Linearization and quadratization techniques for multilinear 0–1 optimization problems

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Multilinear 0–1 optimization

\[
\min \sum_{e \in E} a_e \prod_{i \in e} x_i + \sum_{i \in V} c_i x_i \\
\text{s. t. } x_i \in \{0, 1\} \quad i \in V
\]

- \( V = \{1, \ldots, n\} \), \( E = \) set of subsets \( e \) of \( V \) with \( |e| \geq 2 \) and \( a_e \neq 0 \),
- \( V \) and \( E \) define a hypergraph \( H \).

Example:

\[
f(x_1, x_2, x_3) = 9x_1x_2x_3 + 8x_1x_2 - 6x_2x_3 + x_1 - 2x_2 + x_3
\]
Applications: Computer Vision

Image restoration problems modelled as energy minimization

\[ E(l) = \sum_{p \in \mathcal{P}} D_p(l_p) + \sum_{S \subseteq \mathcal{P}, |S| \geq 2} \sum_{p_1, \ldots, p_s \in S} V_{p_1, \ldots, p_s}(l_{p_1}, \ldots, l_{p_s}), \]

where \( l_p \in \{0, 1\} \quad \forall p \in \mathcal{P}. \)

(Image from "Corel database" with additive Gaussian noise.)
Applications

- Constraint Satisfaction Problem
- Data mining, classification, learning theory...
- Joint supply chain design and inventory management
- Production management
- ...

Problem definition  | Linearizations  | Quadratizations  | Current work and perspectives
General idea

Problem definition
Linearizations
Quadratizations
Current work and perspectives

Higher-degree 0–1 problem

Equivalent linear 0–1 problem
IP resolution techniques

Equivalent quadratic 0–1 problem
QP resolution techniques
Standard Linearization (SL)

$$\min \sum_{e \in E} a_e \prod_{i \in e} x_i + \sum_{i \in V} c_i x_i$$

Standard Linearization (Fortet (1959), Glover and Woolsey (1973))

$$y_e = \prod_{i \in e} x_i$$

$$-y_e + x_i \geq 0$$ \quad \forall i \in e, \forall e \in E \quad (1)$$

$$y_e - \sum_{i \in e} x_i \geq 1 - |e|$$ \quad \forall e \in E \quad (2)$$
SL main drawback and contributions

SL drawback: The continuous relaxation given by the SL is very weak!

Contributions:

- Characterization of cases for which SL provides a perfect formulation (Buchheim, Crama, Rodríguez-Heck (2017), discovered independently by Del Pia, Khajavirad (2017)).

- Definition of a class of valid inequalities strengthening the SL formulation, working especially well for simplified computer vision instances (Crama, Rodríguez-Heck (2017)).
General idea

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Higher-degree 0-1 problem

Equivalent linear 0-1 problem
| IP resolution techniques |
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Equivalent quadratic 0-1 problem
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Definition: Quadratization

Given a multilinear polynomial $f(x)$ on $\{0, 1\}^n$, we say that $g(x, y)$ is a **quadratization** of $f$ if $g(x, y)$ is a quadratic polynomial depending on $x$ and on $m$ auxiliary variables $y_1, \ldots, y_m$, such that

$$f(x) = \min \{ g(x, y) : y \in \{0, 1\}^m \} \quad \forall x \in \{0, 1\}^n.$$  

Then,

$$\min \{ f(x) : x \in \{0, 1\}^n \} = \min \{ g(x, y) : x \in \{0, 1\}^n, y \in \{0, 1\}^m \}.$$  

Which quadratizations are “good”?

- Small number of auxiliary variables.
- Good optimization properties: submodularity (intuitive measure: small number of positive quadratic terms).
Termwise quadratizations

Multilinear expression of a pseudo-Boolean function:

\[ f(x) = -35x_1 x_2 x_3 x_4 x_5 + 50x_1 x_2 x_3 x_4 - 10x_1 x_2 x_4 x_5 + 5x_2 x_3 x_4 + 5x_4 x_5 - 20x_1 \]

Idea: quadratize monomial by monomial, using different sets of auxiliary variables for each monomial.

- Negative case well solved (one auxiliary variable, submodular quadratization).
- Positive monomials much more difficult: just improved the best bound for number of variables!

\[- \prod_{i=1}^{n} x_i = \min_{y \in \{0, 1\}} -y \left( \sum_{i=1}^{n} x_i - (n - 1) \right).\]

Why is this a quadratization? \( f(x) = -x_1 x_2 x_3 x_4 \)

- If \( x_i = 1 \) for all \( i \), then \( \min_{y \in \{0, 1\}} -y \), reached for \( y = 1 \), value \(-1\).
- If there is an \( i \) with \( x_i = 0 \), then \( y \) has a nonnegative coefficient, minimum reached for \( y = 0 \).
Positive monomial: Literature

Ishikawa (2011)

\[
\prod_{i=1}^{n} x_i = \min_{y_1,\ldots,y_k \in \{0,1\}} \sum_{i=1}^{k} y_i (c_{i,n}(-S_1 + 2i) - 1) + aS_2,
\]

\(S_1, S_2: \) elementary linear and quadratic symmetric polynomials in \(n\) variables,

\(k = \lfloor \frac{n-1}{2} \rfloor \) and \(c_{i,n} = \begin{cases} 
1, & \text{if } n \text{ is odd and } i = k, \\
2, & \text{otherwise.}
\end{cases}\)

- **Number of variables:** best published bound for positive monomials.
- **Submodularity:** \(\binom{n}{2}\) positive quadratic terms, but very good computational results.
1st improvement: $\lceil \frac{n}{4} \rceil$ variables

Theorem 1 (E. Boros, Y. Crama, E. R-H)

For all integers $n, m$, if $n \geq 2$, $\frac{n}{4} \leq m \leq \frac{n}{2}$, and $N = n - 2m$ then

$$g(x, y) = \frac{1}{2} (X - Ny_1 - 2Y) (X - Ny_1 - 2Y - 1)$$

is a quadratization of the positive monomial $P_n = \prod_{i=1}^{n} x_i$ using $m$ auxiliary variables, where $X = \sum_{i=1}^{n} x_i$ and $Y = \sum_{j=2}^{m} y_j$. 
2nd improvement: \([\lfloor \log(n) \rfloor - 1\) variables

Theorem 2 (E. Boros, Y. Crama, E. R-H)

Let \(n \leq 2^{k+1}\), \(K = 2^{k+1} - n\) and \(X = \sum_{i=1}^{n} x_i\). Then,

\[
g(x, y) = \frac{1}{2}(K + X - \sum_{i=1}^{k} 2^i y_i)(K + X - \sum_{i=1}^{k} 2^i y_i - 1)
\]

is a quadratization of the positive monomial \(f(x) = P_n(x) = \prod_{i=1}^{n} x_i\) using \(k\) auxiliary variables.

Proof idea:

- \(g(x, y) \geq 0\) (half-product of consecutive integers).
- If \(X \leq n - 1\): \(K + X\) even: make 1st factor zero, \(K + X\) odd: make 2nd factor zero.
- If \(X = n\): 1st factor is at least 2, 2nd factor is at least 1.
Positive monomial: new quadratizations

Smallest number of variables known until now:
- \( \lceil \log(n) \rceil - 1 \) variables

Two other quadratizations, more variables but maybe better optimization properties (?)
- \( \lceil \log(n-1) \rceil \) variables.
- \( \lceil \frac{n}{4} \rceil \) variables.

→ Quadratizations being tested by a group at Cornell University.
Current work: computational

Instance sets:

- random polynomials,
- computer vision inspired polynomials,
- supply chain & inventory management.

Methods to compare:

- Standard linearization
- Termwise quadratizations

<table>
<thead>
<tr>
<th>Pos. Mon. ((P_n))</th>
<th>Neg. Mon. ((N_n))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ishikawa (\left\lfloor \frac{n}{4} \right\rfloor)</td>
<td>1-var. quadrat.</td>
</tr>
<tr>
<td>(\left\lfloor \log(n - 1) \right\rfloor)</td>
<td>1-var. quadrat.</td>
</tr>
<tr>
<td>(\left\lfloor \log(n) \right\rfloor - 1)</td>
<td>1-var. quadrat.</td>
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</tbody>
</table>
Current work: theoretical

Open questions:

Conjecture 1
We need at least $m = \lceil \log(n) \rceil - 1$ variables to quadratize the positive monomial.

Conjecture 2
There is a trade-off between having small number of variables and good optimization properties, more precisely, the “most submodular” quadratizations of the positive monomial have $n - 1$ positive quadratic terms and use $m = n - 2$ variables.
Some references

Y. Crama and E. Rodríguez Heck. A class of new valid inequalities for multilinear 0–1 optimization problems. *Discrete Optimization*. Published online, 2017.


Some references II


