RADIAL PULSATIONS OF STARS

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ABSTRACT

It is shown that the period τ of the fundamental mode of radial pulsation of a star is given to a good approximation by the expression

$$\tau = 2\pi\sqrt{\frac{I}{-(3\bar{\Gamma}_1 - 4)\Omega}}$$

where I denotes the moment of inertia of the star about its center, Ω its gravitational potential, and $\bar{\Gamma}_1$, the mean value of Γ_1 with respect to pressure. A general method of obtaining higher approximations for the period of the fundamental mode as well as for the periods of the higher modes is given and is applied to the standard model. The results obtained for the standard model indicate that, in certain cases, our approximate method yields more accurate values for the periods than have been obtained by the trial-and-error method of integrating the pulsation equation.

1. The differential equation which governs the adiabatic and radial oscillations of a gaseous star is

$$Xr\frac{d^2\xi}{dr^2} + \frac{d\xi}{dr}\left[4X + r\frac{dX}{dr}\right] + \xi\left[\left(\sigma^2 + 4\frac{Gm(r)}{r^3}\right)r\rho + 3\frac{dX}{dr}\right] = 0\,, \tag{1}$$

where ξ denotes the ratio $\delta r/r$ of the radial displacement to the radius. $X = \Gamma_1 P$, P being the sum of the gaseous pressure p_G and the pressure of radiation p_R. The quantity Γ_1 is the adiabatic exponent defined by $\delta P/P = \Gamma_1 \delta\rho/\rho$ and has the well-known value

$$\Gamma_1 = \beta + \frac{(4-3\beta)^2(\gamma-1)}{\beta + 12(\gamma-1)(1-\beta)}\,, \tag{2}$$

where $\beta = p_G/P$ and $\gamma = c_p/c_v$, c_p and c_v denoting, respectively, the specific heats at constant pressure and constant volume for the matter; σ is equal to $2\pi\nu = 2\pi/\tau$, ν being the frequency and τ the period of oscillation; ρ is the density at r and $m(r)$ is the mass interior to r.

By multiplying equation (1) by r^3 we can put it in the self-adjoint form

$$\frac{d}{dr}\left(Xr^4\frac{d\xi}{dr}\right) + \xi\left[\sigma^2 r^4\rho + 4Gm(r)r\rho + 3r^3\frac{dX}{dr}\right] = 0\,. \tag{3}$$

The boundary conditions,

$$\delta r = 0 \quad \text{at} \quad r = 0$$

and

$$\delta P = -\Gamma_1 P\left(3\xi + r\frac{d\xi}{dr}\right) = 0 \quad \text{at} \quad r = R\,, \tag{4}$$

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are met if ξ is regular in the range of integration. The condition of the regularity of ξ is satisfied only for a discrete set of values σ_i, which determine the periods of the free modes of vibration. The solutions ξ_i corresponding to these σ_i are orthogonal with respect to ρr^4:

$$\int_0^R \rho r^4 \xi_i \xi_k dr = 0\,, \qquad i \neq k\,. \tag{5}$$

It is known[3] that equation (3) is the Eulerian equation of the variational problem of minimizing the integral

$$J = \int_0^R \left\{ X r^4 \xi'^2 - \left[4Gm(r)\rho r + 3r^3 \frac{dX}{dr} \right] \xi^2 - \sigma^2 \xi^2 \rho r^4 \right\} dr\,. \tag{6}$$

The conditions of this free variational problem are

$$2Xr^4\xi' = 0 \quad \text{at} \quad r = 0 \quad \text{and at} \quad r = R\,,$$

both of which are satisfied by equations (4). The solution ξ_0 of the fundamental mode can as well be obtained by minimizing

$$\frac{\int_0^R \left\{ X r^4 \xi'^2 - \xi^2 \left[4Gm(r)\rho r + 3r^3 \frac{dX}{dr} \right] \right\} dr}{\int_0^R \xi^2 \rho r^4 dr}\,, \tag{6'}$$

and the minimum value of this integral is equal to σ_0^2. If we introduce the value of X in this expression and use the relation

$$\frac{dP}{dr} = -\frac{Gm(r)\rho}{r^2}\,,$$

we have

$$\sigma_0^2 = \min \frac{\int_0^R \Gamma_1 P r^4 \xi'^2 dr + \int_0^R (3\Gamma_1 - 4)\xi^2 Gm(r) r \rho dr - 3\int_0^R P r^3 \frac{d\Gamma_1}{dr} \xi^2 dr}{\int_0^R \xi^2 \rho r^4 dr}\,. \tag{7}$$

We shall, in the first instance, consider the case where p_R is negligible. Then $\Gamma_1 = \gamma$, $d\Gamma_1/dr = 0$, and

$$\sigma_0^2 = \min \frac{\gamma \int_0^R (r\xi')^2 P dV + (3\gamma - 4)\int_0^R \xi^2 \frac{Gm(r)dm(r)}{r}}{\int_0^R \xi^2 r^2 dm(r)}\,. \tag{8}$$

We also have

$$-d\Omega = \frac{Gm(r)dm(r)}{r} \quad \text{and} \quad dU = \frac{1}{\gamma - 1} P dV\,,$$

[3] Cf., e.g., Courant and Hilbert, *Methoden der mathematischen Physik*, I, 344.

where Ω and U denote, respectively, the gravitational potential energy and the internal energy of the star. On the other hand, if we denote by I the moment of inertia of the star about its center, we have

$$dI = r^2 dm(r) .$$

It is well known that for a sphere I is equal to $\frac{2}{3}I_1$, if I_1 is the moment of inertia about one diameter of the sphere. If Ω and U are taken to be zero when the elements of the star are at infinite distances from one another, then they are connected by the relation

$$-\Omega = 3(\gamma - 1)U .$$

Thus, we can re-write equation (8) as

$$\sigma_0^2 = \min \frac{\gamma(\gamma-1)\int_0^{-\Omega/3(\gamma-1)} (r\xi')^2 dU - (3\gamma-4)\int_0^{\Omega} \xi^2 d\Omega}{\int_0^I \xi^2 dI} . \tag{9}$$

2. We proceed now to obtain approximate solutions of equation (9). As a first approximation, we suppose ξ to be a constant. Then equation (9) becomes

$$\sigma_0^2 = -\frac{(3\gamma-4)\Omega}{I} . \tag{10}$$

From our principle of minimum (eq. [9]) it is obvious that the value σ_0^2 given by equation (10) is either equal to or greater than the true value. For instance, for a star of constant density, ξ is a constant, and equation (10) is exact. We shall also compare equation (10) with the results obtained previously for different models.

T. E. Sterne[4] has found that, for a model in which the distribution of density is given by $\rho = R^2\bar{\rho}/3r^2$,

$$(\sigma_0^2)_{\text{exact}} = \frac{2\pi G\gamma\bar{\rho}}{3}\left[-\tfrac{1}{2} + \tfrac{1}{2}\sqrt{1+8a}\right]\left[2 + \tfrac{1}{2} + \tfrac{1}{2}\sqrt{1+8a}\right] ,$$

where

$$a = \left(3 - \frac{4}{\gamma}\right) .$$

For this model we obtain from equation (10)

$$(\sigma_0^2)_{\text{app}} = 4\pi(3\gamma-4)G\bar{\rho} = 4\pi G\bar{\rho}a\gamma .$$

Hence

$$\frac{(\sigma_0^2)_{\text{exact}}}{(\sigma_0^2)_{\text{app}}} = \frac{1}{3}\left[\frac{2.5 + 0.5\sqrt{1+8a}}{0.5 + 0.5\sqrt{1+8a}}\right] .$$

[4] *M.N.*, **97**, 582, 1937.

This ratio varies from 1 to 0.73 as γ varies from 4/3 to 5/3. In the most unfavorable case $\tau_{app} = \sqrt{0.73}\,\tau_{exact} = 0.85\,\tau_{exact}$. The degree of approximation of our formula (10) is shown in Table 1 for a set of other models.

TABLE 1

Model	References	Computed Values of σ_0^2	Expressions of σ_0^2 Deduced from Equation (10)	Ratio: $\frac{\tau_{app}}{\tau_{comp}}$
Polytrope $n=1.5$ $\frac{\rho_c}{\bar{\rho}}=6.0003$	T. G. Cowling, *M.N.*, **94**, 779, 1934	$\sigma_0^2=4\pi G\rho_c\,\frac{3}{20}$ $\gamma=\frac{5}{3}$	$\sigma_0^2=\frac{4\pi G\bar{\rho}(3\gamma-4)}{3.5\times 0.315}$	0.996
Polytrope $n=2$ $\frac{\rho_c}{\bar{\rho}}=11.4$	J. C. P. Miller, *M.N.*, **90**, 59, 1930	$\sigma_0^2=\frac{4\pi G}{3}\rho_c\times 0.10567$ $\gamma=1.428$	$\sigma_0^2=\frac{4\pi G\bar{\rho}(3\gamma-4)}{3\times 0.220}$	0.964
Cowling model $\frac{\rho_c}{\bar{\rho}}=37$	T. G. Cowling, *M.N.*, **96**, 42, 1936	$\sigma_0^2=\frac{4\pi G}{3}\rho_c\times 0.2595$ $\gamma=5/3$	$\sigma_0^2=\frac{4\pi G\bar{\rho}(3\gamma-4)}{3\times 0.08547}$	0.907
Polytrope $n=4$ $\frac{\rho_c}{\bar{\rho}}=623$	J. C. P. Miller, *M.N.*, **90**, 59, 1930	$\sigma_0^2=\frac{4\pi G}{5}\rho_c\times 0.02856$ $\gamma=1.428$	$\sigma_0^2=\frac{4\pi G\bar{\rho}(3\gamma-4)}{3\times 0.01233}$	0.682

3. If the radiation pressure is not negligible but β is a constant (standard model), then Γ_1 is different from γ, but it is still constant throughout the star, and $d\Gamma_1/dr = 0$. In this case equation (10) takes the form

$$\sigma_0^2 = -\frac{(3\Gamma_1 - 4)\Omega}{I}. \qquad (10')$$

For the standard model,

$$\Omega = -\frac{3}{2}\cdot\frac{GM^2}{R},$$

and by numerical integration, we find that

$$I = 0.113MR^2.$$

Hence

$$\sigma_0^2 = \frac{(3\Gamma_1 - 4)4\pi G\bar{\rho}}{0.226},$$

or

$$\tau_{app} = \frac{1}{[\rho_c(3\Gamma_1 - 4)]^{1/2}} \times \sqrt{\frac{0.226 \times 3.1416 \times 54.18}{6.67\cdot 10^{-8}}} = \frac{24{,}015}{[\rho_c(3\Gamma_1 - 4)]^{1/2}}.$$

From numerical integrations of the equation (1), Eddington concluded,[5] that for the standard model

$$\tau_{\mathrm{Edd}} = \frac{25{,}080}{[\rho_c(3\Gamma_1 - 4)]^{1/2}}\,,$$

giving

$$\frac{\tau_{\mathrm{app}}}{\tau_{\mathrm{Edd}}} = 0.957\,.$$

4. We shall now consider equation (7) for the case where the radiation pressure is important and is variable through the star. We notice that

$$\int_0^R \xi^2 r^3 P \frac{d\Gamma_1}{dr}\,dr = -\int_0^R \Gamma_1 \frac{d}{dr}(\xi^2 r^3 P)dr + [\xi^2 r^3 P \Gamma_1]_0^R\,.$$

The second term on the right-hand side vanishes. Using this relation in equation (7), we obtain

$$\sigma_0^2 = \min \frac{4\int_0^\Omega \xi^2 d\Omega + \int_0^R \Gamma_1 P(3\xi + r\xi')^2 dV}{\int_0^I \xi^2 dI}\,. \tag{11}$$

We now assume again that ξ is constant and obtain

$$\sigma_0^2 = \frac{4\Omega + 9\int_0^R \Gamma_1 P dV}{I}\,.$$

If we define $\overline{\Gamma}_1$ by

$$\overline{\Gamma}_1 = \frac{\int_0^R \Gamma_1 P dV}{\int_0^R P dV}$$

and use the relation

$$-\Omega = 3\int_0^R P dV\,,$$

this approximation becomes

$$\sigma_0^2 = -\frac{(3\overline{\Gamma}_1 - 4)\Omega}{I}\,. \tag{10''}$$

To test this formula we can use the model calculated by J. Wasiutynski, which is composed of a convective core and radiative envelope. The value of β at the center is equal to 0.8, and the value of γ is taken equal to $\frac{5}{3}$. If the molecular weight is taken to be

[5] *The Internal Constitution of the Stars*, formula 130.4.

unity, the mass is $10.263 M_\odot$. One of us[6] has computed the fundamental period for this model and has found $\tau_{comp} = 8.5$ days. The mean value $\bar{\Gamma}_1$ is equal to 1.53, and $\rho_c = 1.621 \cdot 10^{-3}$ gr/cm³ $= 40.3\bar{\rho}$ for the special case integrated. With these values, we obtain from equation (10″)

$$\tau_{app} = 7.729 \text{ days} \qquad \text{or} \qquad \frac{\tau_{app}}{\tau_{comp}} = 0.909 .$$

5. Denoting by E the total energy $\Omega + U$ of the star and using the relation between U and Ω when the radiation pressure is negligible, we can write equation (10) as

$$\tau = 2\pi\sqrt{\frac{I}{3(\gamma - 1)(-E)}} = 2\pi\sqrt{\frac{2I_1}{9(\gamma - 1)(-E)}} .$$

This relation is of the same form as the expression for the free period of torsional oscillation of a body of moment of inertia I_1 and subject to a restoring force $K\theta$, $\tau = 2\pi\sqrt{I_1/K}$. Thus, the restoring force in the fundamental mode of the radial pulsation of a star is proportional to the negative of the total energy of the star. When the latter quantity vanishes, the star becomes labile and the period infinite.

When the pressure of radiation cannot be neglected, we can define with S. Chandrasekhar[7] an exponent Γ_3 by

$$\Gamma_3 = 1 + \frac{(4 - 3\beta)(\gamma - 1)}{\beta + 12(\gamma - 1)(1 - \beta)} .$$

Then

$$E = -\int_0^R \frac{3\Gamma_1 - 4}{\Gamma_3 - 1} P dV = \frac{1}{3}\overline{\left(\frac{3\Gamma_1 - 4}{\Gamma_3 - 1}\right)} \cdot \Omega = \frac{(3\gamma - 4)}{3(\gamma - 1)}\bar{\beta} \cdot \Omega ,$$

where the means are defined just as $\bar{\Gamma}_1$. In terms of these quantities our approximation becomes

$$\tau = 2\pi\sqrt{\frac{\overline{\left(\frac{3\Gamma_1 - 4}{\Gamma_3 - 1}\right)} \cdot I}{3(\overline{3\Gamma_1 - 4})(-E)}} ,$$

which can again be interpreted as before. Another form for equation (10) is

$$\sigma_0^2 = \frac{3\alpha\int_0^R \frac{\gamma P}{\rho} dm(r)}{\int_0^R r^2 dm(r)} , \qquad (12)$$

where $\alpha = (3 - 4/\gamma)$. Now the velocity of sound is given by $c^2 = (\gamma P)/\rho$ and, if we define average values $\bar{c}$ and $\bar{R}$ by the equalities

$$(\bar{c})^2 = \frac{1}{M}\int_0^M c^2 dm(r) , \qquad (\bar{R})^2 = \frac{1}{M}\int_0^M r^2 dm(r) ,$$

[6] P. Ledoux, *Astrophysica Norvegica*, 3, 193, 1940.

[7] *Introduction to the Study of Stellar Structure*, p. 56, 1939.

then equation (12) becomes

$$\sigma_0^2 = 3\alpha \left(\frac{\bar{c}}{\bar{R}}\right)^2 \quad \text{and} \quad \tau = 2\pi \frac{\bar{R}}{\bar{c}}\sqrt{\frac{1}{3\alpha}} .$$

The period of the star is thus seen to be proportional to the time required for a sound wave to travel through the weighted radius $\bar{R}$ with the weighted velocity $\bar{c}$. If the restoring forces during the pulsation depended only on the compression of the material such a relationship would be expected. We know, however, that gravity must also influence the pulsation, and, in fact, its effect on the period appears explicitly in the factor $\alpha^{-(1/2)}$.

We shall make one further remark about our formula (10). If we say that one star is more centrally condensed than another of the same radius[8] when

$$\frac{m_1(r)}{M_1} > \frac{m_2(r)}{M_2} \qquad 0 \leq r \leq R ,$$

then

$$\frac{|\Omega_1|}{M_1^2} > \frac{|\Omega_2|}{M_2^2} \quad \text{and} \quad \frac{I_1}{M_1} < \frac{I_2}{M_2} \quad \text{and} \quad \sigma_1^2 > \sigma_2^2 \frac{M_1}{M_2} .$$

Thus, besides the well-known proportionality of τ to the inverse square root of the mean density and its dependence upon γ, it depends also upon the distribution of the density and in such a way that, to a first approximation, τ increases when the central condensation of the star decreases. This is in good agreement with Table 1 in a previous paper.[6]

6. We shall now apply the Ritz method to obtain higher approximations for the fundamental period as well as for periods of higher modes of vibration. We assume that

$$\xi = \sum_{i=0}^{n} a_i r^{2i} . \tag{13}$$

On substituting the foregoing form for ξ in equation (6), we find that the a_i's are determined by $(n + 1)$ equations of the form

$$\frac{\partial J}{\partial a_j} = 0 , \qquad j = 0, 1, \ldots , n .$$

We shall consider only the case where p_R is small or is a constant fraction of the pressure, so that we can always write the factor $(3\Gamma_1 - 4)$ outside the integrals. The general case can clearly be treated by the same method. Let

$$I_j = \int_0^R r^{2j} dI \qquad \text{with} \qquad dI = r^2 dm(r) ,$$

$$\Omega_j = -\int_0^R r^{2j} d\Omega \qquad \text{with} \qquad d\Omega = -\frac{Gm(r)dm(r)}{r} ,$$

[8] This implies no loss of generality since we can always apply a Lane transformation.

and

$$A_j = \int_0^R 4Pr^{4+2j}dV \qquad \text{with} \qquad dV = 4\pi r^2 dr \,.$$

Then

$$\frac{\partial J}{\partial a_j} = \sum_{i=0}^{n} a_i\{ij\Gamma_1 A_{i+j-2} + (3\Gamma_1 - 4)\Omega_{i+j} - \sigma^2 I_{i+j}\} = 0\,, \qquad j = 0, \ldots, n\,. \tag{14}$$

But by a general theorem[9] we have

$$A_j = \frac{4}{2j+7}\Omega_{j+2}$$

and hence equation (14) becomes

$$\left.\begin{array}{l}\displaystyle\sum_{i=0}^{n} a_i\left\{(3\Gamma_1 - 4)\left[\frac{4ij}{\alpha(2i+2j+7)} + 1\right]\Omega_{i+j} - \sigma^2 I_{i+j}\right\} = 0\,, \\ \qquad\qquad\qquad\qquad j = 0, 1, \ldots, n\end{array}\right\} \tag{15}$$

with

$$\alpha = \left(3 - \frac{4}{\Gamma_1}\right).$$

The condition of compatibility of these $(n+1)$ homogeneous equations is

$$|C_{ji}| = 0\,, \tag{16}$$

where

$$C_{ji} = \left\{(3\Gamma_1 - 4)\left[\frac{4ij}{\alpha(2i+2j+7)} + 1\right]\Omega_{i+j} - \sigma^2 I_{i+j}\right\}.$$

For the standard model we have

$$I_j = \frac{R^{2j+2}M}{(6.8969)^{2j+2}(2.01813)}\int_0^{6.8969} z^{2j+4}u^3 dz = \frac{R^{2j+2}M}{2.01813} I'_j\,, \qquad j = 0, 1, \ldots, n$$

$$\Omega_0 = \frac{3}{2}\frac{GM^2}{R} = \frac{GM^2}{R}\cdot\frac{\Omega'_0}{(2.018)^2}\,,$$

$$\Omega_j = \frac{GM^2R^{2j-1}}{(2.018)^2(6.8969)^{2j-1}}\int_0^{6.8969} z^{2j+1}\left(-z^2\frac{du}{dz}\right)u^3 dz = \frac{GM^2R^{2j-1}}{(2.018)^2}\Omega'_j\,.$$

If Σ^2 is defined by

$$\sigma^2 = \frac{4\pi G(3\Gamma_1 - 4)}{3(2.018)}\cdot\bar{\rho}\Sigma^2\,, \tag{17}$$

[9] Chandrasekhar, *op. cit.*, p. 66, Theorem 2.

equation (16) can be written as

$$|C'_{ji}| = 0 ,$$

where

$$C'_{ji} = \left\{ \frac{4ij}{a(2i + 2j + 7)} + 1 \right\} \Omega'_{j+i} - \Sigma^2 I'_{i+j} .$$

It will further be convenient to define Ω''_{j+i} by

$$\Omega''_{j+i} = \left\{ \frac{4ij}{a(2i + 2j + 7)} + 1 \right\} \Omega'_{j+i} ,$$

so that

$$C'_{ji} = \Omega''_{j+i} - \Sigma^2 I'_{j+i} .$$

One verifies easily that the determinant $|C'_{ji}|$ is equivalent to

$$\begin{vmatrix}
\Sigma^2 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 \\
0 & \Sigma^2 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & \Sigma^2 & 0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & \Sigma^2 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & \Sigma^2 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\Omega'_n & \cdots & \cdots & \cdots & \Omega'_2 & \Omega'_1 & \Omega'_0 & I'_0 & I'_1 & I'_2 & \cdots & \cdots & \cdots & I'_n \\
\Omega''_{1+n} & \cdots & \cdots & \cdots & \Omega''_{1+2} & \Omega''_{1+1} & \Omega'_1 & I'_1 & I'_2 & I'_3 & \cdots & \cdots & \cdots & I'_{n+1} \\
\Omega''_{2+n} & \cdots & \cdots & \cdots & \Omega''_{2+2} & \Omega''_{1+2} & \Omega'_2 & I'_2 & I'_3 & I'_4 & \cdots & \cdots & \cdots & I'_{n+2} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\Omega''_{n+n} & \cdots & \cdots & \cdots & \cdots & \cdots & \Omega'_n & I'_n & \cdots & \cdots & \cdots & \cdots & \cdots & I'_{2n}
\end{vmatrix} \tag{18}$$

The values of the first I' and Ω' involved in this determinant are

$$\begin{aligned}
I'_0 &= 0.22813 & \qquad \Omega'_0 &= 6.10909 \\
I'_1 &= 0.04849 & \qquad \Omega'_1 &= 0.78217 \\
I'_2 &= 0.01507 & \qquad \Omega'_2 &= 0.16235 \\
I'_3 &= 0.00601 & \qquad \Omega'_3 &= 0.04758 \\
I'_4 &= 0.00282 & \qquad \Omega'_4 &= 0.01781
\end{aligned}$$

Our approximation (10′) obtained previously for the fundamental mode of vibration corresponds to $n = 0$, $\xi =$ constant, and the equation for Σ^2 is the vanishing of the first central determinant of (18),

$$\Sigma_0^2 = \frac{\Omega'_0}{I'_0} = \frac{3 \times 2.018}{4\pi G\bar{\rho}} \cdot \frac{\Omega_0}{I_0} .$$

It follows from equation (17) that

$$\sigma_0^2 = \frac{(3\Gamma_1 - 4)\Omega_0}{I_0} ,$$

which is identical to our equation (10′). The second approximation will consist in taking $\xi = a_0 + a_1 r^2$ and equating the second central determinant to zero. Of the two roots of this second-degree equation in Σ^2, the smaller Σ_0^2 gives a second approximation to the period of the fundamental mode, while the larger, Σ_1^2, yields the first approximation for the first mode. These roots are

$$\Sigma^2 = \frac{1}{\alpha}\left\{ 24.4355\alpha + 9.6741 \mp \sqrt{249.2068\alpha^2 - 45.3571\alpha + 93.5891} \right\} , \tag{19}$$

which are easily related to σ^2 by equation (17). The third approximation corresponds to $n = 2$ or $\xi = a_0 + a_1 r^2 + a_2 r^4$. If we write

$$z = 188\Sigma^2 - \left(4567 + \frac{4194}{\alpha}\right) , \tag{20}$$

the equation of the third order in Σ^2, formed by equating the third central determinant to zero, is

$$z^3 + 3Hz + G = 0 ,$$

where

$$H = -\left[7.3581 + \frac{4.4885}{\alpha} + \frac{8.8183}{\alpha^2}\right] 10^6$$

$$G = -\left[2.7826 + \frac{2.045}{\alpha} + \frac{6.875}{\alpha^2} + \frac{3.7171}{\alpha^3}\right] 10^{10} .$$

Hence $G^2 + 4H^3 < 0$, and if we define θ and q by

$$\cos\theta = -\frac{G}{2\sqrt{-H^3}} , \qquad q = 2\sqrt{-H} ,$$

the solutions are

$$z = q\cos\theta , \qquad q\cos\left(\frac{2\pi}{3} + \theta\right) , \qquad q\cos\left(\frac{2\pi}{3} - \theta\right) .$$

Having determined z, we easily obtain Σ^2 by equation (20).

Table 2 provides a comparison between the values of σ^2 given by equations (10′), (19), and (20) with the values obtained by numerical integrations of equation (1) for the standard model. In these numerical integrations the quantity that is generally determined is ω which is related to σ by

$$\sigma^2 = \frac{\pi}{3}\,\frac{G\Gamma_1(6.8969)^3\bar{\rho}\omega^2}{(2.018)} . \tag{21}$$

The corresponding quantities in equations (17) and (21) are therefore $4\alpha\Sigma^2$ and $(6.8969)^3\omega^2$, and these are tabulated in Table 2. One sees from this table that the second approximation already determines the period with high accuracy. Clearly the degree of our approximations depends on the value of Γ_1 or α. We notice that, in agreement with the principle of minimum (eq. [7]), the quantity $4\alpha\Sigma_0^2$ decreases in a monotonic fashion with successive approximations. By the same principle we know, furthermore, that the exact value of $4\alpha\Sigma_0^2$ is less than that given by the highest-order approximation. We are, therefore, inclined to conclude that whenever $(6.8969)^3\omega^2$ is greater than the value of $4\alpha\Sigma_0^2$ obtained in one of our approximations, the former is in error. That these values might be in error is likely because in the numerical integrations of the differential equa-

TABLE 2

α	Fundamental Mode of Vibration				First Mode of Vibration			Second Mode of Vibration
	Computed $(6.8969)^3\omega_0^2$	First App. for $4\alpha\Sigma_0^2$	Second App. for $4\alpha\Sigma_0^2$	Third App. for $4\alpha\Sigma_0^2$	Computed $(6.8969)^3\omega_1^2$	First App. for $4\alpha\Sigma_1^2$	Second App. for $4\alpha\Sigma_1^2$	First App. for $4\alpha\Sigma_2^2$
0.1	10.334	10.712	10.199	10.173	98.42	87.394	61.034	225.647
.2	19.684	21.423	19.364	19.233	....	97.126	66.232	240.54
.3	....	32.135	27.540	27.107	....	108.498	72.072	255.977
.4	36.087	42.846	34.839	33.809	152.879	120.747	78.579	271.916
.5	....	53.558	41.359	39.457	....	133.696	85.675	288.319
0.6	51.179	64.272	47.370	44.226	....	147.31	93.238	305.439

TABLE 3

α	$\frac{a_1}{a_0}R^2$	$\frac{a_2}{a_0}R^4$
0.1	+0.14943	1.08865
.2	+ .58381	2.00679
.3	+ .78223	3.99502
.4	+ .73228	7.12026
.5	+ .39007	11.41735
0.6	−0.27970	16.79257

tion (1), the condition $\delta P = 0$ is imposed not at the surface but at about $\frac{5}{7}$ of the radius.[10] It would seem, therefore, that our approximate method is likely to yield in certain cases more exact values for the periods than can be obtained by the more laborious method of integrating equation (1).

We can also determine the values of a_1/a_0 and a_2/a_0, which appear in our third approximation $\xi = a_0 + a_1r^2 + a_2r^4$, by solving two of the three corresponding equations (15). These values are given in Table 3 for the fundamental mode of vibration.

The corresponding solutions ξ are plotted in Figure 1 as solid lines and the solutions corresponding to the computed $(6.8969)^3\omega_0^2$, as taken from Eddington and Edgar, are shown by the dashed lines. The dotted curve represents a solution obtained for $\alpha = 0.4$ and a value of $4\alpha\Sigma_0^2$ a little smaller than our third approximation ($\sim$33.770). This solu-

[10] Cf., e.g., Edgar, *M.N.*, **93**, 426–29, remarks under Tables 1, 2, 3, and 4.

tion has been computed by means of series near the center and the surface of the star and an intermediate numerical integration. The solution in series in the outer layers of the star is particularly easy to obtain if one supposes that the mass is constant in these layers. This hypothesis is a very good approximation for a rather important part of the star, and it affects only the coefficients of the fifth and higher powers of $(1 - r/R)$ in the series which gives us the solution in this region. This series is of the form

$$\xi = \sum_{i=0}^{\infty} a_i \left(1 - \frac{r}{R}\right)^i \left\{ 1 + B \left[\log\left(1 - \frac{r}{R}\right) + \sum_{\substack{j=0 \\ j\neq 3}}^{\infty} \beta_j \left(1 - \frac{r}{R}\right)^{j-3} \right] \right\}, \qquad (22)$$

where β_0, β_1, β_2 are negative. We determine a_0 and B by equating ξ and $d\xi/dr$ obtained from equation (22) to the values obtained by numerical integration at the point where

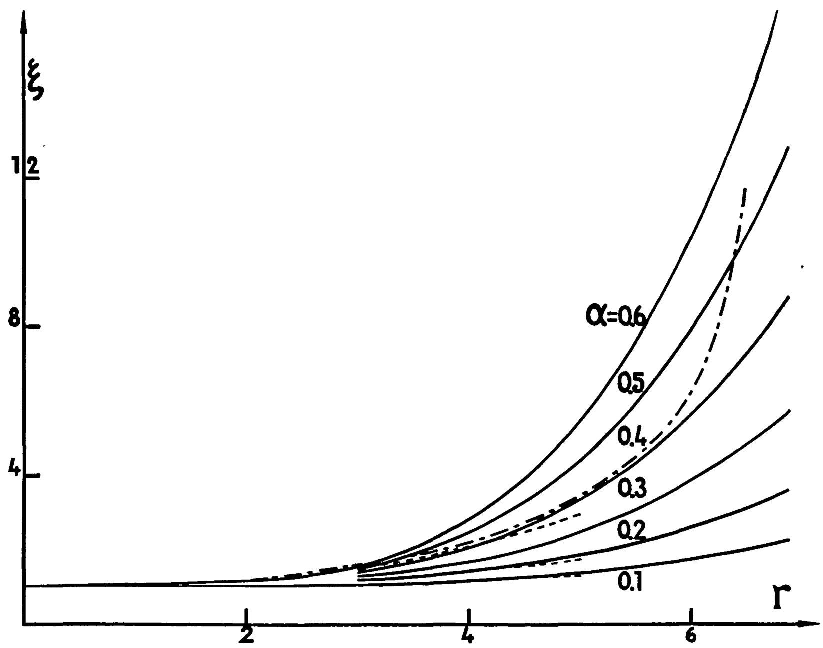

FIG. 1

this numerical integration stops. If B is equal to zero, our solution remains finite, and the chosen value of $4a\Sigma_0^2$ is the correct one. If B is positive, then our solution tends to $-\infty$ toward the surface, and the chosen value of $4a\Sigma_0^2$ is too large. On the contrary, if B is negative—which happens in the particular case integrated here—our solution tends to $+\infty$ toward the surface and the chosen value of $4a\Sigma_0^2$ is too small. But, in fact, when one gets near the solution which remains finite at $r = R$, it becomes very easy to jump from one solution to another during the numerical integration, so that this lower limit for $4a\Sigma_0^2$ is hardly significant. But in any case, as far as the values of the amplitude themselves are concerned, this solution cannot be very different from the true solution in the main part of the star. Thus, although it is well known that the variational method yields the proper values with much greater accuracy than the proper functions, the comparison of this solution with our approximate one, proves that the latter does not deviate too much from the true solution.

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