

A weak local irregularity property in S^ν spaces

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Abstract. It has been shown that, from the prevalence point of view, elements of the S^ν spaces are almost surely multifractal, while the Hölder exponent at almost every point is almost surely equal to the maximum Hölder exponent. We show here that typical elements of S^ν are very irregular by proving that they almost surely satisfy a weak irregularity property: there exists a local irregularity exponent which is constant for almost every element of S^ν and equal to the lowest Hölder exponent.

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1. Introduction

In this paper we are interested in generic results of regularity in some spaces of irregular functions well suited for multifractal studies from both the practical and the theoretical point of view: the S^ν spaces. By generic, we mean prevalent: the notion of prevalence provides an infinite dimensional extension of the notion of translation invariant “Lebesgue measure zero” (see section 2.2). In these settings, a “negligible” set becomes a shy set, while its complement is called a prevalent set. From this perspective, typical elements of some space satisfy a property if the set of the elements for which this property is satisfied is prevalent.

The Hölder regularity is a popular notion of regularity. It has been introduced to study smoothness properties of functions such as the Weierstraß function (see e.g. [16]). Indeed, many “historical” functions share the same property [23]: there exist $H \in (0, 1)$ and a constant $C > 0$ such that the function f satisfies on some interval I ,

$$\forall x, y \in I, |f(x) - f(y)| \leq C|x - y|^H, \quad (1)$$

and

$$\forall x, y \in I, \quad \sup_{(u,v) \in [x,y]^2} |f(u) - f(v)| \geq \frac{1}{C} |x - y|^H. \quad (2)$$

The first inequality can be seen as a regularity condition, while the second is related to the irregularity of the function. It has been shown in [8] that this behavior is the typical behavior of functions belonging to the uniform Hölder space $\Lambda^H(\mathbf{R}^d)$ in the sense of prevalence, i.e. the set

$$\{f \in \Lambda^H(\mathbf{R}^d) : f \text{ satisfies (1) and (2)}\}$$

is prevalent.

However, in many cases, functions do not satisfy such inequalities. It is in particular true for the so-called multifractal functions, which we now define. A bounded function f defined on \mathbf{R}^d belongs to the pointwise Hölder space $\Lambda^\alpha(x_0)$ if the following inequality is satisfied for some constant C and a polynomial P of degree at most α :

$$|f(x) - P(x)| \leq C|x - x_0|^\alpha, \quad (3)$$

in a neighborhood of x_0 . These spaces are embedded and one introduces the Hölder exponent of f at x_0 as follows,

$$h_f(x_0) = \sup\{\alpha \geq 0 : f \in \Lambda^\alpha(x_0)\}.$$

This exponent is a pointwise notion of regularity.

If the function f is very irregular, the computation of h_f for every point is an insuperable task. For such functions, one rather tries to determine the associated spectrum of singularities, also called multifractal spectrum:

$$d_f : [0, \infty] \rightarrow \{-\infty\} \cup [0, d] \quad h \mapsto \dim_{\mathcal{H}}\{x \in \mathbf{R}^d : h_f(x) = h\},$$

where $\dim_{\mathcal{H}}$ denotes the Hausdorff dimension [12, 31]. If $d_f(h) > 0$ for at least two different values h , the function f is said to be multifractal. It can be shown that in several functional spaces, the set of multifractal functions is prevalent [13, 3]. Computing the Hölder spectrum can also be very difficult, since it involves intricate limits. However, there exist heuristic methods, called multifractal formalisms that lead to an estimation of d_f . If the estimation is the exact spectrum, one says that the multifractal formalism holds for the function f . Such methods were originally introduced in the context of fully developed turbulence [14] and are now used in many fields of science (see e.g. [1, 2, 15, 22, 30, 26, 34]). There is no formalism that holds for any function; some of them work for typical functions (e.g. self-similar functions [19, 33]) or lead to upper bounds for d_f [18, 21, 4]. Moreover, most of the multifractal formalisms are based on the so-called box counting method and rely on a Legendre transform, so that the estimation is at best the concave hull of the spectrum of singularities [24] (see [28] for examples of formalisms that do not require the convexity constraint).

To overcome this problem, Jaffard has introduced a multifractal formalism based on the S^ν spaces [20]. The set of functions of S^ν for which the associated multifractal formalism holds is prevalent, so that the typical

elements of S^ν are multifractal [3]. However, one has the following regularity result: the set of functions f for which the Hölder exponent $h_f(x)$ is equal to the maximum exponent h_{\max} almost everywhere (with respect to the Lebesgue measure) is prevalent. To sum up the situation, contrary to the case of Hölder spaces studied in [8], typical elements of S^ν do not satisfy inequalities (1) and (2), but a regularity condition is satisfied in a generic sense.

In this paper, we aim at showing that the typical elements of S^ν also satisfy an irregularity condition. More precisely, we show that the set of elements of S^ν satisfying a weak local irregularity property is prevalent. To this end, we introduce a concept of weak uniform irregularity and define the irregularity exponent $\mathcal{H}_f(x_0)$ of a function at a given point x_0 , which is a local notion. We prove that, in the sense of prevalence, there exists a minimum Hölder exponent h_{\min} (depending on S^ν and in non-trivial cases lower than h_{\max}) such that almost every function of S^ν satisfies $\mathcal{H}_f(x) = h_{\min}$ for any x . We thus have the following heuristic interpretation: for non-trivial spaces S^ν , typical elements of S^ν are very irregular, since, somehow, the lowest Hölder exponent is met near any considered point, while the typical Hölder exponent is the largest one.

The paper is organized as follows. In the next section, we recall some definitions about multifractal analysis and the S^ν spaces. Next, we define the local irregularity exponent. To prove our main result, we will need wavelet criteria for the irregularity; these are stated in Section 4. The prevalent result is obtained in the last section.

2. The S^ν -based multifractal formalism

The multifractal analysis aims to study the smoothness of very irregular functions f . Instead of trying to characterize the regularity of f at a every point, one rather tries to determine the associated spectrum of singularities. The S^ν spaces have been introduced in order to provide an efficient multifractal formalism, i.e. a method that allows the computation of the multifractal spectrum in many practical cases. Although this technique does not always lead to the right spectrum, it has been shown that the S^ν -based multifractal formalism holds for almost every element of S^ν . Here the term “almost every” has to be clearly defined, since one can not use the usual Lebesgue measure in infinite dimensional settings.

2.1. S^ν spaces

Let us briefly recall some definitions and notations (for more precisions, see e.g. [32, 11, 29]). Under some general assumptions, there exist a function ϕ and $2^d - 1$ functions $(\psi^{(i)})_{1 \leq i < 2^d}$ called wavelets such that

$$\{\phi(x - k) : k \in \mathbf{Z}^d\} \cup \{\psi^{(i)}(2^j - k) : 1 \leq i < 2^d, k \in \mathbf{Z}^d, j \in \mathbf{N}\}$$

form a basis of $L^2(\mathbf{R}^d)$. A (complex) function $f \in L^2(\mathbf{R}^d)$ can be decomposed as follows,

$$f = \sum_{k \in \mathbf{Z}^d} C_k \phi(\cdot - k) + \sum_{j=1}^{\infty} \sum_{k \in \mathbf{Z}^d} \sum_{i=1}^{2^d-1} c_{j,k}^{(i)} \psi^{(i)}(2^j \cdot - k),$$

where

$$C_k = \int_{\mathbf{R}^d} f(x) \phi(x - k) dx$$

and

$$c_{j,k}^{(i)} = 2^{dj} \int_{\mathbf{R}^d} f(x) \psi^{(i)}(2^j x - k) dx. \quad (4)$$

Let us remark that we do not choose the (usual) L^2 normalization for the wavelets, but rather an L^∞ normalization, which is better fitted to the study of the Hölder regularity. Expressions such as (4) can make sense in more general settings (e.g. if f is a distribution). Hereafter, we will assume that the wavelets belong to $C^\gamma(\mathbf{R}^d)$ with $\gamma \geq \alpha + 1$, and that the functions $\{\partial^s \phi\}_{|s| \leq \gamma}$, $\{\partial^s \psi\}_{|s| \leq \gamma}$ have fast decay. Moreover, for the sake of simplicity, when dealing with the S^ν spaces, we will suppose that the application f is defined on the torus $\mathbf{T}^d = \mathbf{R}^d / \mathbf{Z}^d$ [20]. Let

$$\Lambda = \{(i, j, k) : 1 \leq i < 2^d, j \in \mathbf{N}, k \in \{0, \dots, 2^j - 1\}^d\} .$$

If $(i, j, k) \in \Lambda$, the periodized wavelets

$$\psi_p^{(i)}(2^j \cdot - k) = \sum_{l \in \mathbf{Z}^d} \psi^{(i)}(2^j(\cdot - l) - k)$$

form a basis of the one-periodic functions of $L^2([0, 1]^d)$ [9]. Such a setting is by no mean a restriction since we are interested in local properties; moreover “real-life” data are compactly supported. We will denote $(c_{j,k}^{(i)})_{(i,j,k) \in \Lambda}$ or $(c_\lambda)_{\lambda \in \Lambda}$ the wavelet coefficients of a function belonging to $L^2([0, 1]^d)$.

Let us now introduce the S^ν spaces.

Definition 1. For a sequence $c = (c_\lambda)_{\lambda \in \Lambda}$, $C > 0$ and $h \in \mathbf{R}$, let us set

$$E_j(C, h)[c] = \{(i, k) : |c_{j,k}^{(i)}| \geq C2^{-jh}\}.$$

The wavelet profile ν_c of c is defined as

$$\nu_c(h) = \lim_{\epsilon \rightarrow 0^+} \limsup_{j \rightarrow \infty} \frac{\log_2 \#E_j(1, h + \epsilon)[c]}{j},$$

with $h \in \mathbf{R}$.

The wavelet profile gives, in some way, the asymptotic behavior of the number of coefficients of c larger than a given order of magnitude. It can be shown that if c represents the wavelet coefficients of a function f , ν_c does not depend on the chosen wavelet basis [20]. Therefore one can set

$E_j(C, h)[f] = E_j(C, h)[c]$ and $\nu_f = \nu_c$. Clearly, ν_f is non-decreasing, right-continuous and there exists $h_{\min} > 0$ such that

$$\nu_f(h) \in \begin{cases} [0, d] & \text{if } h \geq h_{\min} \\ \{-\infty\} & \text{if } h < h_{\min} \end{cases} .$$

A function with such properties is called an admissible profile.

Definition 2. *Let ν be an admissible profile; f belongs to S^ν if one has $\nu_f(h) \leq \nu(h)$ for any $h \in \mathbf{R}$.*

There exists a distance dist on S^ν such that (S^ν, dist) is a complete separable metric space [4, 3].

Let us now introduce the multifractal formalism associated to the S^ν spaces. Let

$$h_{\max} = \inf_{h \geq h_{\min}} \frac{h}{\nu_f(h)}$$

and define the function d_f^ν as

$$d_f^\nu(h) = \begin{cases} h \sup_{h' \in (0, h]} \frac{\nu_f(h')}{h'} & \text{if } h \leq h_{\max} \\ 1 & \text{otherwise} \end{cases} .$$

This formalism provides an upper bound for the singularity spectrum: for all $f \in S^\nu$, one has $d_f(h) \leq d_f^\nu(h)$ for any $h \geq 0$. Moreover, it allows to recover the increasing part of many non-concave spectra [24].

2.2. Prevalence of multifractal functions in S^ν

In infinite dimensional Banach spaces, there is no σ -finite translation invariant measure. In [5], Christensen turned a characterization of Lebesgue measure zero Borel sets into a definition in the setting of complete metric vector spaces (see also [17]).

Definition 3. *Let E be a complete metric vector space. A Borel set $B \subset E$ is Haar-null if there exists a Borel probability measure μ , strictly positive on some compact set $K \subset E$ such that $\mu(B + x) = 0$ for any $x \in E$.*

A subset of E is Haar-null if it is included in a Haar-null Borel subset of E . The complement of a Haar-null set is called a prevalent set.

In [3], it has been proved that, as soon as ν is non-trivial, typical elements of S^ν are multifractal. More precisely, it can be shown that the set

$$\{f \in S^\nu : d(h) = d_f^\nu(h) \text{ if } h \leq h_{\max} \text{ and } d(h) = -\infty \text{ otherwise}\}$$

is prevalent. Moreover, the following regularity result also holds: the set

$$\{f \in S^\nu : \text{the Hölder exponent is Lebesgue-almost everywhere } h_{\max}\}$$

is prevalent.

3. Local Hölder regularity and irregularity

We introduce here a notion of local irregularity, which will be used to state that a generic weak irregularity condition holds for elements of S^ν . To this end, we first need to recall different concepts of global regularity, based on the uniform Hölder spaces.

If Ω is an open subset of \mathbf{R}^d , for any $h \in \mathbf{R}^d$, we will denote by Ω_h the set

$$\Omega_h = \{x \in \mathbf{R}^d : [x, x + ([\alpha] + 1)h] \subset \Omega\},$$

where the value $\alpha > 0$ will be implied by the context and $[\alpha]$ denotes the greatest integer lower than α (with this definition, if α is integer, $[\alpha] = \alpha - 1$). We will need the classical notion of finite difference.

Definition 4. Let $x, h \in \mathbf{R}^d$ and $f : \mathbf{R}^d \rightarrow \mathbf{C}$; the first order difference of f is $\Delta_h^1 f(x) = f(x + h) - f(x)$. For $n \geq 2$, the difference of order n is defined by

$$\Delta_h^n f(x) = \Delta_h^{n-1} \Delta_h^1 f(x).$$

3.1. Uniform Hölder and irregularity spaces

We can now introduce the uniform Hölder spaces $\Lambda^\alpha(\Omega)$ and the irregularity spaces $I^\alpha(\Omega)$.

Definition 5. Let Ω be an open subset of \mathbf{R}^d and $\alpha > 0$; a bounded function f defined on \mathbf{R}^d belongs to $\Lambda^\alpha(\Omega)$ if there exist $C, r_0 > 0$ such that for any $r \leq r_0$,

$$\sup_{|h| \leq r} \|\Delta_h^{[\alpha]+1} f\|_{L^\infty(\Omega_h)} \leq Cr^\alpha. \quad (5)$$

A function f is said to be uniformly Hölderian on Ω if for some $\alpha > 0$, f belongs to $\Lambda^\alpha(\Omega)$. In the above definition, we did not use the polynomial characterization of the Hölder spaces as in (3), but rather the finite differences characterization, which is equivalent (see e.g. [25]) but allows to introduce the irregularity spaces in a more natural way, since the polynomial refers to the regular part of the considered function.

Let us remark that the statement $f \notin \Lambda^\alpha(\Omega)$ means that for any $C > 0$, there exists a sequence $(r_n)_n$ (depending on C) decreasing to zero for which

$$\sup_{|h| \leq r_n} \|\Delta_h^{[\alpha]+1} f\|_{L^\infty(\Omega_h)} \geq Cr_n^\alpha.$$

We need a stronger notion of irregularity.

Definition 6. Let Ω be an open subset of \mathbf{R}^d and $\alpha > 0$; a bounded function f defined on \mathbf{R}^d belongs to $\Lambda_w^\alpha(\Omega)$ if there exist $C > 0$ and a sequence $(r_n)_n$ decreasing to zero such that

$$\sup_{|h| \leq r_n} \|\Delta_h^{[\alpha]+1} f\|_{L^\infty(\Omega_h)} \leq Cr_n^\alpha. \quad (6)$$

for any n .

A function f is said to be weakly uniformly Hölder on Ω with exponent α if f belongs to $\Lambda_w^\alpha(\Omega)$. Of course, since the sequence is decreasing to zero, the constant C is of no importance in (6). We are naturally led to the following notion of irregularity.

Definition 7. *Let Ω be an open subset of \mathbf{R}^d and $\alpha > 0$; a bounded function f defined on \mathbf{R}^d belongs to $I^\alpha(\Omega)$ if and only if $f \notin \Lambda_w^\alpha(\Omega)$.*

It is easy to check that $f \in I^\alpha(\Omega)$ means that there exist $C, r_0 > 0$ such that for any $r \leq r_0$, we have

$$\sup_{|h| \leq r} \|\Delta_h^{[\alpha]+1} f\|_{L^\infty(\Omega_h)} \geq Cr^\alpha,$$

which has to be compared with (5). The utility of these spaces is shown in [7, 8, 6].

3.2. Local irregularity exponents

As for the (pointwise) Hölder exponent h_f , one can define the uniform Hölder exponent of a bounded f on an open set Ω as follows:

$$H_f(\Omega) = \sup\{\alpha > 0 : f \in \Lambda^\alpha(\Omega)\}.$$

In the same order of idea, let us introduce a similar exponent, corresponding to the irregular counterpart of the uniform Hölder spaces.

Definition 8. *The upper uniform Hölder exponent of a bounded function f on an open set Ω is defined as*

$$\mathcal{H}_f(\Omega) = \inf\{\alpha > 0 : f \in I^\alpha(\Omega)\} = \sup\{\alpha > 0 : f \in \Lambda_w^\alpha(\Omega)\}.$$

We obviously have $H_f(\Omega) \leq \mathcal{H}_f(\Omega)$. The equality is satisfied for functions such as the Weierstraß function [16] and many space-filling functions [23]. Moreover, the set of elements of $\Lambda^H(\mathbf{R}^d)$ for which this equality is satisfied, i.e. the set $\Lambda^H(\mathbf{R}^d) \cap I^H(\mathbf{R}^d)$, is prevalent [7]. Typical elements of $\Lambda^H(\mathbf{R}^d)$ are thus monofractal in a strong sense.

To define the local irregularity exponent, we use the same approach as in [27].

Definition 9. *A sequence $(\Omega_n)_n$ of open subsets of \mathbf{R}^d is decreasing to $x_0 \in \mathbf{R}^d$ if*

- $m < n$ implies $\Omega_n \subset \Omega_m$,
- $|\Omega_n| \rightarrow 0$ as $n \rightarrow \infty$,
- $\bigcap_n \Omega_n = \{x_0\}$.

The following lemma is needed.

Lemma 1. *If $(\Omega_n)_n$ and $(\Omega'_n)_n$ are two sequences of open sets that decrease to x_0 , then*

$$\sup_{n \in \mathbf{N}} \mathcal{H}_f(\Omega_n) = \sup_{n \in \mathbf{N}} \mathcal{H}_f(\Omega'_n).$$

Proof. Let us first remark that if U and V are two open sets satisfying $U \subset V$, then $\mathcal{H}_f(V) \leq \mathcal{H}_f(U)$. Let us now suppose that

$$\sup_{n \in \mathbf{N}} \mathcal{H}_f(\Omega_n) > \sup_{n \in \mathbf{N}} \mathcal{H}_f(\Omega'_n).$$

There exists an index n_1 such that $\mathcal{H}_f(\Omega_{n_1}) > \sup_{n \in \mathbf{N}} \mathcal{H}_f(\Omega'_n)$. Now let $r > 0$ be such that $B(x_0, r) \subset \Omega_{n_1}$; since $(\Omega'_n)_n$ is decreasing to x_0 , there exists an index n_2 such that $\Omega'_{n_2} \subset B(x_0, r)$. One thus have

$$\mathcal{H}_f(\Omega'_{n_2}) \geq \mathcal{H}_f(\Omega_{n_1}) > \sup_{n \in \mathbf{N}} \mathcal{H}_f(\Omega'_n),$$

which leads to a contradiction. \square

Definition 10. *If f is a bounded function, the local irregularity exponent of f at x_0 is*

$$\mathcal{H}_f(x_0) = \sup_n \mathcal{H}_f(\Omega_n),$$

where $(\Omega_n)_n$ is a sequence of open sets decreasing to x_0 .

In the same way, the local regularity exponent of a bounded function f at x_0 is defined as $H_f(x_0) = \sup_n H_f(\Omega_n)$, where $(\Omega_n)_n$ is a sequence of open sets decreasing to x_0 . We still have $H_f(x_0) \leq \mathcal{H}_f(x_0)$.

4. Wavelets, irregularity spaces and irregularity exponents

In this section, we establish some technical results that will be needed in the last section. We first give necessary and sufficient conditions for a function to belong to $I^\alpha(\Omega)$. Next, we “characterize” the local irregularity exponent in terms of wavelet coefficients, under a strong uniform regularity hypothesis.

In what follows, we will assume that the multiresolution analysis is compactly supported (see [10]). The following result is shown in [21]: in \mathbf{R} , if the wavelet basis belongs to $C^M(\mathbf{R})$, there exists a fast decaying function Ψ_M such that $\psi = \Delta_{1/2}^M \Psi_M$. In \mathbf{R}^d , we will use the tensor product wavelet basis (see [32, 11]),

$$\psi^{(i)}(x) = \Psi^{(1)}(x_1) \cdots \Psi^{(d)}(x_d),$$

where $\Psi^{(i)}$ ($i \in \{1, \dots, d\}$) are either ψ or ϕ , but at least one of them must equal ψ . We will always suppose that $\Psi^{(1)} = \psi$. We will also use the following notations: given $j \in \mathbf{N}$, Ω an open subset of \mathbf{R}^d and a family of wavelets $\psi_{j,k}^{(i)}$, we set

$$\Gamma_j(\Omega) = \{(i, k) : \text{supp}(\psi_{j,k}^{(i)}) \subset \Omega\}$$

and

$$\|c_\lambda\|_j^\Omega = \sup_{\Gamma_j(\Omega)} |c_{j,k}^{(i)}|.$$

Finally, the natural number M will always refers to the quantity $M = [\alpha] + 1$.

4.1. Wavelet and uniform irregularity

The uniform regularity of a function is related to the decay rate of its wavelet coefficients (see [32]). Let f be a bounded function and $\alpha \in (0, 1)$; f belongs to $\dot{\Lambda}^\alpha(\Omega)$, where $\dot{\Lambda}^\alpha(\Omega)$ denotes the homogeneous version of the Hölder space $\Lambda^\alpha(\Omega)$ (roughly speaking, one has to work on the homogeneous version because wavelets have vanishing moments [32]) if and only if there exists $C > 0$ such that for any $j \geq 0$,

$$\|c_\lambda\|_j^\Omega \leq C2^{-\alpha j}. \quad (7)$$

The following result gives a sufficient and a necessary condition for a function of $\Lambda^\alpha(\Omega)$ to belong to $I^\alpha(\Omega)$.

Theorem 1. *Let $\alpha > 0$, $f \in \Lambda^\alpha(\Omega)$. If there exists $C > 0$ and $\gamma > 1$ such that*

$$\max\left\{\sup_{j \leq l \leq j + \log_2 j} \|c_\lambda\|_l^\Omega, 2^{-jM} \sup_{j - \log_2 j \leq l \leq j} (2^{lM} \|c_\lambda\|_l^\Omega)\right\} \geq C2^{-j\alpha} j^\gamma, \quad (8)$$

for any $j \geq 0$, then $f \in I^\alpha(\Omega)$.

Now, if f belongs to $I^\alpha(\Omega)$, there exist $C > 0$ and $\beta \in (0, 1)$ such that for any integer $j \geq 0$,

$$\max\left\{\sup_{j \leq l \leq j + \log_2 j} \|c_\lambda\|_l^\Omega, 2^{-jM} j^\beta \sup_{j - \log_2 j \leq l \leq j} (2^{lM} \|c_\lambda\|_l^\Omega)\right\} \geq C2^{-j\alpha}. \quad (9)$$

Proof. To prove the first part of the theorem, let us suppose that $f \in \Lambda_w^\alpha(\mathbf{R}^d)$ and let $C > 0$. As shown in [8], there exists some increasing sequence of integers $(j_n)_n$ such that for any $n \in \mathbf{N}$ and any $j \geq j_n$,

$$\sup_{|h| \leq 2^{-j}} \|\Delta_h^M f\|_{L^\infty(\Omega_h)} \leq C2^{-j_n\alpha}. \quad (10)$$

Let us show that this inequality leads to a contradiction.

By definition of the wavelet coefficients, we have for any $j \geq 0$ and any $(i, k) \in \Gamma_j(\Omega)$,

$$\begin{aligned} c_{j,k}^{(i)} &= 2^{jd} \int_{\mathbf{R}^d} f(x) \Psi^{(1)}(2^j x_1 - k_1) \cdots \Psi^{(d)}(2^j x_d - k_d) dx \\ &= 2^{jd} \int_{\mathbf{R}^d} f(x) \Delta_{1/2}^M \Psi_M(2^j x_1 - k_1) \cdots \Psi^{(d)}(2^j x_d - k_d) dx \\ &= 2^{jd} \int_{\mathbf{R}^d} \Delta_{1/2^{j+1}e_1}^M f(x) \Psi_M(2^j x_1 - k_1) \cdots \Psi^{(d)}(2^j x_d - k_d) dx, \end{aligned}$$

where $e_1 = (1, 0, \dots, 0)$. By definition of $\Gamma_j(\Omega)$, the properties of Ψ_M allow to write, for any $n \in \mathbf{N}$, any $j \geq j_n$ and any $(i, k) \in \Gamma_j(\Omega)$,

$$\begin{aligned} |c_{j,k}^{(i)}| &\leq 2^{jd} \int_{\Omega} |\Delta_{1/2^{j+1}e_1}^M f(x)| |\Psi_M(2^j x_1 - k_1) \cdots \Psi^{(d)}(2^j x_d - k_d)| dx \\ &\leq C2^{jd} 2^{-j_n\alpha} \int_{\mathbf{R}^d} |\Psi_M(2^j x_1 - k_1) \cdots \Psi^{(d)}(2^j x_d - k_d)| dx \\ &= C2^{-j_n\alpha} \|\Psi_M \otimes \cdots \otimes \Psi^{(d)}\|_{L^1(\mathbf{R}^d)}, \end{aligned}$$

thanks to inequality (10).

For $n \in \mathbf{N}$, let $l_n = j_n + \gamma[\log_2 j_n]$; for n sufficiently large, one has $l_n - \log_2 l_n \geq j_n$. Therefore, the following relations hold for n sufficiently large (and any $(i, k) \in \Gamma_j(\Omega)$):

$$\sup_{l_n \leq j \leq l_n + \log_2 l_n} |c_{j,k}^{(i)}| \leq C 2^{-j_n \alpha} \leq C 2^{-l_n \alpha} l_n^{\gamma'}$$

and

$$\sup_{l_n - \log_2 l_n \leq j \leq l_n} 2^{jM} |c_{j,k}^{(i)}| \leq C 2^{l_n M} 2^{-j_n \alpha} \leq C 2^{l_n M} 2^{-l_n \alpha} l_n^{\gamma'},$$

for some $\gamma' < \gamma$, which is in contradiction with inequality (8).

To prove the second part of the theorem we will use the following result (see [32]): Let $f \in \Lambda^\gamma(\Omega)$; since $\Lambda^\gamma(\Omega) \subset B_{\infty, \infty}^0(\Omega) \cap \dot{\Lambda}^\gamma(\Omega)$, the wavelet characterizations of these two functional spaces lead to the existence of a constant $C > 0$ that does not depend on the function such that for any $h \in \mathbf{R}^d$ and any $x \in \Omega_h$, one has

$$|\Delta_h^{[\gamma]+1} f(x)| \leq C \sup_{j \in \mathbf{N}} \|c_\lambda\|_j^\Omega, \quad (11)$$

and

$$|\Delta_h^{[\gamma]+1} f(x)| \leq C |h|^\gamma \sup_{j \in \mathbf{N}} \{2^{j\gamma} \|c_\lambda\|_j^\Omega\}. \quad (12)$$

Let us assume $f \in I^\alpha(\Omega)$ and suppose that property (9) is not satisfied. In this case, for any $C > 0$ and any $\beta \in (0, 1)$, there exists an increasing sequence of integers $(j_n)_n$ such that, for any $n \in \mathbf{N}$,

$$\max \left\{ \sup_{j_n \leq l \leq j_n + \log_2 j_n} \|c_\lambda\|_l^\Omega, \frac{2^{-j_n M}}{2^{-j_n M} j_n^\beta} \sup_{j_n - \log_2 j_n \leq l \leq j_n} (2^{lM} \|c_\lambda\|_l^\Omega) \right\} \leq C 2^{-j_n \alpha}. \quad (13)$$

Let us fix $C > 0$, $x \in \Omega_h$ and let $n_0 \in \mathbf{N}$, $h \in \mathbf{R}^d$ be such that $|h| \leq 2^{-j_{n_0}}$. We just have to show that $f \in \Lambda_w^\alpha(\Omega)$. We will use the following notations:

$$f_{-1} = \sum_{k \in \mathbf{Z}^d} C_k \phi(\cdot - k), \quad f_j = \sum_{i=1}^{2^d-1} \sum_{k \in \mathbf{Z}^d} c_{j,k}^{(i)} \psi(2^j \cdot - k),$$

with $j \geq 0$. Since f is uniformly Hölder, f_j and $\sum_{j \geq -1} f_j$ converge uniformly on any compact set and

$$\Delta_h^M f = \sum_{j \geq -1} \Delta_h^M f_j.$$

We first consider the function $g_1 = \sum_{j=0}^{j_{n_0}} f_j$. Let us fix $\gamma \in ([\alpha] + 1 - \beta, [\alpha] + 1)$. The regularity of the wavelets and property (12) imply the existence of a constant $C > 0$ independent of n_0 , x and h such that

$$|\Delta_h^M g_1(x)| \leq C |h|^\gamma \sup_{l \leq j_{n_0}} (2^{l\gamma} \|c_\lambda\|_l^\Omega). \quad (14)$$

Since $f \in \Lambda^\alpha(\Omega)$, there exists a constant such that, for any $n \in \mathbf{N}$,

$$\sup_{l \leq j_{n_0} - \log_2 j_{n_0}} (2^{l\gamma} \|c_\lambda\|_l^\Omega) \leq C 2^{(j_{n_0} - \log_2 j_{n_0})(\gamma - \alpha)} = C' \frac{2^{j_{n_0}(\gamma - \alpha)}}{j_{n_0}^{\gamma - \alpha}}.$$

On the other hand, we have, using relation (13),

$$\begin{aligned} & \sup_{j_{n_0} - \log_2 j_{n_0} \leq l \leq j_{n_0}} (2^{l\gamma} \|c_\lambda\|_l^\Omega) \\ &= \sup_{j_{n_0} - \log_2 j_{n_0} \leq l \leq j_{n_0}} (2^{l(\gamma - M)} 2^{lM} \|c_\lambda\|_l^\Omega) \\ &\leq C 2^{(j_{n_0} - \log_2 j_{n_0})(\gamma - M)} 2^{j_{n_0}(M - \alpha)} / j_{n_0}^\beta \\ &= C' j_{n_0}^{M - \gamma} 2^{j_{n_0}(\gamma - \alpha)} / j_{n_0}^\beta \\ &\leq C' 2^{j_{n_0}(\gamma - \alpha)}. \end{aligned}$$

This implies that for n_0 sufficiently large (since $0 < M - \gamma \leq \beta$),

$$\sup_{l \leq j_{n_0}} (2^{l\gamma} \|c_\lambda\|_l^\Omega) \leq C 2^{j_{n_0}(\gamma - \alpha)},$$

and hence, using inequality (14),

$$|\Delta_h^M g_1(x)| \leq C 2^{-j_{n_0}\alpha}, \quad (15)$$

since h has been chosen adequately.

Let us now consider $g_2 = \sum_{j > j_{n_0}} f_j$. Property (11) applied to g_2 directly gives the following relation:

$$|\Delta_h^M g_2| \leq C \sup_{l > j_{n_0}} (\|c_\lambda\|_l^\Omega). \quad (16)$$

Once again, since $f \in \Lambda^\alpha(\Omega)$, we have

$$\sup_{l \geq j_{n_0} + \log_2 j_{n_0}} \|c_\lambda\|_l^\Omega \leq C 2^{-(j_{n_0} + \log_2 j_{n_0})\alpha} = C' \frac{2^{-j_{n_0}\alpha}}{j_{n_0}^\alpha}.$$

Now, relation (13) implies

$$\sup_{j_{n_0} \leq l \leq j_{n_0} + \log_2 j_{n_0}} \|c_\lambda\|_l^\Omega \leq C 2^{-j_{n_0}\alpha}.$$

These last inequalities lead to the following relation for n_0 sufficiently large,

$$|\Delta_h^M g_2(x)| \leq C 2^{-j_{n_0}\alpha}, \quad (17)$$

thanks to relation (16).

Putting relations (15) and (17) together, we obtain

$$|\Delta_h^M f(x)| = |\Delta_h^M (g_1 + g_2)(x)| \leq C 2^{-j_{n_0}\alpha},$$

for n_0 sufficiently large, that is $f \in \Lambda_w^\alpha(\Omega)$, which is impossible, since f belongs to $I^\alpha(\Omega)$. \square

Let us remark that this result is far from being a characterization of the space $I^\alpha(\Omega)$, since the hypothesis $f \in \Lambda^\alpha(\Omega)$ is essential to prove the sufficiency of the condition.

4.2. Wavelets and local irregularity exponents

The preceding result leads to the following corollary about the irregularity exponent.

Corollary 1. *Let $\alpha > 0$; if $f \in \Lambda^\alpha(\Omega)$, then the irregularity exponent of f on Ω equals α if and only if*

$$\lim_{j \rightarrow \infty} -j^{-1} \log_2 \max \left\{ \sup_{j \leq l \leq j + \log_2 j} \|c_\lambda\|_l^\Omega, \right. \\ \left. 2^{-jM} \sup_{j - \log_2 j \leq l \leq j} (2^{lM} \|c_\lambda\|_l^\Omega) \right\} = \alpha.$$

Proof. Theorem 1 directly yields that if $f \in \Lambda^\alpha(\Omega)$, the irregularity exponent of f on Ω equals α if and only if

$$\limsup_{j \rightarrow \infty} \frac{\log_2 \max \left\{ \sup_{j \leq l \leq j + \log_2 j} \|c_\lambda\|_l^\Omega, 2^{-jM} \sup_{j - \log_2 j \leq l \leq j} (2^{lM} \|c_\lambda\|_l^\Omega) \right\}}{-j} = \alpha.$$

We have to prove that the upper limit can be replaced with a limit. Since $f \in \Lambda^\alpha(\Omega)$, relation (7) implies that

$$\liminf_{j \rightarrow \infty} \frac{\log_2 \max \left\{ \sup_{j \leq l \leq j + \log_2 j} \|c_\lambda\|_l^\Omega, 2^{-jM} \sup_{j - \log_2 j \leq l \leq j} (2^{lM} \|c_\lambda\|_l^\Omega) \right\}}{-j}$$

is always larger than

$$\liminf_{j \rightarrow \infty} \frac{\log_2 \max \left\{ \sup_{j \leq l \leq j + \log_2 j} 2^{-l\alpha}, 2^{-jM} \sup_{j - \log_2 j \leq l \leq j} 2^{l(M-\alpha)} \right\}}{-j} = \alpha,$$

which is sufficient to conclude. \square

We can now state the local version of the previous result.

Theorem 2. *Let $\alpha > 0$; if $f \in \Lambda^\alpha(\Omega)$ and $x_0 \in \mathbf{R}^d$, then*

$$\mathcal{H}(x_0) = \alpha$$

if and only if

$$\lim_{r \rightarrow 0} \lim_{j \rightarrow \infty} -j^{-1} \log_2 \max \left\{ \sup_{j \leq l \leq j + \log_2 j} \|c_\lambda\|_l^{B(x_0, r)}, \right. \\ \left. 2^{-jM} \sup_{j - \log_2 j \leq l \leq j} (2^{lM} \|c_\lambda\|_l^{B(x_0, r)}) \right\} = \alpha.$$

The usefulness of the previous result may not seem obvious at first glance, but the prevalent result obtained in the next section relies on this theorem.

5. A prevalent result on the S^ν spaces

We prove here a prevalent result that gives more insight into the regularity of the elements of S^ν . To obtain it, we first need to introduce the method we will use; we will also need some properties obtained in [3].

5.1. The stochastic process technique

Our result concerning the prevalence relies on the stochastic process technique. Let us recall that a random element X on a complete metric space E is a measurable mapping X defined on a probability space (Ω, \mathcal{A}, P) with values in E . Given a random element X on E , one can define a probability measure on E by the formula

$$P_X(A) = P\{X \in A\}.$$

Replacing measure μ in definition 3 of a Haar-null set with P_X , we see that in order to prove that a set is Haar-null, it is sufficient to check that for any $f \in E$,

$$P_X(A + f) = 0.$$

We will define a stochastic process with a random wavelet series associated in a proper way to ν . To this end, for each $j \geq 0$, let us define as in [3],

$$F_j(h) = \begin{cases} 0 & \text{if } h < h_{\min} \\ 2^{-jd} \sup\{j^2, 2^{j\nu(h)}\} & \text{if } h \geq h_{\min} \end{cases}, \quad (18)$$

with

$$h_{\min} = \inf\{h : \nu(h) \geq 0\}.$$

Since F_j is non-decreasing and piecewise continuous, it is the repartition function of some probability distribution associated to a probability law ρ_j supported on $[h_{\min}, \infty]$.

The following remark is made in [3]: If ρ_j is the probability distribution whose repartition function F_j is defined by (18), then there exists some sequence of random numbers $(c_\lambda)_{\lambda \in \Lambda}$ with independent phase and moduli such that for any j , ρ_j is the common law of $-\log_2 |c_{j,k}^{(i)}|/j$ and satisfies the two following conditions:

$$\lim_{j \rightarrow \infty} \frac{2^{jd} \rho_j((-\infty, h])}{j} = \nu(h), \quad (19)$$

for any $h \in \mathbf{R}$ and $h \geq h_{\min}$ implies

$$2^{jd} \rho_j((-\infty, h]) \geq j^2. \quad (20)$$

Starting from these results, we will use the following random wavelet series associated to ν :

$$X_\nu = \sum_{\lambda \in \Lambda} c_\lambda \psi_\lambda. \quad (21)$$

It is shown in [4] that the metric topology d on S^ν makes it a Polish space, which is a very good framework for prevalence. Moreover, as proved in [3], the measure P_X is a Borel measure (relatively to this topology). In other words, we can use the stochastic process technique with X_ν .

5.2. Prevalent irregularity properties in S^ν

We are now ready to prove the following result.

Theorem 3. *The following set is prevalent in S^ν ,*

$$\{f : \forall x \in \mathbf{T}^d, \text{ the local irregularity exponent of } f \text{ at } x \text{ is } \mathcal{H}_f(x) = h_{\min}\},$$

where

$$h_{\min} = \inf\{h : \nu(h) \geq 0\}.$$

From what precedes, it is sufficient to show the following result.

Proposition 1. *Let X_ν be the random wavelet series defined by (21). Then, for any $f \in S^\nu$, the local irregularity exponent $\mathcal{H}_{f+X_\nu}(x)$ of $f + X_\nu$ at x is equal to h_{\min} almost surely.*

Proof. Since $S^\nu \subset \Lambda^{h_{\min}}(\mathbf{T}^d)$, for any $f \in S^\nu$, $f + X_\nu \in \Lambda^{h_{\min}}(\mathbf{T}^d)$ almost surely. Let us now fix $m \in \mathbf{N}$ and define, for any $r \in \mathbf{Z}^d$,

$$T_{r,m} = \prod_{n=1}^d \left(\frac{r_n}{2^m}, \frac{r_n + 1}{2^m} \right),$$

so that $\mathbf{T}^d = \cup_r \overline{T_{r,m}}$. We aim at showing that the equality

$$\mathcal{H}_{f+X_\nu}(T_{r,m}) = h_{\min}$$

holds almost surely for any r, m ; in this case, Theorem 2 directly yields the required result.

For $\lambda \in \Lambda$, we will denote c_λ the wavelet coefficients associated to X_ν and d_λ the wavelet coefficients associated to $f \in S^\nu$. Let us first remark that if, for some fixed $\lambda \in \Lambda$, $\Re(c_\lambda \overline{d_\lambda}) \leq 0$ (where we denote $\Re(c_\lambda \overline{d_\lambda})$ the real part of the complex number $c_\lambda \overline{d_\lambda}$) i.e. c_λ is in the complex half-plane opposite to d_λ , then

$$|c_\lambda - d_\lambda| \geq |c_\lambda|.$$

Therefore, for any $\lambda \in \Lambda$,

$$P(f : |c_\lambda - d_\lambda| \geq |c_\lambda|) \geq P(f : \Re(c_\lambda \overline{d_\lambda}) \leq 0) \geq 1/2. \quad (22)$$

If we define for any $N \in \mathbf{N}$ and $\lambda = (i, j, k) \in \Lambda$,

$$A_{N,\lambda} = \{f : \exists k' \in k + [0, N]^d : |c_{j,k'}^{(i)} - d_{j,k'}^{(i)}| \geq |c_{j,k'}^{(i)}|\},$$

then, thanks to the independence of the wavelet coefficients of X_ν and inequality (22), we have

$$P(A_{N,\lambda}) \geq 1 - 2^{-N^d}$$

for any N and λ .

Now, for $n \in \mathbf{N}$, let us set

$$B_{r,m,n} = \{f : \|c_\lambda\|_j^{T_{r,m}} \geq 2^{-j(h_{\min} + 1/n)}\}.$$

If for any i , we have $\text{supp}(\psi^{(i)}) \subset [0, M]^d$, then $\text{supp}(\psi_\lambda) \subset T_{r,m}$ if and only if, for any $l \in \{1, \dots, d\}$, we have $r_l 2^{j-m} \leq k_l \leq (r_l + 1) 2^{j-m} - M$. Therefore, for any fixed $j \geq m$,

$$\begin{aligned} P(B_{r,m,n}) &= 1 - (1 - 2^{j(\nu(h_{\min}+1/n)-d)})^{2^{d(j-m)}} \\ &\geq 1 - \exp(-2^{-md} 2^{j\nu(h_{\min}+1/n)}). \end{aligned}$$

Let us choose $N = 2^{j-m}$; for any m, n and any $j \geq m$, we have

$$\begin{aligned} P(f : \bigcap_r \|c_\lambda - d_\lambda\|_j^{T_{r,m}} \geq 2^{-j(h_{\min}+1/n)}) \\ &\geq (P(f : (\bigcup_{(i,k) \in \Gamma_j(\Omega)} A_{2^{j-m}, \lambda} \cap B_{2r, m+1, n})))^{2^m} \\ &\geq (1 - 2^{-2^{d(j-m)}} - \exp(-2^{-md} 2^{j\nu(h_{\min}+1/n)}))^{2^{md}}. \end{aligned}$$

Moreover, for any n ,

$$\sum_m \sum_{j \geq m} (1 - (1 - 2^{-2^{d(j-m)}} - \exp(-2^{-md} 2^{j\nu(h_{\min}+1/n)}))^{2^{md}}) < \infty.$$

Therefore, the Borel-Cantelli lemma implies that for any n , there exists $m_0 \in \mathbf{N}$ such that, for any $m \geq m_0$, any r and any $j \geq m$, the inequality

$$\|c_\lambda - d_\lambda\|_j^{T_{r,m}} \geq 2^{-j(h_{\min}+1/n)} \quad (23)$$

holds almost surely. Using inequality (23), we see that for any $n \in \mathbf{N}$,

$$\begin{aligned} \max\{ \sup_{j \leq l \leq j + \log_2 j} \|c_\lambda - d_\lambda\|_l^{T_{r,m}}, \\ 2^{-j([h_{\min}] + 1)} \sup_{j - \log_2 j \leq l \leq j} (2^{l([h_{\min}] + 1)} \|c_\lambda - d_\lambda\|_l^{T_{r,m}}) \} \end{aligned}$$

is larger than $2^{-j(h_{\min}+1/n)}$ for any r, m almost surely. Moreover, since $X_\nu + f \in \Lambda^{h_{\min}}(\mathbf{T}^d)$ almost surely, we also almost surely have that, for any j and any r, m ,

$$\begin{aligned} \max\{ \sup_{j \leq l \leq j + \log_2 j} \|c_\lambda - d_\lambda\|_l^{T_{r,m}}, \\ 2^{-j([h_{\min}] + 1)} \sup_{j - \log_2 j \leq l \leq j} (2^{l([h_{\min}] + 1)} \|c_\lambda - d_\lambda\|_l^{T_{r,m}}) \} \end{aligned}$$

is lower than $2^{-j h_{\min}}$. These two bounds allows us to say that for any $n \in \mathbf{N}$, the quantity

$$\begin{aligned} \lim_{j \rightarrow \infty} -j^{-1} \log_2 \max\{ \sup_{j \leq l \leq j + \log_2 j} \|c_\lambda - d_\lambda\|_l^{T_{r,m}}, \\ 2^{-j([h_{\min}] + 1)} \sup_{j - \log_2 j \leq l \leq j} (2^{l([h_{\min}] + 1)} \|c_\lambda - d_\lambda\|_l^{T_{r,m}}) \} \end{aligned}$$

almost surely belongs to $[h_{\min}, h_{\min} + 1/n]$, for any r, m . Since this relation is valid for any n , corollary 1 allows to conclude. \square

We can sum-up the regularity results of the S^ν spaces as follows:

Corollary 2. *If the admissible profile ν is not reduced to a single point, the set*

$$\{f \in S^\nu \quad : \quad h_f(x) = h_{\max} \text{ and } \mathcal{H}_f(x) = h_{\min} \\ \text{with } h_{\max} \neq h_{\min} \text{ for Lebesgue-almost every } x\}$$

is prevalent.

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