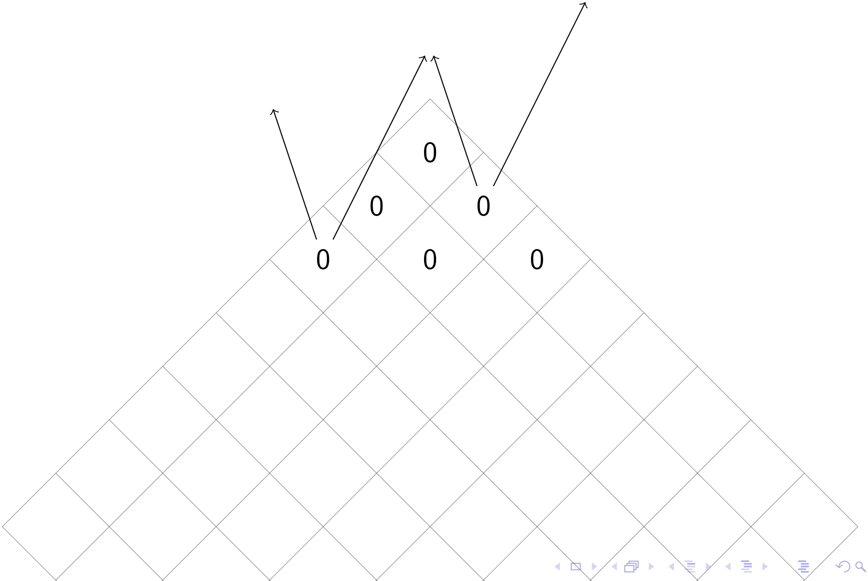
If we let $I = \{(10,2),\ (2,10),\ (32,5),\ (5,32),\ (10,10)\}$, we get



If we let $I = \{(10,2),\ (2,10),\ (32,5),\ (5,32),\ (10,10)\}$, we get



Cellular automata — kind of space-time diagram with bounded memory, the rules are $(1, 2)$ and $(3, 1)$



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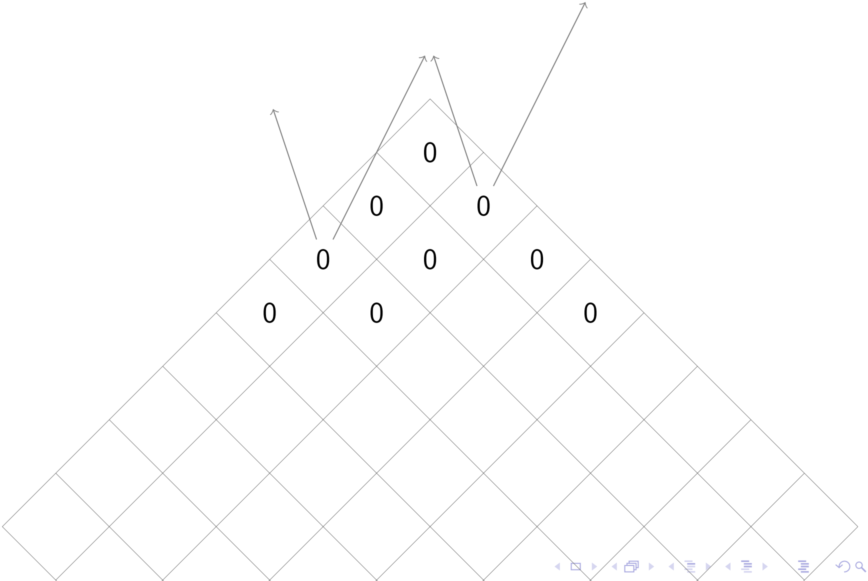

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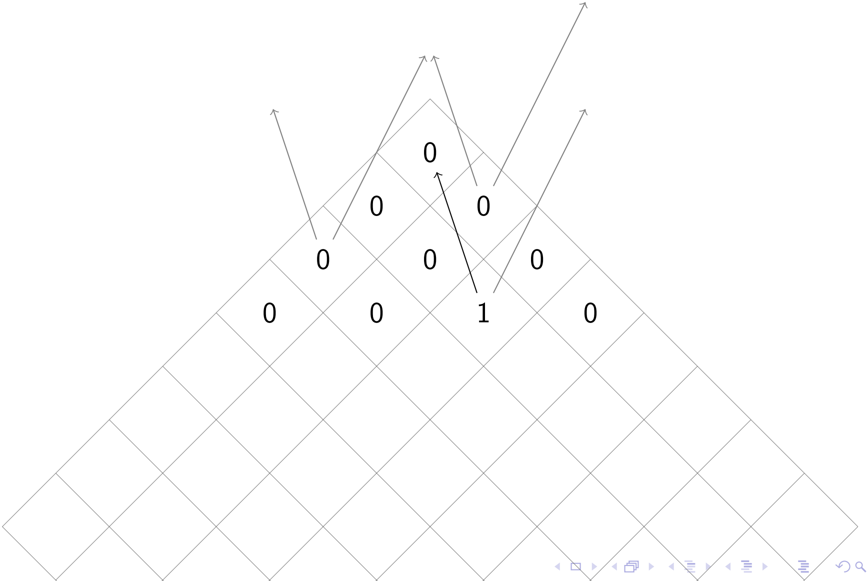

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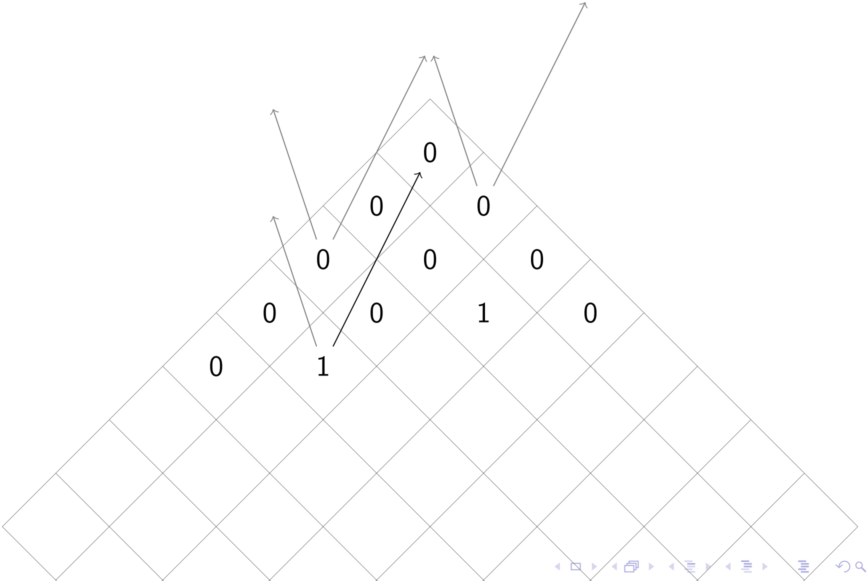

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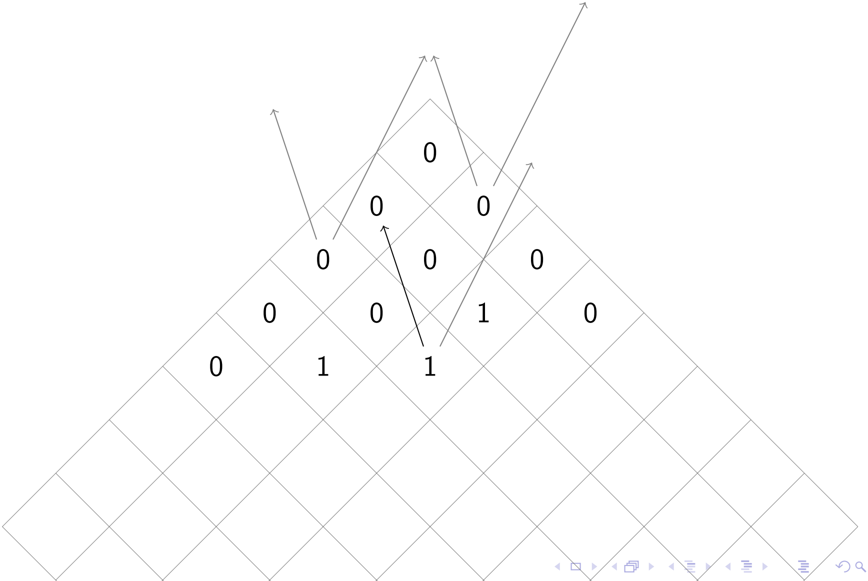

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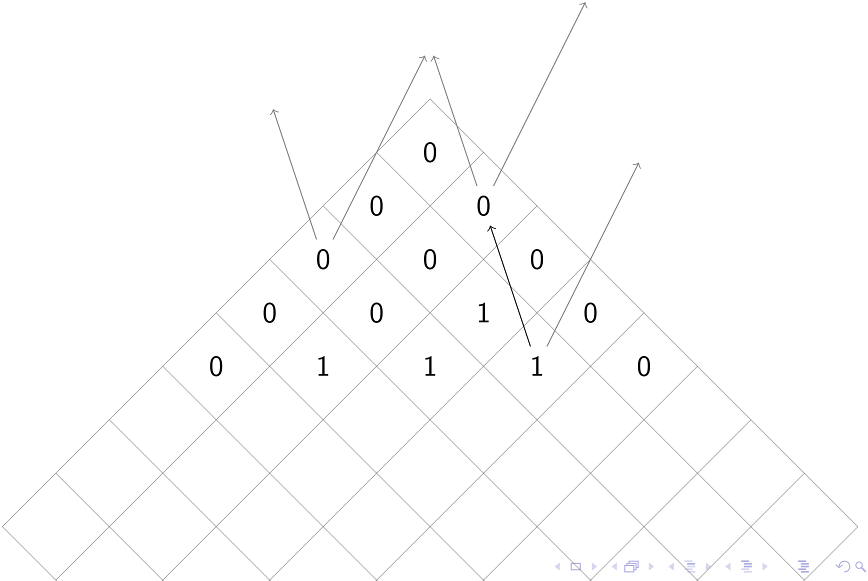

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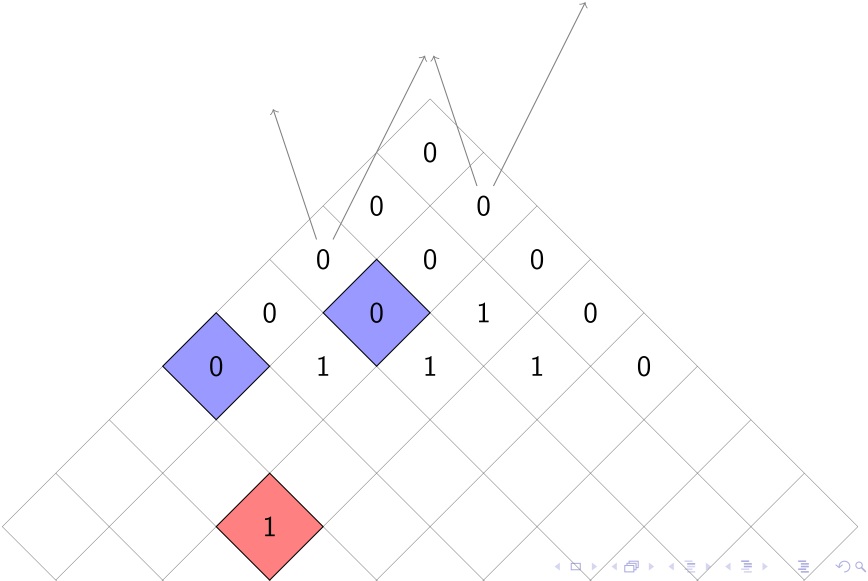

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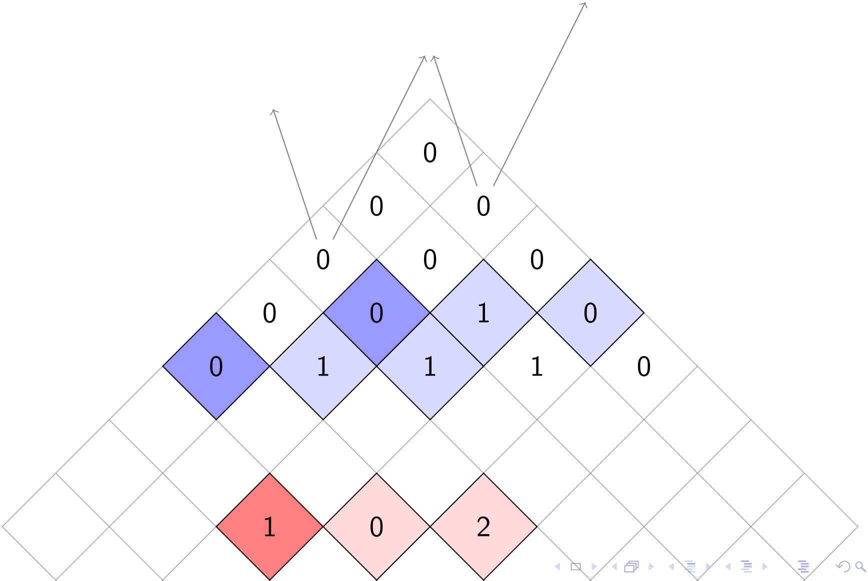

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