A class of valid inequalities for multilinear 0–1 optimization problems

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Multilinear 0-1 optimization

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$$\min \sum_{S \in \mathcal{S}} a_S \prod_{i \in S} x_i + l(x)$$
$$\text{s. t. } x_i \in \{0,1\} \qquad i = 1, \ldots, n$$

- $\mathcal{S}$: subsets of $\{1, \ldots, n\}$ with $a_S \neq 0$ and $|S| \geq 2$,
- $l(x)$ linear part.

Standard Linearization (SL)

Standard Linearization

$$\begin{aligned} \min \sum_{S \in \mathcal{S}} a_S y_S + l(x) & \\ \text{s. t. } y_S \leq x_i & \qquad \forall i \in S, \forall S \in \mathcal{S} \\ y_S \geq \sum_{i \in S} x_i - (|S| - 1) & \qquad \forall S \in \mathcal{S} \end{aligned}$$

$(y_S = \prod_{i \in S} x_i)$

- for variables $x_i, y_S \in \{0, 1\}$, the convex hull of feasible solutions is P^*_{SL},
- for continous variables $x_i, y_S \in [0, 1]$, the set of feasible solutions is P_{SL}.

SL drawback: The continuous relaxation given by the SL is very weak!

The 2-link inequalities

Definition

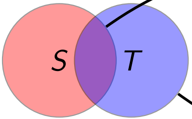
For $S, T \in \mathcal{S}$ and y_S, y_T such that $y_S = \prod_{i \in S} x_i$, $y_T = \prod_{i \in T} x_i$,

- the **2-link** associated with (S, T) is the linear inequality

$$\boxed{\mathbf{y_S \leq y_T - \sum_{i \in T \backslash S} x_i + |T \backslash S|}}$$

- P_{SL}^{2links} is the polytope defined by the SL inequalities and the 2-links.

Interpretation



$$y_S = 1 \Rightarrow \forall i \in S, x_i = 1$$

$$y_T = 0 \text{ and } y_S = 1 \Rightarrow \exists j \in T \backslash S, x_j = 0$$

Theoretical contributions

Theorem 1: A complete description for the case of two monomials

For the case of two nonlinear monomials, $P^*_{SL} = P^{2links}_{SL}$, i.e., the **standard linearization** and the **2-links** provide a **complete description** of P^*_{SL}.

Proof idea (Theorem 1):

- Consider the extended formulation (with variables in $[0, 1]$)

$$
\begin{aligned}
& y_{S\cap T} \leq x_i, && \forall i \in S \cap T, && (1)\\
& y_{S\cap T} \geq \sum_{i\in S\cap T} x_i - (|S \cap T| - 1), && && (2)\\
& y_S \leq y_{S\cap T}, && && (3)\\
& y_S \leq x_i, && \forall i \in S\setminus T, && (4)\\
& y_S \geq \sum_{i\in S\setminus T} x_i + y_{S\cap T} - |S\setminus T|, && && (5)\\
& y_T \leq y_{S\cap T}, && && (6)\\
& y_T \leq x_i, && \forall i \in T\setminus S, && (7)\\
& y_T \geq \sum_{i\in T\setminus S} x_i + y_{S\cap T} - |T\setminus S|, && && (8)
\end{aligned}
$$

- Notice that the two polytopes P^0 and P^1 obtained by fixing variable $y_{S\cap T}$ to 0 and 1, resp., are integral.
- Compute $conv(P^0 \cup P^1)$ using Balas (1974) and see that it is P^{2links}_{SL}.

Theoretical contributions

Theorem 2: Facet-defining inequalities for the case of two monomials

For the case of two nonlinear monomials defined by S, T with $|S \cap T| \geq 2$, the 2-links are **facet-defining** for P^*_{SL}.

Proof idea (Theorem 2): Since P^*_{SL} is full-dimensional (dim $n+2$), find $n+1$ affinely independent points in the faces defined by the 2-links.

Computational experiments: are the 2-links helpful for the general case?

Objectives

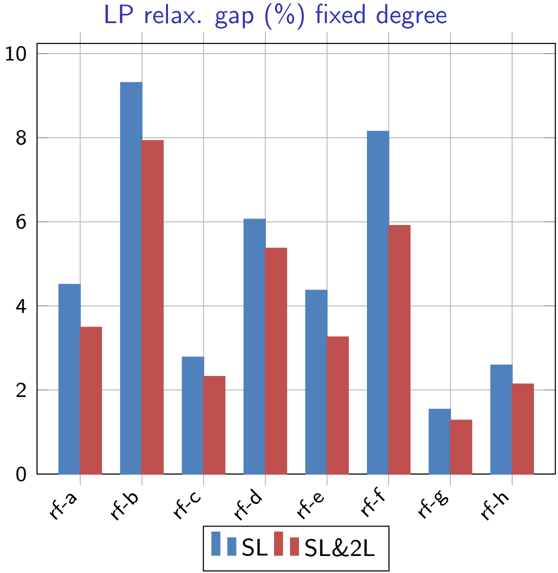
- compare the **bounds** obtained when optimizing over P_{SL} and P_{SL}^{2links},
- compare the **computational performance** of **exact resolution methods**.

Software used: CPLEX 12.06.

Inequalities used

- SL: standard linearization (model),
- cplex: CPLEX automatic cuts,
- 2L: 2-links.

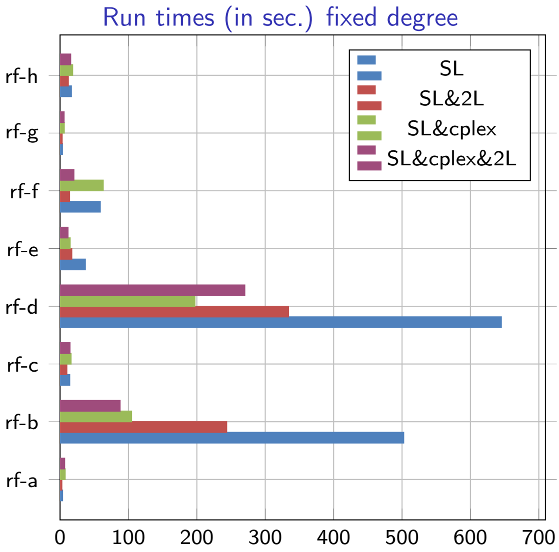
Random instances: bound improvement



Fixed degree:

inst.	d	n	m
rf-a	3	400	800
rf-b	3	400	900
rf-c	3	600	1100
rf-d	3	600	1200
rf-e	4	400	550
rf-f	4	400	600
rf-g	4	600	750
rf-h	4	600	800

Random instances: computation times results



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rf-f	4	400	600
rf-g	4	600	750
rf-h	4	600	800

Instances inspired from image restoration: definition

Image restoration

$$\begin{matrix} \color{red}{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & \color{red}{0} & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & \color{red}{1} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{matrix} \longrightarrow \begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}$$

Base images:

- top left rect. (tl),
- centre rect. (cr),
- cross (cx).

Perturbations:

- none (n),
- low (l),
- high (h).

$\boxed{\text{Up to } n = 225 \text{ variables and } m = 1598 \text{ terms}}$

Image restoration instances: bounds results

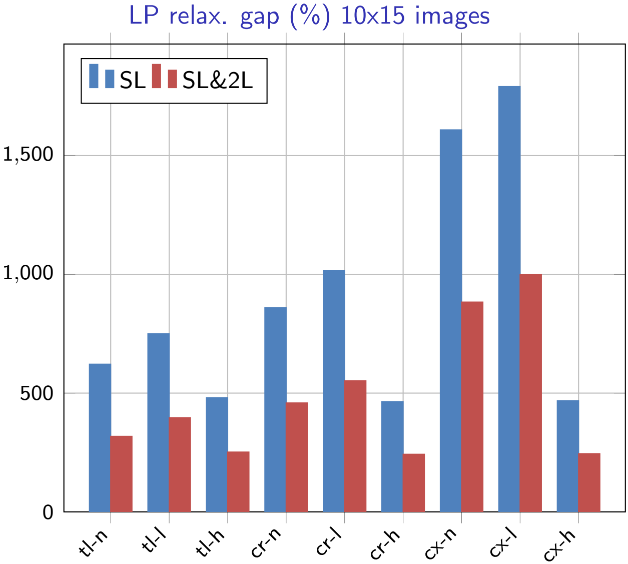


Image restoration instances: bounds results

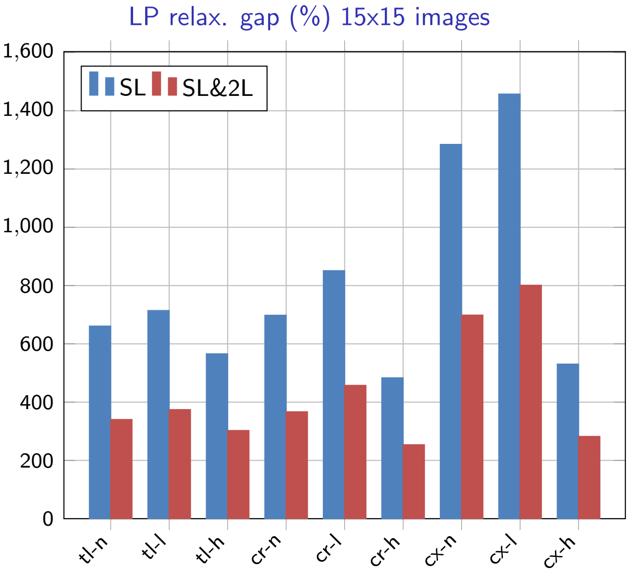


Image restoration instances: computation times results

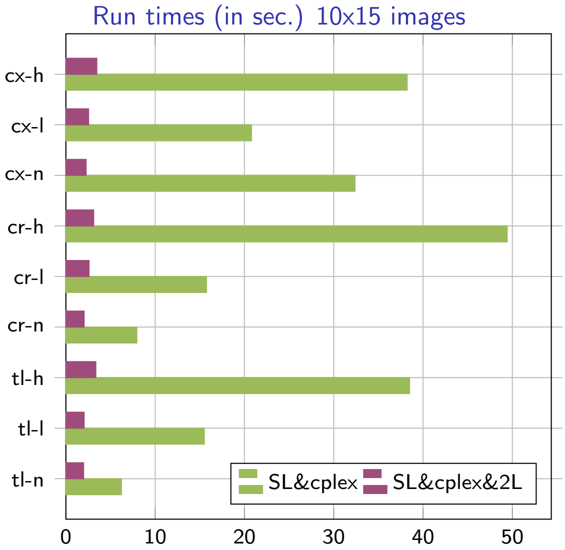
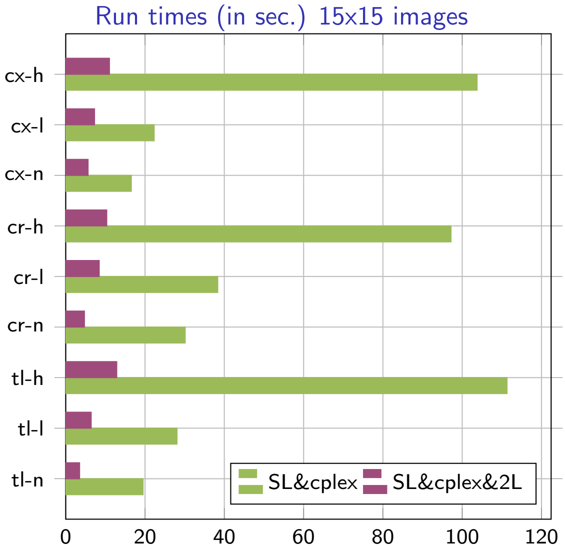


Image restoration instances: computation times results



Perspectives

Summary

- SL + 2-links = a complete description (two nonlinear monomials).
- 2-links help computationally for the general case.

Extensions and perspectives

- Characterization of when P_{SL} has integer vertices (joint work with C. Buchheim, discovered independently by A. del Pia and A. Khajavirad).
- "3-link", "4-link" inequalities... worth?
- Other reduction methods, to the quadratic case.

Some references I

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When is SL a complete description?

Summary

- SL + 2-links = a complete description (two nonlinear monomials).
- 2-links help computationally for the general case.

Question:
Can we characterize when the SL alone is a complete description of the convex hull P^*_{SL}?

Joint work with C. Buchheim.
Characterization independently discovered by A. Del Pia and A. Khajavirad.

SL complete description

Multilinear 0-1 optimization

$$\min \sum_{S \in \mathcal{S}} a_S \prod_{i \in S} x_i + l(x)$$
$$\text{s. t. } x_i \in \{0,1\} \qquad i = 1, \dots, n$$

Standard linearization constraints

$$\min \sum_{S \in \mathcal{S}} a_S y_S + l(x)$$
$$\text{s. t. } y_S \le x_i \qquad \forall i \in S, \forall S \in$$
$$y_S \ge \sum_{i \in S} x_i - (|S| - 1) \qquad \forall S \in$$

Subsets $\mathcal{S}$ define a hypergraph H.
We write $P_{SL} = P_{SL}^{(H)}$.
Matrix of constraints M_H.

SL complete description

Theorem 3

Given a hypergraph H, the following statements are equivalent:

(a) $P_{SL}^{(H)}$ is an integer polytope.

(b) M_H is balanced.

(c) H is Berge-acyclic.

Derived from a more general result taking into account the sign pattern of the monomials.

SL complete description: signed case

Theorem 4

Given a hypergraph $H = (V, E)$ and a sign pattern $s \in \{-1, 1\}^E$, the following statements are equivalent:

(a) For all $f \in \mathcal{P}(H)$ with sign pattern s, every vertex of P_H maximizing L_f is integer.

(b) $M_{H(s)}$ is balanced.

(c) $H(s)$ has no negative special cycle.

(d) $P_{H(s)}$ is an integer polytope.

$P_{H(s)}$ is defined by constraints

$$y_S \le x_i \qquad \forall i \in S, \forall S \in \mathcal{S}, sgn(a_S) = +1$$

$$y_S \ge \sum_{i \in S} x_i - (|S| - 1) \qquad \forall S \in \mathcal{S}, sgn(a_S) = -1$$